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Image Quality Enhancement Model and its Application Based on PDE

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Partial differential equations application in digital image processing is developing rapidly in recent years. The method aim to establish a mathematical model with the partial differential equation, and the model will change the image based on partial differential equation and eventually get the expected effect on the image. Through the processing method with partial differential equation, the image can get the effect unattainable by traditional method. Based on the research result both domestic and abroad, this paper mainly illustrates the establish, analysis and realization of an image enhancement model with PDE. By testing the PDE model with the real image enhancement process, the validity of this method has been verified.

1. Overview

Nature can be described in mathematical language, equations is a strong tool of analyzing scientific phenomena. Human cannot get any knowledge of nature or transform the nature to a human adapted environment without the help of mathematical tool, such as partial differential equation. Partial differential equation to have a history of less than 300 years, its development is very rapid, from the partial differential equation of separation of variables method to the traveling wave method, with the advent of the computer, finite difference method arises and has been applied to many fields in recent years, Cheng Wang and Suming Zhu (2015) reported. Partial differential equation can describe a lot of physical and chemical phenomena, Pavlidis, T (2012) reported. Algorithms for graphics and image processing. Springer Science & Business Media.
In recent years, as people's constantly research and the understanding of partial differential equations arises, partial differential equation is also gradually applied to the field of digital image. And its application in the field of digital image enhancing also emerges.

2. The establishment of the partial differential equation

The establishment process of Partial differential equation model can be generally concluded as follows:
(1) Work out the target energy functional.
(2) Solve the functional extremum problems. With Eluer formula, transform the functional extremum problem into solving a partial differential equation.
(3) Evolve curve and surface to according to the solution of partial differential equations, finite difference method and programming will be needed in the process of solving partial differential equation.
(4) Observe and analysis of the research results, modify models such as morbid equation which must be regularized.

2.1 Catte model of image enhancement

Among all the PDE models, the man-made structure of energy function and model established based on this function have a large proportion. Its main point is to construct an energy function, through minimize the energy functional of the structure, the researchers can achieve the purpose of realization of digital image processing such as image enhancement. The partial differential equation derived from the minimize energy function with variational method is the core of the digital image processing. The PDE model described in this article are all man-made structure of an energy functional model which was confirmed (Demirel, H. and Anbarjafari, G. (2011)).

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Catte F, Lions and Morel put forward a kind of anisotropic diffusion model, the model use $g(|\nabla u_\sigma|)$, $u_\sigma = G_\sigma \times u$ as diffusion coefficient, to reduce the noise caused by the estimated error. The improved Catte diffusion model is as follows:

$$\frac{\partial u}{\partial t} = div(g(|\nabla G_\sigma \times u|)\nabla u) \quad (1)$$

$$u(x, y, 0) = u_0(x, y) \quad (2)$$

Which G_σ is Gaussian kernel function, $G_\sigma = \frac{1}{4\pi\sigma} e^{\frac{x^2+y^2}{4\sigma}}$, $|\nabla G_\sigma \times u|$ means the gradient of image with scale of σ , $u(x, y, 0)$ is the original image, Celik, T., & Tjahjadi, T (2012) reported.

In the model, the gradient ∇u is calculating with diffusion coefficient which get from the Gaussian filter convolution image. Dispersion coefficient $|\nabla G_\sigma \times u|$ prevents the excessive proliferation in border, effectively solve the well-posedness of the problems in the equations. The model's success depends on the choice of an appropriate value for the regularization parameter, but how to select an accurate Gaussian kernel scale σ is a problem, if σ is too small, the result can be inaccurate estimates, the model will get more spread diffusion than before, if σ large will lead to a smooth too fast.

2.2 Other PDE model of image enhancement

Image after Catte model filter and enhanced, highly easy to loss on the edge of the detail, which make the image lost peak shape edge features and the edge of the model become too narrow. Therefore, Lin Zhouchen, Shi Qingyun proposed a denoising model which keep the reality of anisotropic diffusion model, advocately using $|\nabla u_\sigma|^2 + u_{xx}^2 + u_{yy}^2$ to replace $|\nabla u|$ in Catte model, the PDE equation is as follows:

$$\frac{\partial u}{\partial t} = div(g(|\nabla G_\sigma \times u|^2 + (G_\sigma \times u_{xx})^2 + (G_\sigma \times u_{yy})^2)\nabla u) \quad (3)$$

$$u(x, y, 0) = u_0(x, y) \quad (4)$$

The equations above use the second degree derivative, compared with Catte model, the model spread quickly on a flat area, and enhance the image at the border or edge, especially reserves the weak boundary.
But this model is only suitable for the removal of uniform distribution of noise and Gaussian noise which is not too strong, model operator is quite reserve on the edge of the peak shape, if the Gaussian noise is very strong, certain isolated point may be caused on the processed image. Domestic scholars Liu Gui-Lan argues that replace the classic anisotropic diffusion coefficient with diffusion tensor, this method is same as the PDE model above, both of these two method will achieve the same effect.

3. The PDE model in image enhancement

3.1 Conversion Function in the Process of Image Enhancement

Assume the original image is $I_A(x, y)$, the enhanced image is $I_B(x, y)$, image conversion function is $f(D)$. Easy to know function $f(D)$ monotonous, original image grey scale set to D_A , after the $f(D)$ enhanced image grey scale is D_B . D_A is in the range of $[D_{A\min}, D_{A\max}]$, D_B is in the range of

$[D_{B\min}, D_{B\max}]$. $h_A(D_A)$ represents the original image grey scale D_A for the number of pixels to account for the proportion of the total number of pixels, $h_B(D_B)$ represents the original image grey scale D_B for the number of pixels to account for the proportion of the total number of pixels. From the map, the grey scale of the original image $I_A(x,y)$ falls in the set of $[D_A, D_A + \Delta D_A]$ equal to the number of enhanced image grey scale set $[D_B, D_B + \Delta D_B]$, then the following equation is valid:

$$h_A(D_A)\Delta D_A = h_B(D_B)\Delta D_B \quad (5)$$

The equation above is the main idea of cumulative frequency conversion function in image enhancement. According to $\frac{\Delta D_B}{\Delta D_A} = f(D)$, we can get the equation as follow:

$$f(D) = \frac{h_A(D_A)}{h_B(D_B)} \quad (6)$$

Based on the Newton-Leibniz formula, we can get:

$$f(D) = \int_{D_{A\min}}^{D} \frac{h_A(\zeta)}{h_B(\zeta)} d\zeta + D_{B\min} \quad (7)$$

The enhanced image $I(x,y)$'s grey scale is uniform distributed among the set of $[D_{\min}, D_{\max}]$, then:

$$h_B(D) = \frac{1}{D_{B\max} - D_{B\min}} \quad (8)$$

The transformation function is as follow when combined the above two equation:

$$f(D) = (D_{b\max} - D_{b\min})\int_{D_{A\min}}^{D} h_A(\zeta)d\zeta + D_{b\min} \quad (9)$$

Thus the image enhancement of cumulative frequency transition function is the equation above with the agreed restriction:

$$H_A(D) = \int_{D_{A\min}}^{D} h_A(\zeta)d\zeta \quad (10)$$

3.2 The piecewise linear stretch histogram equalization method

The main idea of the piecewise linear stretch histogram equalization method is to divide the original image grey scale to none even N sections. The dividing principle is satisfy when within each segment Γ_n the sum of the histogram equals to:

$$\sum_{i\in\Gamma_n} h(i) = \frac{1}{N}, n = 1,\ldots,N \quad (11)$$

The enhanced image grey scales will be evenly divided into N sections, then paits the starting point and end point of every grey scale sections to get the corresponding conversion function of each section, Hammond, D. K et al (2011) reported.

3.3 The PDE method of Image Enhancement

For the gray scale image, image enhancement is gray scale image contrast enhancement. Traditional digital image processing method includes cumulative frequency transition function method and the piecewise linear stretch histogram equalization method. Only on the ground of image enhancement, the PDE method and cumulative frequency transition function methods get the same conclusion. However, the PDE method used in image enhancement also realize the image denoising at the same time. This is the advantage of PDE method which was confirmed (Demirel, H. and Anbarjafari, G. (2011)).
Image enhancement of the PDE method need to minimize the energy functional:

$$E(I)=\frac{1}{2}\int\left[\frac{I(x,y)}{D_{d\max}}-\frac{1}{2}\right]^2 dxdy-\frac{1}{4}\iint|I(x,y)-I(u,v)|dxdydudv \quad (12)$$

The aim of minimize $E(I)$ is to minimize the first term and maximize the second term, Arosio, L., et al(2013) reported. Then purpose of minimize the first term is to make the overall grey scale after the enhancement equals $\frac{D_{b\max}}{2}$, the purpose of maximize the second term is to increase the contrast after the enhancement compared to the original image. Both of the two process have the effect to enhance the image quality.

According to equation (9) and (10), the energy functional gradient descent flow of minimized equation (12) is as follows:

$$\frac{\partial I(x,y,t)}{\partial t}=[1-\frac{I(x,y,t)}{D_{b\max}}]A_\Omega-A(I(x,y,t)) \quad (13)$$

A_Ω is the total number of image pixels, $A(I(x,y,t))$ is the number of grey scales which bigger than $I(x,y,t)$. When stablizied, the left side of equation (13) equals to 0, then:

$$I(x,y,\infty)=D_{b\max}\frac{A_\Omega-A(I)}{A_\Omega}=D_{b\max}(H(I)) \quad (14)$$

Assume $D_{b\min}=0$, the equation (14) is same as equation (9), thus the PDE method have the same result to the image enhancement method with the cumulative frequency and conversion function. But that doesn't mean the PDE method has no actual meaning, this will elaborate later.

Then the equation (13) change to:

$$\frac{\partial I(x,y,t)}{\partial t}=[(D_{b\max}-D_{b\min})H(I(x,y,t))+D_{\min}]-I(x,y,t) \quad (15)$$

Among equation (15), $[(D_{b\max}-D_{b\min})H(I(x,y,t))+D_{\min}]$ is the image enhanced with the cumulative frequency for conversion function method, equation (15) can be summerized as:

$$\frac{\partial I(x,y,t)}{\partial t}=f(I(x,y,t))-I(x,y,t) \quad (16)$$

$f(I(x,y,t)$ represents the user specified grey scale transformation function. An simple analysis of the equation (16) shows, that when $f(.)$ is bigger than $I(x,y,t)$, the $I(x,y,t)$ will increase, when $f(.)$ is smaller than $I(x,y,t)$, the $I(x,y,t)$ will decrease, that's to say, the $I(x,y,t)$ will "follow" the change of $f(I(x,y,t))$, and finally equals to $f(I(x,y,t))$.

The advantage of PDE method is the noise removal along with the image quality enhancement, which shown as follows:

$$\frac{\partial I(x,y,t)}{\partial t}=div(\frac{\nabla I}{|\nabla I|})+\alpha[f(I(x,y,t))-I(x,y,t)] \quad (17)$$

On the gradient descent flow of equation (17) can enhance the image quality, but also de-noising the image, selection of parameters α will balance the image de-noising and image enhancement. Traditional image enhancement method enhance the image quality after denoising or before it. The enhancement of image quality before the image de-noising will increase the noise of the image, making the latter de-nosing ineffective. The enhancement of image quality before the image de-noising hardly can protect the weak edge or boarder of the image, the original weak edge or boarder will not be enhanced effectively. Thus both of the traditional methods of image enhancement can't achieve the result of the PDE method. Here again to see the advantages of the proposed PDE image quality enhancement method.

4. The application of image quality enhancement with PDE model

The reason of the image blur is because of the mean integral operation. In addition, the convolution of fuzzy kernel function will make the image look more blurrer, such as Gaussian voncolution kernels. Then the deconvolution or inverse operation such as differential, can make the image clear. In the traditional method of image sharpening, Laplace operator will sharpening the image. In fact, this method is equivalent to isotropic reverse diffusion equation. Koenderink and Perona introduce the linear and nonlinear diffusion equation in the digital image processing, Koenderink illustrates the initial image $u_0(x, y)$ convolution with Gaussian filter in different scales, the result is equivalent to the thermal diffusion equation with a constant conduction coefficient, for two-dimensional image, the filtering process can be expressed as the follow:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{18}$$

The Gaussian filter variance is proportional to the time solution of the equation. Due to the thermal diffusion equation of heat conduction coefficient is constant, the diffusion is isotropic diffusion. In 1990, Perona and Malik announced the anisotropic diffusion smoothing, the partial differential equation of anisotropic smoothing is:

$$\frac{\partial u}{\partial t} = div[c(|\nabla u|).\nabla u] \tag{19}$$

Above which div is the divergence operator, ∇u is the image gradient, c is diffusion coefficient, usually adopt $c(\nabla u) = \exp(-(|\nabla u| / k)^2)$, k is the gradient threshold. Observe the above equation (18) and (19), and change the spread symbol of the two equation, the anti-diffusion equation can be shown as follows:

$$\frac{\partial u}{\partial t} = (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{20}$$

$$\frac{\partial u}{\partial t} = -div[c(|\nabla u|).\nabla u] \tag{21}$$

Figure 1: Original Lucy Image

Figure 2: Enhanced Lucy Image with histogram equalization method

Figure 3: Enhanced Lucy Image with PDE Equation Method

The progress of image forward diffusion is equivalent to the Gaussian kernel convolution process, and the backward diffusion process is equivalent to the anti-convolution process. Reverse diffusion process also play an important role in other aspects of image processing. It is worth noting that the image can be sharpened only if the image has high signal-to-noise ratio, or else the sharpened image will have a low signal-to-noise ratio, the reason is after sharpening the noise increased greatly compare to the signal. Figure 1 of Lucy is the original image, Figure 2 is the enhanced image of Lucy after the use of histogram equalization method. Figure 3 is the enhanced image of Lucy with the use of PDE equation method proposed in this paper. It's obviously that the image quality after enhancement have a better edge details and overall quality.

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