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# Degradation Modeling Using Stochastic Filtering for Systems under Imperfect Maintenance

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Wiener process with a linear drift has been extensively studied in degradation modeling, mainly due to the existence of an analytical expression of the first hitting time distribution which permits feasible mathematical developments. However, a fundamental problem related to the stationary Wiener process is that it can only describe linearly drifted diffusion processes. This article is devoted to characterizing degradation phenomena with non-stationary Wiener processes. A new treatment is initiated to characterize the efficiency of imperfect maintenance, i.e., extending the improvement factor method on the degradation rate function. A stochastic filtering technique is employed to dynamically update the estimate of the degradation rate. A numerical example is given to illustrate the potential applications in real practice.

## 1. Introduction

Degradation modeling serves as an efficient modus operandi to evaluate reliability and to predict failure events for some highly reliable systems of which the event data are scarce. In the context of degradation modeling, massed degradation measurements can be recorded on each individual within a population, eliminating the necessity to wait until failure to obtain event data. Early work on degradation modeling is referenced by Ray and Phoha (1999), while more recent results are mentioned by Elwany et al. (2011) and Zio and Di Maio (2012). Wiener process with a linear drift is a widely used mathematical tactic to characterize degradation phenomena. A Wiener process $\{Z_t, t \geq 0\}$ with drift coefficient $\lambda > 0$ and variance parameter $\sigma^2$ can be formulated as $Z_t = \lambda t + \sigma B(t)$,, where $\{B(t), t \geq 0\}$ is the standard Brownian motion. Wiener process with a linear drift has stationary, independent and normally distributed increments, i.e., for all $0 \leq s < t$, $Z_t - Z_s$ is independent of $Z_s$ and has normal distribution $N\big(\lambda(t-s), \sigma^2(t-s)\big)$. The stationary Wiener process has mathematical advantages in that the distribution function of the first hitting time can be formulated analytically, known as the inverse Gaussian distribution. However, this statistical process is inadequate in modeling non-linearly drifted diffusion processes. Non-linearly drifted diffusion processes exist pervasively in practice (Bian and Gebraeel 2012, and Son et al. 2013), while a rather limited amount of research has been done on non-linearly drifted diffusion processes. This article investigates a maintenance strategy under which non-stationary Wiener processes are utilized to characterize degradation phenomena. We say that $\{X_t, t \geq 0\}$ is a non-stationary Wiener process with drift function $v(t)$ and variance parameter $\sigma^2$ if $X_t = v(t) + \sigma B(t)$, where $v(t)$ is a non-linear, right-continuous, real-valued function on $t \geq 0$ with $v(0) = 0$. $\{X_t, t \geq 0\}$ has independent, non-stationary and normally distributed random increments, i.e., for all $0 \leq s < t$, the random increment $X_t - X_s$ is independent of $X_s$ and has normal distribution $N\big(v(t) - v(s), \sigma^2(t-s)\big)$. Apparently, the non-stationary Wiener process reduces to the stationary Wiener process with $v(t)$ being a linear function in $t$.

Compared with the As Good As New and As Bad As Old assumptions on maintenance efficiency, it is more realistic in true experience that maintenance actions merely restore a system condition to somewhere between As Bad As Old and As Good As New. This situation is known as the imperfect maintenance. Extensive research on imperfect maintenance has been documented; see Wang (2002) and Lindqvist (2006) for a recent review on various treatments. One of the most popular treatments on

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imperfect maintenance is the improvement factor method in which each imperfect maintenance changes the system time of the failure rate curve to some newer time but not all the way to zero (not new); see Pham and Wang (1996). By assuming that the underlying deteriorating process conforms to the non-stationary Wiener process $\{X_t, t \geq 0\}$, starting from the installation of a new system, the expected degradation up to time $t$ is $E(X_t) = v(t)$. Therefore, the first-order derivative of $v(t)$, $v'(t)$, can be treated as the degradation rate function of the underlying deteriorating process. The concept of the improvement factor method can be extended to the degradation rate function $v'(t)$ to accessing maintenance efficiency. Specifically, we assume that the degradation rate function $v'(t)$, starting from the installation of a new system, changes into $\beta\, v'(t - t_1 + \alpha\, t_1)$ once an imperfect maintenance is released at time $t_1 \geq 0$. Here $0 < \alpha < 1$ is an age-reduction factor, and $\beta > 1$ is a degradation-rate-increase factor. Evaluating maintenance efficiency via degradation rate function instead of hazard rate function has a main advantage in that deriving the hazard rate function via the first hitting time distribution function is mathematically intractable, especially having introduced the non-linear drift function.

Another distinguishing feature of the proposed maintenance strategy is the utilization of a stochastic filtering technique to dynamically update the degradation rate function, once a new piece of monitoring information is obtained. The bulk of the documented research on the improvement factor method postulates a constant age-reduction factor and a constant hazard-rate-increase factor during a system's whole operational life cycle. Nevertheless, the assumption of constant improvement factors will be inappropriate and problematical in many circumstances. Intuitively, as the operational condition varies wildly, each maintenance action has a different degree of impact on the degradation rate. By invoking the stochastic filtering technique, we can dynamically access the maintenance efficiency and better predict the degradation trend. Diagnosis, cost models and the use of state space models are common in the field of condition monitoring (CM), as is usually found in the literature; see Ece and Basaran (2011) and Wang and Wang (2012).

The remainder of this paper is organized as follows. Section 2 develops the framework of degradation modeling using a stochastic filtering technique. A simulation study is given in Section 3 to illustrate the applicability and ascendency of the advanced strategy. Section 4 concludes the paper and points out possible topics for future research.

## 2. Model formulation

Non-stationary Wiener process $\{X_t, t \geq 0\}$ with drift function $v(t)$ and variance parameter $\sigma^2$ has independent and normally distributed increments. Define random variable $T_1$ to be the first hitting time of the degradation process $\{X_t, t \geq 0\}$ to failure threshold $l$, which is a pre-determined constant. The analytical form of the first hitting time distribution function of the non-stationary Wiener process can be obtained in very few cases, e.g., Buonocore et al. (2011). Si et al. (2012) developed a closed-form expression approximating the first hitting time distribution function for the non-stationary Wiener process under some mild assumptions, and the results will be employed in this paper. In order to illustrate the proposed maintenance strategy, we assume that the drift function in the non-stationary Wiener process is formulated to be $v(t) = \lambda t^{\theta}$, with $\lambda > 0$ and$\theta > 0$. The probability density function of the first hitting time $T_1$ can be approximated by

$$\tilde{f}(t; v(\cdot), \sigma, l) = \frac{1}{\sqrt{2\pi t}}\left[\frac{l - v(t)}{\sigma t} + \frac{v'(t)}{\sigma}\right]\exp\left\{-\frac{[l - v(t)]^2}{2\sigma^2 t}\right\}, \qquad t \geq 0. \tag{1}$$

Preventive maintenance is assumed to be imperfect. CM checking points are regularly arranged with identical interval $\triangle > 0$, i.e., the system will be checked at epochs $i\,\triangle$ $(\mathrm{i} = 1,2,3,\dots)$. Upon each checking epoch $i\,\triangle$ $(\mathrm{i} = 1,2,3,\dots)$, CM information $x_i$ is obtained instantly, and the inspection duration is negligible. Having obtained the CM information, engineers have to decide whether or not to take preventive maintenance at the checking point $i\,\triangle$. (We will not study how the maintenance decision is made.) To simplify matters, we assume that if an imperfect maintenance is released at epoch $t$, the degradation rate function immediately after the maintenance action changes from $v'(t)$ into $bv'(t)$. Here $0 < b \leq 1$ is the degradation-rate-reduction factor, and we do not consider the age-reduction factor. By re-writing $bv'(t) = b\theta\lambda t^{\theta-1}$as $bv'(t) = \theta\tilde{\lambda}t^{\theta-1}$, where $\tilde{\lambda} = b\lambda$, it is equivalent to the statement that the value of the scale parameter immediately after the maintenance action changes from $\lambda$ to $\tilde{\lambda}$. To access the degradation rate at each checking point, it is sufficient to estimate the scale parameter in the degradation rate function at each checking point. We develop an updating procedure, i.e. the Kalman filter, for recursively estimating the scale parameter (i.e., the degradation rate) at each checking point. The Kalman filter is an effective approach for discrete-time state estimation, and its applications have been extensively reported in operations research; see Si et al. (2011) for a recent review.

Particularly, by denoting $\lambda_i$ to be the hidden value of the scale parameter at the $i$th inspection epoch before any maintenance action and $x_i$ to be the CM observation at the $i$th inspection epoch, we have process equation

$$\lambda_i = b_{i-1,i}\lambda_{i-1} + w_i, \tag{2}$$

and measurement equation

$$y_i = \eta_i\lambda_i + \omega_i, \tag{3}$$

where $y_i = x_i - x_{i-1}$ denotes the degradation increment at each checking point and $x_0 = 0$. The measurement noise is assumed to be zero mean Gaussian white noise with variance $\sigma^2 \vartriangle$, i.e., $\omega_i \sim N(0, \sigma^2 \vartriangle)$. The linear relationship between the state variable, $\lambda_i$, and the measurement, $y_i$, and the normal assumption on the measurement noise are rooted in the non-stationary Wiener degradation process. The linear relationship between two consecutive states is rooted in the improvement factor method on the degradation rate function. To characterize the variability in the maintenance efficiency, we assume that the degradation-rate-reduction factor $b$ is a random variable following normal distribution$N(\bar{b}, Q)$. Therefore, if no maintenance action is released at the $(i-1)$th checking point, we have $b_{i-1,i} = 1$ and $w_i = 0$; and if an imperfect maintenance is released at the $(i-1)$th checking point, we have $b_{i-1,i} = \bar{b}$ and $w_i \sim N(0, Q\lambda_{i-1}^2)$. For notational convenience, we say that the process noise is assumed to be zero mean Gaussian white noise with variance $Q_i$, i.e., $w_i \sim N(0, Q_i)$.

At the $i$th checking point, we define $\hat{\lambda}_i^-$ to be the a prior estimate of the scale parameter, $\lambda_i$, given knowledge of the degradation process and maintenance history prior to (excluding) the $i$th checking point and $\hat{\lambda}_i$ to be the a posterior estimate of the scale parameter given new measurement $x_i$. By restricting the optimal estimate, $\hat{\lambda}_i$, to be a linear combination of the a prior estimate, $\hat{\lambda}_i^-$, and the degradation increment, $y_i$, and by setting the cost function to be the mean-square error, we arrive at the Kalman filter. Define $P_i^-$ to be the a prior error covariance matrix and $P_i$ to be the a posterior error covariance matrix at the $i$th checking point. For $i = 0$, $\hat{\lambda}_0$ and $P_0$ are chosen arbitrarily, reflecting our best assessment about the scale parameter prior to any CM information. For $i = 1,2,\ldots$, the recursive estimation of the scale parameter entails the following updating equations:

- state estimate propagation
  $$\hat{\lambda}_i^- = b_{i-1,i}\hat{\lambda}_{i-1}; \tag{4}$$
- error variance propagation
  $$P_i^- = b_{i-1,i}^2 P_{i-1} + Q_i; \tag{5}$$
- Kalman gain
  $$G_i = \frac{P_i^-\eta_i}{P_i^-\eta_i^2 + \sigma^2 \vartriangle}; \tag{6}$$
- state estimate update
  $$\hat{\lambda}_i = b_{i-1,i}\hat{\lambda}_{i-1} + G_i\left(y_i - \eta_i\hat{\lambda}_i^-\right); \tag{7}$$
- error variance update
  $$P_i = (1 - G_i\eta_i)P_i^-. \tag{8}$$

Consequently, the updated estimate and variance of $\lambda_i$are given by, respectively,

$$\hat{\lambda}_i = b_{i-1,i}\hat{\lambda}_{i-1} + \frac{\left(b_{i-1,i}^2 P_{i-1} + Q_i\right)\eta_i}{\left(b_{i-1,i}^2 P_{i-1} + Q_i\right)\eta_i^2 + \sigma^2 \vartriangle}\left(y_i - b_{i-1,i}\hat{\lambda}_{i-1}\right); \tag{9}$$

$$P_i = \frac{\left(b_{i-1,i}^2 P_{i-1} + Q_i\right) \times \sigma^2 \vartriangle}{\left(b_{i-1,i}^2 P_{i-1} + Q_i\right)\eta_i^2 + \sigma^2 \vartriangle}. \tag{10}$$

## 3. A numerical example

To demonstrate the application process and the performance of the proposed algorithms, a simulation study was conducted. The simulated degradation measurements $\{x_1, x_2, \ldots, x_3, \ldots\}$ are generated as follows. At the first checking point, we randomly simulate an observation $x_1$, which follows a normal distribution $N(\lambda \vartriangle^\theta, \sigma^2 \vartriangle)$. If an imperfect maintenance is then released at the first checking point, the scale parameter immediately after the maintenance action changes to $b\lambda$. Here, $b$ is a random variable following normal distribution $N(\bar{b}, Q)$. Therefore, at the second checking point, we randomly simulate a degradation increment $y_2$ from normal distribution $N\left(b\lambda\left[2^\theta - 1\right] \vartriangle^\theta, \sigma^2 \vartriangle\right)$. If no maintenance is taken at the first checking point, at the second checking point we randomly simulate a degradation increment $y_2$ from normal distribution $N\left(\lambda\left[2^\theta - 1\right] \vartriangle^\theta, \sigma^2 \vartriangle\right)$, and the second degradation measurement is $x_2 = x_1 + y_2$. By analogy, at the $i$th checking point, $i = 1,2,3, \ldots$, we randomly simulate a degradation increment $y_i$ from normal distribution $N\left(\prod_{j=1}^{i-1} b_j^{m_j} \lambda\left[i^\theta - (i-1)^\theta\right] \vartriangle^\theta, \sigma^2 \vartriangle\right)$, and the $i$ th degradation measurement is $x_i = x_{i-1} + y_i$. Here $m_i$ is an indicator function: if an imperfect maintenance is released at the $i$th checking point we have $m_i = 1$, and if there is no maintenance action at the $i$th checking point we have $m_i = 0$.

To demonstrate the competence of the proposed degradation-modeling strategy, we use in this section the following data set: $\lambda = 7.46 \times 10^{-5}$, $\theta = 1.63$, $\sigma = 1.762$, $\vartriangle = 600$, $\bar{b} = 0.92$ and $Q = 0.005$. To start with, we need to give the initial values, which are specified as follows: $\lambda_0 = 2$, $P_0 = 0.5$. In this paper, the value of the indicator $m_i$ is generated randomly, with $P(m_i = 1) = 0.28$ for all $i = 1,2,3, \ldots$. We use the Kalman filter to dynamically update the scale parameter $\lambda_i$. To visually demonstrate the competence of the proposed degradation modeling strategy, we plot the deviation $\hat{\lambda}_i - \lambda_i$ in Figure 1. As can be seen, the estimate converges rapidly as the CM points $i$ increases. Figure 2 plots the filter gains$P_i$, showing that the filter gain converges as well.

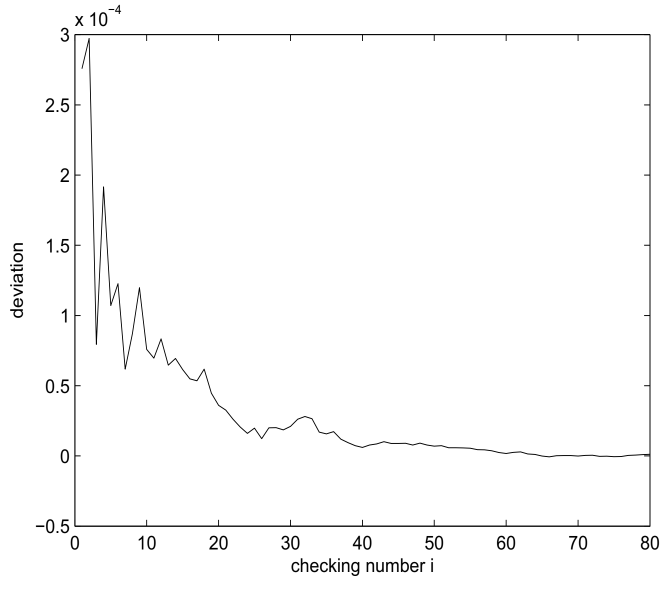


*Figure 1: $\hat{\boldsymbol{\lambda}}_i - \boldsymbol{\lambda}_i$ converges to zero as $\boldsymbol{i}$ increases.*

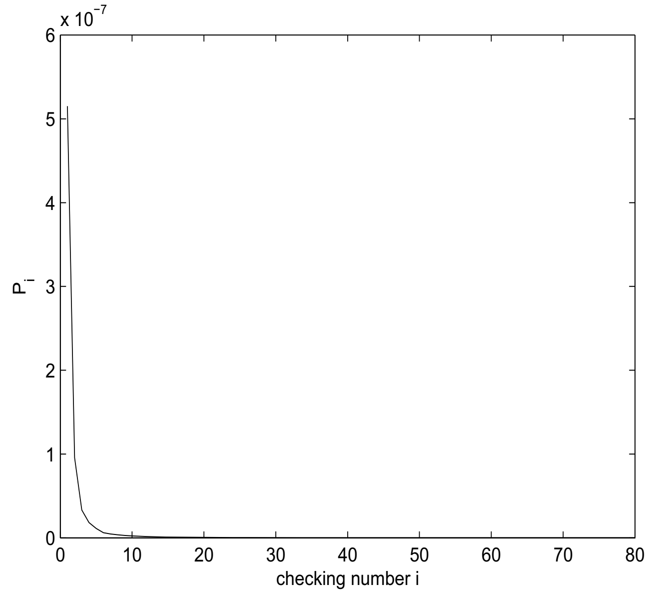


*Figure 2: Filter gain converges to zero as $\boldsymbol{i}$ increases.*

To show the robustness of the algorithm, we run the simulation for 100 times and plot the deviations $\{\hat{\boldsymbol{\lambda}}_{\boldsymbol{i}} - \boldsymbol{\lambda}_{\boldsymbol{i}}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$, with sample size being 100, in Figure 3. In Figure 3, all the deviations converge rapidly to zero, showing the efficiency of the stochastic filtering algorithm in assessing the scale parameter.

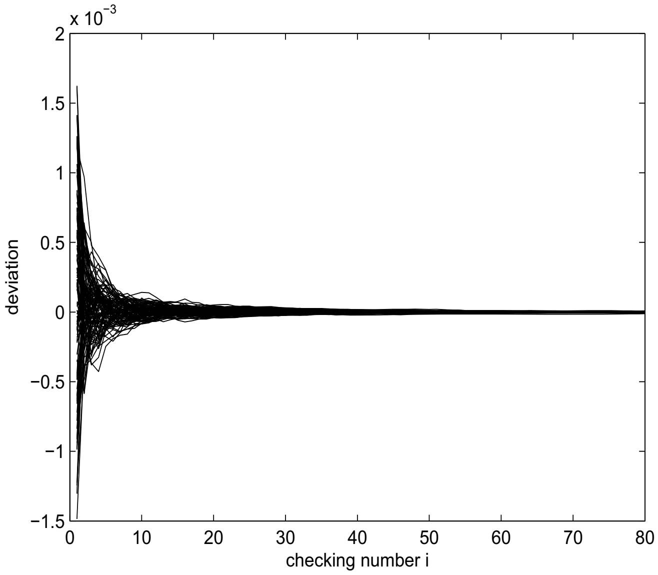


*Figure 3: Evolution paths of $\{\hat{\boldsymbol{\lambda}}_{\boldsymbol{i}} - \boldsymbol{\lambda}_{\boldsymbol{i}}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$ with 100 samples.*

## 4. Conclusion

This study has developed a degradation-modeling method for non-stationary Wiener degradation process, which can be employed in the condition based maintenance under which the maintenance is imperfect. The imperfect-maintenance assumption has rarely been studied in condition based maintenance, due to the complexity of degradation modeling, and this study has advanced an efficient tactic to deal with this problem. Simulation study shows that the stochastic filtering algorithm performs quite well in assessing degradation rate. Particularly, after running the algorithm 30 times, we are able to accurately estimate the degradation rate. To put it in another word, our algorithm only requires a sample with small sample size, showing another advantage of the proposed method.

Future research can be done in many directions. Most of the available research on degradation modeling postulates a deterministic failure threshold and assesses the reliability a component or system by comparing the projected degradation process to this critical threshold. However, since the operational condition from user to user varies wildly, each component or system should be treated individually. Under the circumstances, a probabilistic failure threshold is more adequate and reasonable. Therefore, it is of interest to further investigate how the maintenance strategy will perform when the failure threshold is a random variable. Moreover, for a practical implementation, it is necessary to fit degradation processes to the real data. A procedure of estimating the model parameters from the collected data has to be considered in a future work.

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