BIOMATH
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Biomath Forum
ORIGINAL ARTICLE

On the cyclic DNA codes over the finite rings $\mathbb{Z}_4 + w\mathbb{Z}_4$ and $\mathbb{Z}_4 + w\mathbb{Z}_4 + v\mathbb{Z}_4 + wv\mathbb{Z}_4$

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Received: 8 May 2017, accepted: 16 December 2017, published: 22 December 2017

***Abstract*—The structures of the cyclic DNA codes of odd length over the finite rings $R = \mathbb{Z}_4 + w\mathbb{Z}_4$, $w^2 = 2$ and $S = \mathbb{Z}_4 + w\mathbb{Z}_4 + v\mathbb{Z}_4 + wv\mathbb{Z}_4, w^2 = 2, v^2 = v, wv = vw$ are studied. The links between the elements of the rings R, S and 16 and 256 codons are established, respectively. The cyclic codes of odd length over the finite ring R satisfy reverse complement constraint and the cyclic codes of odd length over the finite ring S satisfy reverse constraint and reverse complement constraint are studied. The binary images of the cyclic DNA codes over the finite rings R and S are determined. Moreover, a family of DNA skew cyclic codes over R is constructed, its property of being reverse complement is studied.**

***Keywords*-DNA codes; cyclic codes; skew cyclic codes.**

I. Introduction

DNA is formed by the strands and each strand is sequence consists of four nucleotides ; Adenine (A), Guanine (G), Thymine (T) and Cytosine (C). Two strands of DNA are linked with Watson-Crick Complement. This is as $\overline{A} = T$, $\overline{T} = A$, $\overline{G} = C$, $\overline{C} = G$. For example if $c = (ATCCG)$ then its complement is $\overline{c} = (TAGGC)$.

A code is called a DNA code if it satisfies some or all of the following conditions:

i) The Hamming contraint, for any two different codewords $c_1, c_2 \in C$, $H(c_1, c_2) \geq d$
ii) The reverse constraint, for any two different codewords $c_1, c_2 \in C$, $H(c_1, c_2^r) \geq d$
iii) The reverse complement constraint, for any two different codewords $c_1, c_2 \in C$, $H(c_1, c_2^{rc}) \geq d$
iv) The fixed GC content constraint, for any codeword $c \in C$ contains the some number of G and C element.

The purpose of the i)-iii) constraints is to avoid undesirable hybridization between different strands.

DNA computing were started by Leonhard Adleman in 1994, in [3]. The special error correct-

Citation: Abdullah Dertli, Yasemin Cengellenmis, On the cyclic DNA codes over the finite rings $\mathbb{Z}_4 + w\mathbb{Z}_4$ and $\mathbb{Z}_4 + w\mathbb{Z}_4 + v\mathbb{Z}_4 + wv\mathbb{Z}_4$, Biomath 6 (2017), 1712167,
http://dx.doi.org/10.11145/j.biomath.2017.12.167

ing codes over some finite fields and finite rings with 4^n elements where $n \in N$ were used for DNA computing applications.

In [12], the reversible codes over finite fields were studied, firstly. It was shown that $C = \langle f(x) \rangle$ is reversible if and only if $f(x)$ is a self reciprocal polynomial. In [1], they developed the theory for constructing linear and additive cyclic codes of odd length over $GF(4)$. In [13], they introduced a new family of polynomials which generates reversible codes over a finite field $GF(16)$.

In [2], the reversible cyclic codes of any length n over the ring $\mathbb{Z}_4$ were studied. A set of generators for cyclic codes over $\mathbb{Z}_4$ with no restrictions on the length n was found. In [17], the cyclic DNA codes over the ring $R = \{0, 1, u, 1+u\}$ where $u^2 = 1$ based on a similarity measure were constructed. In [9], the codes over the ring $F_2 + uF_2, u^2 = 0$ were constructed for using in DNA computing applications.

I. Siap et al. considered the cyclic DNA codes over the finite ring $F_2[u]/\left\langle u^2 - 1 \right\rangle$ in [18]. In [10], Liang and Wang considered the cyclic DNA codes over $F_2 + uF_2, u^2 = 0$. Yıldız and Siap studied the cyclic DNA codes over $F_2[u]/\left\langle u^4 - 1 \right\rangle$ in [20]. Bayram et al. considered codes over the finite ring $F_4 + vF_4, v^2 = v$ in [3]. Zhu and Chan studied the cyclic DNA codes over the non-chain ring $F_2[u,v]/\left\langle u^2, v^2 - v, uv - vu \right\rangle$ in [21]. In [6], Bennenni at al. studied the cyclic DNA codes over $F_2[u]/\left\langle u^6 \right\rangle$. Pattanayak et al. considered the cyclic DNA codes over the ring $F_2[u,v]/ < u^2 - 1, v^3 - v, uv - vu >$ in [15]. Pattanayak and Singh studied the cyclic DNA codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$ in [14].

J. Gao et al. studied the construction of the cyclic DNA codes by cyclic codes over the finite ring $F_4[u]/\left\langle u^2 + 1 \right\rangle$, in [11]. Also, the construction of DNA the cyclic codes has been discussed by several authors in [7,8,16].

We study families of DNA cyclic codes of the finite rings $\mathbb{Z}_4 + w\mathbb{Z}_4$, $w^2 = 2$ and $\mathbb{Z}_4 + w\mathbb{Z}_4 + v\mathbb{Z}_4 + wv\mathbb{Z}_4, w^2 = 2, v^2 = v, wv = vw$. The rest of the paper is organized as follows. In section 2, details about algebraic structure of the finite ring $\mathbb{Z}_4 + w\mathbb{Z}_4$, $w^2 = 2$ are given. We define a Gray map from R to $\mathbb{Z}_4$. In section 3, the cyclic codes of odd length over R satisfy the reverse complement constraint are determined. In section 4, the cyclic codes of odd length over S satisfy the reverse complement constraint and the reverse contraint are examined. A linear code over S is represented by means of two linear codes over R. In section 5, the binary image of cyclic DNA code over R is determined. In section 6, the binary image of cyclic DNA code over S is determined. In section 7, by using a non trivial automorphism, the DNA skew cyclic codes are introduced. In section 8, the design of linear DNA code is presented.

II. Preliminaries

The algebraic structure of the finite ring $R = \mathbb{Z}_4 + w\mathbb{Z}_4$, $w^2 = 2$ is given in [4]. R is the commutative, characteristic 4 ring $\mathbb{Z}_4 + w\mathbb{Z}_4 = \{a + wb : a, b \in \mathbb{Z}_4\}$ with $w^2 = 2$. R can also be thought of as the quotient ring $\mathbb{Z}_4[w]/\left\langle w^2 - 2 \right\rangle$. R is a principal ideal ring with 16 elements and finite chain ring. The units of the ring are

$$1, 3, 1+w, 3+w, 1+2w, 1+3w, 3+3w, 3+2w,$$

and the non-units are

$$0, 2, w, 2w, 3w, 2+w, 2+2w, 2+3w.$$

R has 4 ideals:

$$\begin{aligned}
\langle 0 \rangle &= \{0\}, \\
\langle 1 \rangle &= \langle 3 \rangle = \langle 1+3w \rangle = ... = R, \\
\langle w \rangle &= \{0, 2, w, 2w, 3w, 2+w, 2+2w, 2+3w\}, \\
&= \langle 3w \rangle = \langle 2+w \rangle = \langle 2+3w \rangle, \\
\langle 2w \rangle &= \{0, 2w\}, \\
\langle 2 \rangle &= \langle 2+2w \rangle = \{0, 2, 2w, 2+2w\}.
\end{aligned}$$

We have

$$\langle 0 \rangle \subset \langle 2w \rangle \subset \langle 2 \rangle \subset \langle w \rangle \subset R.$$

Moreover R is a Frobenious ring.

We define $\phi : R \longrightarrow \mathbb{Z}_4^2$ as

$$\phi(a + wb) = (a, b).$$

The Gray map is extended component wise to

$$\begin{aligned} \phi \quad &: \quad R^n \longrightarrow \mathbb{Z}_4^{2n} \\ (\alpha_1, \alpha_2, ..., \alpha_n), \quad &= \quad (a_1, ..., a_n, b_1, ..., b_n), \end{aligned}$$

where $\alpha_i = a_i + b_i w$ with $i = 1, 2, ..., n$. ϕ is a $\mathbb{Z}_4$ module isomorphism.

A linear code C of length n over R is an R-submodule of R^n. An element of C is called a codeword. A code of length n is cyclic if the code is invariant under the automorphism σ which is

$$\sigma(c_0, c_1, ..., c_{n-1}) = (c_{n-1}, c_0, ..., c_{n-2})$$

A cyclic code of length n over R can be identified with an ideal in the quotient ring $R[x]/\langle x^n - 1\rangle$ via the R–modul isomorphism

$$\begin{aligned} R^n \quad &\longrightarrow \quad R[x]/\langle x^n - 1\rangle \\ (c_0, c_1, ..., c_{n-1}) \quad &\longmapsto \quad c_0 + c_1 x + ... + c_{n-1} x^{n-1} \\ &\qquad + \langle x^n - 1\rangle \end{aligned}$$

Theorem 1: Let C be a cyclic code in $R[x]/\langle x^n - 1\rangle$ .Then there exists polynomials $g(x), a(x)$ such that $a(x)|g(x)|x^n - 1$ and $C = \langle g(x), wa(x)\rangle$.

The ring $R[x]/\langle x^n - 1\rangle$ is a principal ideal ring when n is odd. So, if n is odd, then there exists $s(x) \in R[x]/\langle x^n - 1\rangle$ such that $C = \langle s(x)\rangle$, in [4,19].

III. The reversible complement codes over R

In this section, we study the cyclic code of odd length over R satisfies the reverse complement constraint. Let $\{A, T, G, C\}$ represent the DNA alphabet. DNA occurs in sequences with represented by sequences of the DNA alphabet. DNA code of length n is defined as a set of the codewords $(x_0, x_1, ..., x_{n-1})$ where $x_i \in \{A, T, G, C\}$. These codewords must satisfy the four constraints which are mentioned in [21].

Since the ring R is of cardinality 16, we define the map ϕ which gives a one to one correspondence between the elements of R and the 16 codons over the alphabet $\{A, T, G, C\}^2$ by using the Gray map as follows

Elements	Gray images	DNA double pairs
0	$(0,0)$	AA
1	$(1,0)$	CA
2	$(2,0)$	GA
3	$(3,0)$	TA
w	$(0,1)$	AC
$2w$	$(0,2)$	AG
$3w$	$(0,3)$	AT
$1+w$	$(1,1)$	CC
$1+2w$	$(1,2)$	CG
$1+3w$	$(1,3)$	CT
$2+w$	$(2,1)$	GC
$2+2w$	$(2,2)$	GG
$2+3w$	$(2,3)$	GT
$3+w$	$(3,1)$	TC
$3+2w$	$(3,2)$	TG
$3+3w$	$(3,3)$	TT

The codons satisfy the Watson-Crick Complement.

Definition 2: For $x = (x_0, x_1, ..., x_{n-1}) \in R^n$, the vector $(x_{n-1}, x_{n-2}, ..., x_1, x_0)$ is called the reverse of x and is denoted by x^r. A linear code C of length n over R is said to be reversible if $x^r \in C$ for every $x \in C$.

For $x = (x_0, x_1, ..., x_{n-1}) \in R^n$, the vector $(\overline{x}_0, \overline{x}_1, ..., \overline{x}_{n-1})$ is called the complement of x and is denoted by x^c. A linear code C of length n over R is said to be complement if $x^c \in C$ for every $x \in C$.

For $x = (x_0, x_1, ..., x_{n-1}) \in R^n$, the vector $(\overline{x}_{n-1}, \overline{x}_{n-2}, ..., \overline{x}_1, \overline{x}_0)$ is called the reversible complement of x and is denoted by x^{rc}. A linear code C of length n over R is said to be reversible complement if $x^{rc} \in C$ for every $x \in C$.

Definition 3: Let $f(x) = a_0 + a_1 x + ... + a_t x^t \in R[x]$ ($S[x]$) with $a_t \neq 0$ be polynomial. The reciprocal of $f(x)$ is defined as $f^*(x) = x^t f(\frac{1}{x})$. It is easy to see that $\deg f^*(x) \leq \deg f(x)$ and if $a_0 \neq 0$, then $\deg f^*(x) = \deg f(x)$. $f(x)$ is called a self reciprocal polynomial if there is a constant m such that $f^*(x) = m f(x)$.

Lemma 4: Let $f(x), g(x)$ be polynomials in $R[x]$. Suppose $\deg f(x) - \deg g(x) = m$ then,

i) $(f(x)g(x))^* = f^*(x)g^*(x)$

ii) $(f(x)+g(x))^* = f^*(x)+x^m g^*(x)$

Lemma 5: For any $a \in R$, we have $a+\overline{a} = 3+3w$.

Lemma 6: If $a \in \{0,1,2,3\}$, then we have $(3+3w) - \overline{wa} = wa$.

Theorem 7: Let $C = \langle g(x), wa(x)\rangle$ be a cyclic code of odd length n over R. If $f(x)^{rc} \in C$ for any $f(x) \in C$, then $(1+w)(1+x+x^2+...+x^{n-1}) \in C$ and there are two constants $e, d \in \mathbb{Z}_4^*$ such that $g^*(x) = eg(x)$ and $a^*(x) = da(x)$.

Proof: Suppose that $C = \langle g(x), wa(x)\rangle$, where $a(x)|g(x)|x^n - 1 \in \mathbb{Z}_4[x]$. Since $(0,0,...,0) \in C$, then its reversible complement is also in C.

$$\begin{aligned}(0,0,...,0)^{rc} &= (3+3w, 3+3w, ..., 3+3w)\\ &= 3(1+w)(1,1,...,1) \in C.\end{aligned}$$

This vector corresponds of the polynomial

$$\begin{aligned}&(3+3w)+(3+3w)x+...+(3+3w)x^{n-1}\\ &= (3+3w)\frac{x^n-1}{x-1} \in C.\end{aligned}$$

Since $3 \in \mathbb{Z}_4^*$, then $(1+w)(1+x+...+x^{n-1}) \in C$.

Let $g(x) = g_0 + g_1x + ... + g_{s-1}x^{s-1} + g_sx^s$. Note that

$$\begin{aligned}g(x)^{rc} =& (3+3w)+(3+3w)x+...+(3+3w)x^{n-s-2}\\ &+\overline{g}_s x^{n-s-1}+...+\overline{g}_1 x^{n-2}+\overline{g}_0 x^{n-1} \in C.\end{aligned}$$

Since C is a linear code, then

$$3(1+w)(1+x+x^2+...+x^{n-1}) - g(x)^{rc} \in C$$

which implies that $((3+3w)-\overline{g}_s)x^{n-s-1} + ((3+3w)-\overline{g}_{s-1})x^{n-s-2}+...+((3+3w)-\overline{g}_0)x^{n-1} \in C$. By using $(3+3w)-\overline{a} = a$, this implies that

$$x^{n-s-1}(g_s+g_{s-1}x+...+g_0x^s) = x^{n-s-1}g^*(x) \in C$$

Since $g^*(x) \in C$, this implies that

$$g^*(x) = g(x)u(x) + wa(x)v(x)$$

where $u(x), v(x) \in \mathbb{Z}_4[x]$. Since $g_i \in \mathbb{Z}_4$, for $i = 0,1,...,s$, we have that $v(x) = 0$. As $\deg g^*(x) = \deg g(x)$, we have $u(x) \in \mathbb{Z}_4^*$. Therefore there is a constant $e \in \mathbb{Z}_4^*$ such that $g^*(x) = eg(x)$. So, $g(x)$ is a self reciprocal polynomial.

Let $a(x) = a_0 + a_1x + ... + a_tx^t$. Suppose that $wa(x) = wa_0 + wa_1x + ... + wa_tx^t$. Then

$$\begin{aligned}(wa(x))^{rc} =& (3+3w)+(3+3w)x+...\\ &+\overline{wa_t}x^{n-t-1}+...+\overline{wa_1}x^{n-2}\\ &+\overline{wa_0}x^{n-1} \in C\end{aligned}$$

As $(3+3w)\frac{x^n-1}{x-1} \in C$ and C is a linear code, then

$$-(wa(x))^{rc} + (3+3w)\frac{x^n-1}{x-1} \in C$$

Hence, $x^{n-t-1}[(-(\overline{wa_t})+(3+3w))+(-(\overline{wa_{t-1}})+(3+3w))x+...+(-(\overline{wa_0})+(3+3w))x^t]$. By the Lemma 6, we get

$$x^{n-t-1}(wa_t + wa_{t-1}x + ... + wa_0x^t)$$

$x^{n-t-1}wa^*(x) \in C$. Since $wa^*(x) \in C$, we have

$$wa^*(x) = g(x)h(x) + wa(x)s(x)$$

Since w doesn't appear in $g(x)$, it follows that $h(x) = 0$ and $a^*(x) = a(x)s(x)$. As $\deg a^*(x) = \deg a(x)$, then $s(x) \in \mathbb{Z}_4^*$. So, $a(x)$ is a self reciprocal polynomial. ■

Theorem 8: Let $C = \langle g(x), wa(x)\rangle$ be a cyclic code of odd length n over R. If $(1+w)(1+x+x^2+...+x^{n-1}) \in C$ and $g(x), a(x)$ are self reciprocal polynomials, then $c(x)^{rc} \in C$ for any $c(x) \in C$.

Proof: Since $C = \langle g(x), wa(x)\rangle$, for any $c(x) \in C$, there exist $m(x)$ and $n(x)$ in $R[x]$ such that $c(x) = g(x)m(x) + wa(x)n(x)$. By using the Lemma 4, we have

$$\begin{aligned}c^*(x) &= (g(x)m(x) + wa(x)n(x))\\ &= (g(x)m(x))^* + x^s(wa(x)n(x))\\ &= g^*(x)m^*(x) + wa^*(x)(x^sn^*(x))\end{aligned}$$

Since $g^*(x) = eg(x), a^*(x) = da(x)$, we have $c^*(x) = eg(x)m^*(x) + dwa(x)(x^sn^*(x)) \in C$. So, $c^*(x) \in C$.

Let $c(x) = c_0 + c_1x + ... + c_tx^t \in C$. Since C is a cyclic code, we get

$$x^{n-t-1}c(x) = c_0x^{n-t-1}+c_1x^{n-t}+...+c_tx^{n-1} \in C$$

Since $(1+w)+(1+w)x+...+(1+w)x^{n-1} \in C$ and C is a linear code we have

$$-(1+w)\frac{x^n-1}{x-1} - x^{n-t-1}c(x)$$
$$= -(1+w)-(1+w)x+...+(-c_0-(1+w))x^{n-t-1}$$
$$+...+(-c_t-(1+w))x^{n-1} \in C.$$

By using $\overline{a}+(1+w) = -a$, this implies that

$$-(1+w)-...+\overline{c}_0 x^{n-t-1}+...+\overline{c}_t x^{n-1} \in C$$

This shows that $(c^*(x))^{rc} \in C$.

$$((c^*(x))^{rc})^* = \overline{c}_t + \overline{c}_{t-1}x+...+(3+3w)x^{n-1}$$

This corresponds this vector $(\overline{c}_t, \overline{c}_{t-1}, ..., \overline{c}_0, ..., \overline{0})$. Since $(c^*(x)^{rc})^* = (x^{n-t-1}c(x))^{rc}$, so $c(x)^{rc} \in C$. ■

Example 9: Let $x^3-1 = (x+3)(x^2+x+1) \in \mathbb{Z}_4[x]$. Let $C = \langle x^2+x+1+w(x^2+x+1)\rangle$. C is a cyclic DNA code of length 3 over R. The Gray image of C under the Gray map ϕ is a DNA code of length 6, Hamming distance 3. These codewords are as follows

All 16 codewords of C

$CCCCCC$	$TGTGTG$
$GGGGGG$	$GTGTGT$
$TTTTTT$	$GCGCGC$
$AAAAAA$	$CGCGCG$
$GAGAGA$	$CTCTCT$
$AGAGAG$	$TCTCTC$
$TATATA$	$ACACAC$
$ATATAT$	$CACACA$

Example 10: Let $x^7-1 = (x+3)(x^3-2x^2+x-1)(x^3-x^2+2x-1) \in \mathbb{Z}_4[x]$. Let $C =< x^6-3x^5+x^4-3x^3+x^2-3x+1+w(x^6-3x^5+x^4-3x^3+x^2-3x+1) >$. C is a cyclic DNA code of length 7 over R. The Gray image of C under the Gray map ϕ is a DNA code of length 14, Hamming distance 7. These codewords are as follows

All 16 codewords of C

$CCCCCCCCCCCCCC$
$GGGGGGGGGGGGGG$
$TTTTTTTTTTTTTT$
$AAAAAAAAAAAAAA$
$GAGAGAGAGAGAGA$
$AGAGAGAGAGAGAG$
$TATATATATATATA$
$ATATATATATATAT$
$TGTGTGTGTGTGTG$
$GTGTGTGTGTGTGT$
$GCGCGCGCGCGCGC$
$CGCGCGCGCGCGCG$
$CTCTCTCTCTCTCT$
$TCTCTCTCTCTCTC$
$ACACACACACACAC$
$CACACACACACACA$

IV. The reversible and reversible complement codes over S

Throughout this paper, S denotes the commutative ring $\mathbb{Z}_4+w\mathbb{Z}_4+v\mathbb{Z}_4+wv\mathbb{Z}_4 = \{b_1+wb_2+vb_3+wvb_4 : b_j \in \mathbb{Z}_4, 1 \leq j \leq 4\}$ with $w^2 = 2, v^2 = v, wv = vw$, with characteristic 4. S can also be thought of as the quotient ring $\mathbb{Z}_4[w,v]/ < w^2-2, v^2-v, wv-vw >$.

Let

$$\begin{aligned} S &= \mathbb{Z}_4+w\mathbb{Z}_4+v\mathbb{Z}_4+wv\mathbb{Z}_4 \\ &= (\mathbb{Z}_4+w\mathbb{Z}_4)+v(\mathbb{Z}_4+w\mathbb{Z}_4) \\ &= R+vR \end{aligned}$$

We define the Gray map ϕ_1 from S to R as follows

$$\begin{aligned} \phi_1 &: S \longrightarrow R^2 \\ a+vb &\longmapsto (a,b) \end{aligned}$$

where $a,b \in R$. This Gray map is extended compenentwise to

$$\begin{aligned} \phi_1 &: S^n \longrightarrow R^{2n} \\ x &= (x_1,...,x_n) \longmapsto (a_1,...,a_n,b_1,...,b_n) \end{aligned}$$

where $x_i = a_i+vb_i, a_i, b_i \in R$ for $i = 1,2,...,n$.

In this section, we study the cyclic codes of odd length n over S satisfy reverse and reverse complement constraint. Since the ring S is of the cardinality 4^4, then we define the map ϕ_1 which gives a one to one correspondence between the element of S and the 256 codons over the alphabet $\{A,T,G,C\}^4$ by using the Gray map. For example:

$$0=0+v0\longmapsto\phi_1(0)=(0,0)\longrightarrow AAAA$$

$$2wv=0+v(2w)\longmapsto\phi_1(2wv)=(0,2w)\longrightarrow AAAG$$

$$\begin{aligned}1+3v+3wv=1+v(3+3w)&\longmapsto\phi_1(1+v(3+3w))\\&=(1,3+3w)\longrightarrow CATT\end{aligned}$$

Definition 11: Let A_1, A_2 be linear codes.

$$A_1\otimes A_2=\{(a_1,a_2):a_1\in A_1,a_2\in A_2\}$$

and

$$A_1\oplus A_2=\{a_1+a_2:a_1\in A_1,a_2\in A_2\}$$

Let C be a linear code of length n over S. Define

$$\begin{aligned}C_1&=\{a:\exists\ b\in R^n,a+vb\in C\}\\C_2&=\{b:\exists\ a\in R^n,a+vb\in C\}\end{aligned}$$

where C_1 and C_2 are linear codes over R of length n.

Theorem 12: Let C be a linear code of length n over S. Then $\phi_1(C)=C_1\otimes C_2$ and $|C|=|C_1|\,|C_2|$.

Corollary 13: If $\phi_1(C)=C_1\otimes C_2$, then $C=vC_1\oplus(1-v)C_2$.

Theorem 14: Let $C=vC_1\oplus(1-v)C_2$ be a linear code of odd length n over S. Then C is a cyclic code over S if and only if C_1, C_2 are cyclic codes over R.

Proof: Let $(a_0^1,a_1^1,...,a_{n-1}^1)\in C_1,(a_0^2,a_1^2,...,a_{n-1}^2)\in C_2$. Assume that $m_i=va_i^1+(1-v)a_i^2$ for $i=0,1,2,...,n-1$. Then $(m_0,m_1,...,m_{n-1})\in C$. Since C is a cyclic code, it follows that $(m_{n-1},m_0,m_1,...,m_{n-2})\in C$. Note that $(m_{n-1},m_0,...,m_{n-2})=v(a_{n-1}^1,a_0^1,...,a_{n-2}^1)+(1-v)(a_{n-1}^2,a_0^2,...,a_{n-2}^2)$. Hence $(a_{n-1}^1,a_0^1,...,a_{n-2}^1)\in C_1,(a_{n-1}^2,a_0^2,...,a_{n-2}^2)\in C_2$. Therefore C_1, C_2 are cyclic codes over R.

Conversely, suppose that C_1, C_2 are cyclic codes over R. Let $(m_0,m_1,...,m_{n-1})\in C$, where $m_i=va_i^1+(1-v)a_i^2$ for $i=0,1,2,...,n-1$. Then $(a_{n-1}^1,a_0^1,...,a_{n-2}^1)\in C_1,(a_{n-1}^2,a_0^2,...,a_{n-2}^2)\in C_2$. Note that $(m_{n-1},m_0,...,m_{n-2})=v(a_{n-1}^1,a_0^1,...,a_{n-2}^1)+(1-v)(a_{n-1}^2,a_0^2,...,a_{n-2}^2)\in C$. So, C is a cyclic code over S. ■

Theorem 15: Let $C=vC_1\oplus(1-v)C_2$ be a linear code of odd length n over S. Then C is reversible over S iff C_1, C_2 are reversible over R.

Proof: Let C_1, C_2 be reversible codes. For any $b\in C, b=vb_1+(1-v)b_2$, where $b_1\in C_1, b_2\in C_2$. Since C_1 and C_2 are reversible, $b_1^r\in C_1, b_2^r\in C_2$. So, $b^r=vb_1^r+(1-v)b_2^r\in C$. Hence C is reversible.

On the other hand, Let C be a reversible code over S. So for any $b=vb_1+(1-v)b_2\in C$, where $b_1\in C_1, b_2\in C_2$, we get $b^r=vb_1^r+(1-v)b_2^r\in C$. Let $b^r=vb_1^r+(1-v)b_2^r=vs_1+(1-v)s_2$, where $s_1\in C_1, s_2\in C_2$. So C_1 and C_2 are reversible codes over R. ■

Lemma 16: For any $c\in S$, we have $c+\overline{c}=(3+3w)+v(3+3w)$.

Lemma 17: For any $a\in S$, $\overline{a}+3\overline{0}=3a$.

Theorem 18: Let $C=vC_1\oplus(1-v)C_2$ be a cyclic code of odd length n over S. Then C is reversible complement over S iff C is reversible over S and $(\overline{0},\overline{0},...,\overline{0})\in C$.

Proof: Since C is reversible complement, for any $c=(c_0,c_1,...,c_{n-1})\in C, c^{rc}=(\overline{c}_{n-1},\overline{c}_{n-2},...,\overline{c}_0)\in C$. Since C is a linear code, so $(0,0,...,0)\in C$. Since C is reversible complement, so $(\overline{0},\overline{0},...,\overline{0})\in C$. By using the Lemma 17, we have

$$\begin{aligned}3c^r&=3(c_{n-1},c_{n-2},...,c_0)\\&=(\overline{c}_{n-1},\overline{c}_{n-2},...,\overline{c}_0)+3(\overline{0},\overline{0},...,\overline{0})\in C.\end{aligned}$$

So, for any $c\in C$, we have $c^r\in C$.

On the other hand, let C be reversible. So, for any $c=(c_0,c_1,...,c_{n-1})\in C$, $c^r=(c_{n-1},c_{n-2},...,c_0)\in C$. To show that C is reversible complement, for any $c\in C$,

$$\begin{aligned}c^{rc}&=(\overline{c}_{n-1},\overline{c}_{n-2},...,\overline{c}_0)\\&=3(c_{n-1},c_{n-2},...,c_0)+(\overline{0},\overline{0},...,\overline{0})\in C.\end{aligned}$$

So, C is reversible complement. ■

Lemma 19: For any $a, b \in S$,

$$\overline{a+b} = \overline{a} + \overline{b} - 3(1+w)(1+v).$$

Theorem 20: Let D_1 and D_2 be two reversible complement cyclic codes of length n over S. Then $D_1 + D_2$ and $D_1 \cap D_2$ are reversible complement cyclic codes.

Proof: Let $d_1 = (c_0, c_1, ..., c_{n-1}) \in D_1, d_2 = (c_0^1, c_1^1, ..., c_{n-1}^1) \in D_2$. Then,

$$\begin{aligned}(d_1+d_2)^{rc} &= \left(\overline{(c_{n-1}+c_{n-1}^1)}, ..., \overline{(c_1+c_1^1)}, \overline{(c_0+c_0^1)}\right)\\ &= \Big(\overline{c_{n-1}} + \overline{c_{n-1}^1} - 3(1+w)(1+v), ...,\\ &\qquad \overline{c_0} + \overline{c_0^1} - 3(1+w)(1+v)\Big)\\ &= (\overline{c_{n-1}} - 3(1+w)(1+v), ..., \overline{c_0}\\ &\qquad -3(1+w)(1+v)) + \left(\overline{c_{n-1}^1}, ..., \overline{c_0^1}\right)\\ &= \left(d_1^{rc} - 3(1+w)(1+v)\frac{x^n-1}{x-1}\right)\\ &\qquad + d_2^{rc} \in D_1 + D_2.\end{aligned}$$

This shows that $D_1 + D_2$ is reversible complement cyclic code. It is clear that $D_1 \cap D_2$ is reversible complement cyclic code. ■

V. Binary images of cyclic DNA codes over R

The 2-adic expansion of $c \in \mathbb{Z}_4$ is $c = \alpha(c) + 2\beta(c)$ such that $\alpha(c) + \beta(c) + \gamma(c) = 0$ for all $c \in \mathbb{Z}_4$

c	$\alpha(c)$	$\beta(c)$	$\gamma(c)$
0	0	0	0
1	1	0	1
2	0	1	1
3	1	1	0

The Gray map is given by

$$\begin{aligned}\Psi &: \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2\\ c &\longmapsto \Psi(c) = (\beta(c), \gamma(c))\end{aligned}$$

for all $c \in \mathbb{Z}_4$ in [14]. Define

$$\begin{aligned}\breve{O} &: R \longrightarrow \mathbb{Z}_2^4\\ a + bw &\longmapsto \breve{O}(a+wb) = \Psi(\phi(a+wb))\\ &= \Psi(a, b)\\ &= (\beta(a), \gamma(a), \beta(b), \gamma(b))\end{aligned}$$

Let $a + wb$ be any element of the ring R. The Lee weight w_L of the element of the ring R is defined as follows

$$w_L(a + wb) = w_L(a, b)$$

where $w_L(a, b)$ described the usual Lee weight on $\mathbb{Z}_4^2$. For any $c_1, c_2 \in R$ the Lee distance d_L is given by $d_L(c_1, c_2) = w_L(c_1 - c_2)$.

The Hamming distance $d(c_1, c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codewords $c_1 - c_2$.

AA	$\longrightarrow$	0000	CG	$\longrightarrow$	0111
CA	$\longrightarrow$	0100	CT	$\longrightarrow$	0110
GA	$\longrightarrow$	1100	GC	$\longrightarrow$	1101
TA	$\longrightarrow$	1000	GG	$\longrightarrow$	1111
AC	$\longrightarrow$	0001	GT	$\longrightarrow$	1110
AG	$\longrightarrow$	0011	TC	$\longrightarrow$	1001
AT	$\longrightarrow$	0010	TG	$\longrightarrow$	1011
CC	$\longrightarrow$	0101	TT	$\longrightarrow$	1010

Lemma 21: The Gray map $\breve{O}$ is a distance preserving map from (R^n, Lee distance) to ($\mathbb{Z}_2^{4n}$, Hamming distance). It is also $\mathbb{Z}_2$-linear.

Proof: For $c_1, c_2 \in R^n$, we have $\breve{O}(c_1 - c_2) = \breve{O}(c_1) - \breve{O}(c_2)$. So, $d_L(c_1, c_2) = w_L(c_1 - c_2) = w_H(\breve{O}(c_1 - c_2)) = w_H(\breve{O}(c_1) - \breve{O}(c_2)) = d_H(\breve{O}(c_1), \breve{O}(c_2))$. So, the Gray map $\breve{O}$ is distance preserving map. For any $c_1, c_2 \in R^n, k_1, k_2 \in \mathbb{Z}_2$,we have $\breve{O}(k_1c_1 + k_2c_2) = k_1\breve{O}(c_1) + k_2\breve{O}(c_2)$. Thus, $\breve{O}$ is $\mathbb{Z}_2$-linear. ■

Proposition 22: Let σ be the cyclic shift of R^n and v be the 4-quasi-cyclic shift of $\mathbb{Z}_2^{4n}$. Let $\breve{O}$ be the Gray map from R^n to $\mathbb{Z}_2^{4n}$. Then $\breve{O}\sigma = v\breve{O}$.

Proof: Let $c = (c_0, c_1, ..., c_{n-1}) \in R^n$, we have $c_i = a_{1i} + wb_{2i}$ with $a_{1i}, b_{2i} \in \mathbb{Z}_4, 0 \leq i \leq n-1$. By applying the Gray map, we have

$$\breve{O}(c) = \begin{pmatrix}\beta(a_{10}), \gamma(a_{10}), \beta(b_{20}), \gamma(b_{20}), \beta(a_{11}),\\ \gamma(a_{11}), \beta(b_{21}), \gamma(b_{21}), ..., \beta(a_{1n-1}),\\ \gamma(a_{1n-1}), \beta(b_{2n-1}), \gamma(b_{2n-1})\end{pmatrix}.$$

Hence

$$v(\breve{O}(c)) = \begin{pmatrix}\beta(a_{1n-1}), \gamma(a_{1n-1}), \beta(b_{2n-1}), \gamma(b_{2n-1}),\\ \beta(a_{10}), \gamma(a_{10}), \beta(b_{20}), \gamma(b_{20}), ..., \beta(a_{1n-2}),\\ \gamma(a_{1n-2}), \beta(b_{2n-2}), \gamma(b_{2n-2})\end{pmatrix}.$$

On the other hand,

$$\sigma(c) = (c_{n-1}, c_0, c_1, ..., c_{n-2}).$$

We have

$$\breve{O}(\sigma(c)) = \begin{pmatrix} \beta(a_{1n-1}), \gamma(a_{1n-1}), \beta(b_{2n-1}), \\ \gamma(b_{2n-1}), \beta(a_{10}), \gamma(a_{10}), \beta(b_{20}), \gamma(b_{20}), ..., \\ \beta(a_{1n-2}), \gamma(a_{1n-2}), \beta(b_{2n-2}), \gamma(b_{2n-2}) \end{pmatrix}.$$

Therefore, $\breve{O}\sigma = \upsilon\breve{O}$. ■

Theorem 23: If C is a cyclic DNA code of length n over R then $\breve{O}(C)$ is a binary quasi-cyclic DNA code of length $4n$ with index 4.

VI. Binary image of cyclic DNA codes over S

We define

$$\widetilde{\Psi} \quad : \quad S \longrightarrow \mathbb{Z}_4^4$$
$$a_0 + wa_1 + va_2 + wva_3 \quad \longmapsto \quad (a_0, a_1, a_2, a_3)$$

where $a_i \in \mathbb{Z}_4$, for $i = 0, 1, 2, 3$.

Now, we define $\Theta : S \longrightarrow \mathbb{Z}_2^8$ as

$$\begin{aligned} &a_0 + wa_1 + va_2 + wva_3 \\ &\longmapsto \Theta(a_0 + wa_1 + va_2 + wva_3) \\ &= \Psi(\widetilde{\Psi}(a_0 + wa_1 + va_2 + wva_3)) = \\ &(\beta(a_0),\gamma(a_0),\beta(a_1),\gamma(a_1),\beta(a_2),\gamma(a_2),\beta(a_3),\gamma(a_3)), \end{aligned}$$

where Ψ is the Gray map $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$.

Let $a_0 + wa_1 + va_2 + wva_3$ be any element of the ring S. The Lee weight w_L of the element of the ring S is defined as

$$w_L(a_0+wa_1+va_2+wva_3) = w_L((a_0, a_1, a_2, a_3))$$

where $w_L((a_0, a_1, a_2, a_3))$ described the usual Lee weight on $\mathbb{Z}_4^4$. For any $c_1, c_2 \in S$, the Lee distance d_L is given by $d_L(c_1, c_2) = w_L(c_1 - c_2)$.

The Hamming distance $d(c_1, c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codewords $c_1 - c_2$.

The binary images of cyclic DNA codes;

$$\begin{array}{ccc} AAAA & \longrightarrow & 00000000 \\ AACA & \longrightarrow & 00000100 \\ AAGA & \longrightarrow & 00001100 \\ AATA & \longrightarrow & 00001000 \\ \vdots & \vdots & \vdots \end{array}$$

Lemma 24: The Gray map Θ is a distance preserving map from (S^n, Lee distance) to ($\mathbb{Z}_2^{8n}$, Hamming distance). It is also $\mathbb{Z}_2$-linear.

Proof: It is proved as in the proof of the Lemma 21. ■

Proposition 25: Let σ be the cyclic shift of S^n and $\acute{\upsilon}$ be the 8-quasi-cyclic shift of $\mathbb{Z}_2^{8n}$. Let Θ be the Gray map from S^n to $\mathbb{Z}_2^{8n}$. Then $\Theta\sigma = \acute{\upsilon}\Theta$.

Proof: It is proved as in the proof of the Proposition 22. ■

Theorem 26: If C is a cyclic DNA code of length n over S then $\Theta(C)$ is a binary quasi-cyclic DNA code of length $8n$ with index 8.

Proof: Let C be a cyclic DNA code of length n over S. So, $\sigma(C) = C$. By using the Proposition 25, we have $\Theta(\sigma(C)) = \acute{\upsilon}(\Theta(C)) = \Theta(C)$. Hence $\Theta(C)$ is a set of length $8n$ over the alphabet $\mathbb{Z}_2$ which is a quasi-cyclic code of index 8. ■

VII. Skew cyclic DNA codes over R

We will use a non trivial automorphism, for all $a + wb \in R$, it is defined by

$$\theta \quad : \quad R \longrightarrow R$$
$$a + wb \quad \longmapsto \quad a - wb$$

The ring $R[x, \theta] = \{a_0 + a_1x + ... + a_{n-1}x^{n-1} : a_i \in R, n \in N\}$ is called skew polynomial ring. It is non commutative ring. The addition in the ring $R[x, \theta]$ is the usual polynomial and multiplication is defined as $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$. The order of the automorphism θ is 2.

Definition 27: A subset C of R^n is called a skew cyclic code of length n if C satisfies the following conditions,

$i)$ C is a submodule of R^n,

$ii)$ If $c = (c_0, c_1, ..., c_{n-1}) \in C$, then $\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), ..., \theta(c_{n-2})) \in C$

Let $f(x) + \langle x^n - 1\rangle$ be an element in the set $\breve{R}_n = R[x, \theta] / \langle x^n - 1\rangle$ and let $r(x) \in R[x, \theta]$. Define multiplication from left as follows,

$$r(x)(f(x) + \langle x^n - 1\rangle) = r(x)f(x) + \langle x^n - 1\rangle$$

for any $r(x) \in R[x, \theta]$.

Theorem 28: $\breve{R}_n$ is a left $R[x, \theta]$-module where multiplication defined as in above.

Theorem 29: A code C over R of length n is a skew cyclic code if and only if C is a left $R[x,\theta]$-submodule of the left $R[x,\theta]$-module $\check{R}_n$.

Theorem 30: Let C be a skew cyclic code over R of length n and let $f(x)$ be a polynomial in C of minimal degree. If $f(x)$ is monic polynomial, then $C = \langle f(x) \rangle$, where $f(x)$ is a right divisor of $x^n - 1$.

For all $x \in R$, we have

$$\theta(x) + \theta(\overline{x}) = 3 - 3w.$$

Theorem 31: Let $C = \langle f(x) \rangle$ be a skew cyclic code over R, where $f(x)$ is a monic polynomial in C of minimal degree. If C is reversible complement, the polynomial $f(x)$ is self reciprocal and

$$(3+3w)\frac{x^n-1}{x-1} \in C.$$

Proof: Let $C = \langle f(x) \rangle$ be a skew cyclic code over R, where $f(x)$ is a monic polynomial in C. Since $(0,0,...,0) \in C$ and C is reversible complement, we have $\left(\overline{0},\overline{0},...,\overline{0}\right) = (3+3w, 3+3w, ..., 3+3w) \in C$.

Let $f(x) = 1 + a_1 x + ... + a_{t-1}x^{t-1} + x^t$. Since C is reversible complement, we have $f^{rc}(x) \in C$. That is

$$\begin{aligned} f^{rc}(x) = {} & (3+3w) + (3+3w)x + ... + (3+3w)x^{n-t-2} \\ & + (2+3w)x^{n-t-1} + \overline{a}_{t-1}x^{n-t} + ... \\ & + \overline{a}_1 x^{n-2} + (2+3w)x^{n-1}. \end{aligned}$$

Since C is a linear code, we have

$$f^{rc}(x) - (3+3w)\frac{x^n-1}{x-1} \in C.$$

This implies that

$$\begin{aligned} & -x^{n-t-1} + (\overline{a}_{t-1} - (3+3w))x^{n-t} + ... \\ & + (\overline{a}_1 - (3+3w))x^{n-2} - x^{n-1} \in C. \end{aligned}$$

Multiplying on the right by x^{t+1-n}, we have

$$\begin{aligned} & -1 + (\overline{a}_{t-1} - (3+3w))\theta(1)x + ... \\ & + (\overline{a}_1 - (3+3w))\theta^{t-1}(1)x^{t-1} - \theta^t(1)x^t \in C. \end{aligned}$$

By using $a + \overline{a} = 3 + 3w$, we have

$$\begin{aligned} & -1 - a_{t-1}x - a_{t-2}x^2 - ... - a_1 x^{t-1} - x^t \\ & = 3f^*(x) \in C. \end{aligned}$$

Since $C = \langle f(x) \rangle$, there exist $q(x) \in R[x,\theta]$ such that $3f^*(x) = q(x)f(x)$. Since $\deg f(x) = \deg f^*(x)$, we have $q(x) = 1$. Since $3f^*(x) = f(x)$, we have $f^*(x) = 3f(x)$. So, $f(x)$ is self reciprocal. ■

Theorem 32: Let $C = \langle f(x) \rangle$ be a skew cyclic code over R, where $f(x)$ is a monic polynomial in C of minimal degree. If $(3+3w)\frac{x^n-1}{x-1} \in C$ and $f(x)$ is self reciprocal, then C is reversible complement.

Proof: Let $f(x) = 1 + a_1 x + ... + a_{t-1}x^{t-1} + x^t$ be a monic polynomial of the minimal degree.

Let $c(x) \in C$. So, $c(x) = q(x)f(x)$, where $q(x) \in R[x,\theta]$. By using Lemma 4, we have $c^*(x) = (q(x)f(x))^* = q^*(x)f^*(x)$. Since $f(x)$ is self reciprocal, so $c^*(x) = q^*(x)ef(x)$, where $e \in \mathbb{Z}_4 \backslash \{0\}$. Therefore $c^*(x) \in C = \langle f(x) \rangle$. Let $c(x) = c_0 + c_1 x + ... + c_t x^t \in C$. Since C is a cyclic code, we get

$$c(x)x^{n-t-1} = c_0 x^{n-t-1} + c_1 x^{n-t} + ... + c_t x^{n-1} \in C.$$

The vector corresponding to this polynomial is

$$(0,0,...,0,c_0,c_1,...,c_t) \in C.$$

Since $(3+3w, 3+3w, ..., 3+3w) \in C$ and C linear, we have

$$\begin{aligned} & (3+3w, 3+3w, ..., 3+3w) - (0,0,...,0,c_0,c_1,...,c_t) \\ & = (3+3w, ..., 3+3w, (3+3w)-c_0, ..., (3+3w)-c_t) \in C. \end{aligned}$$

By using $a + \overline{a} = 3 + 3w$, we get

$$(3+3w, 3+3w, ..., 3+3w, \overline{c}_0, ..., \overline{c}_t) \in C,$$

which is equal to $(c(x)^*)^{rc}$. This shows that $((c(x)^*)^{rc})^* = c(x)^{rc} \in C$. ■

VIII. DNA CODES OVER S

Definition 33: Let f_1 and f_2 be polynomials with $\deg f_1 = t_1, \deg f_2 = t_2$ and both dividing $x^n - 1 \in R[x]$.

Let $m = \min\{n - t_1, n - t_2\}$ and $f(x) = vf_1(x) + (1-v)f_2(x)$ over S. The set $L(f)$ is called a Γ-set, where the automorphism $\Gamma : S \longrightarrow S$ is defined as follows:

$$a + wb + vc + wvd \longmapsto a + b + w(b+d) - vc - wvdc.$$

$$L(f) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_t & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \Gamma(a_0) & \Gamma(a_1) & \cdots & \cdots & \Gamma(a_t) & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_0 & a_1 & \cdots & \cdots & a_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \Gamma(a_0) & \Gamma(a_1) & \cdots & \cdots & \Gamma(a_t) & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \end{bmatrix} \quad (1)$$

The set $L(f)$ is defined as

$$L(f) = \{E_0, E_1, ..., E_{m-1}\},$$

where

$$E_i = \begin{cases} x^i f & \text{if } i \text{ is even} \\ x^i \Gamma(f) & \text{if } i \text{ is odd} \end{cases}$$

$L(f)$ generates a linear code C over S denoted by $C = \langle f \rangle_\Gamma$. Let $f(x) = a_0 + a_1 x + ... + a_t x^t$ be over S and S-submodule generated by $L(f)$ is generated by the matrix in Eq. (1).

Theorem 34: Let f_1 and f_2 be self reciprocal polynomials dividing $x^n - 1$ over R with degree t_1 and t_2, respectively. If $f_1 = f_2$, then $f = vf_1 + (1-v)f_2$ and $|\langle L(f) \rangle| = 256^m$. $C = \langle L(f) \rangle$ is a linear code over S and $\Theta(C)$ is a reversible DNA code.

Proof: It is proved as in the proof of the Theorem 5 in [5]. ■

Corollary 35: Let f_1 and f_2 be self reciprocal polynomials dividing $x^n - 1$ over R and $C = \langle L(f) \rangle$ be a cyclic code over S. If $\frac{x^n-1}{x-1} \in C$, then $\Theta(C)$ is a reversible complement DNA code.

Example 36: Let $f_1(x) = f_2(x) = x - 1$ dividing $x^7 - 1$ over R. Hence,

$$C = \langle vf_1(x) + (1-v)f_2(x) \rangle_\Gamma = \langle x - 1 \rangle_\Gamma$$

is a Γ-linear code over S and $\Theta(C)$ is a reversible complement DNA code, because of

$$\frac{x^7 - 1}{x - 1} \in C.$$

Acknowledgement 37: We wish to express sincere thanks to Steven Dougherty who gave helpful comments.

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