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Submanifolds of a (k, μ)-Contact Manifold

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ABSTRACT

The object of the present paper is to study submanifolds of (k, μ)-contact manifolds. We find the necessary and sufficient conditions for a submanifolds of (k, μ)-contact manifolds to be invariant and anti-invariant. Also, we examine the integrability of the distributions involved in the definition of CR-submanifolds of (k, μ)-contact manifolds.

RESUMEN

El objeto del presente artículo es estudiar subvariedades de variedades (k, μ)-contacto. Encontramos las condiciones necesarias y suficientes para que subvariedades de variedades (k, μ)-contacto sean invariantes y anti-invariantes. También examinamos la integrabilidad de las distribuciones involucradas en la definición de subvariedades CR de variedades (k, μ)-contacto.

Keywords and Phrases: (k, μ)-contact manifold; invariant submanifold; anti-invariant submanifold.

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1 Introduction

In 1995 Blair, Koufogiorgos and Papantoniou [4] introduced the notion of contact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution and such type of manifolds are called (k,μ)-contact manifolds.

To study the geometry of an unknown manifold, it is sometime convenient and yet interesting to first imbed it into a rather known manifold and then study its geometry side by side that of the ambient manifold. This approach gave birth to the introduction of submanifold theory. The study of complex submanifolds of a Kähler manifold from differential geometric points of view was initiated by Calabi [7] in the early 1950's and it was continued by several geometers like Blair, Ogiue [5], Chen, Verheyen [8], Yano, Kon [19], Yano, Ishihara [18], Kon [11] and many others. From then onwards submanifolds of a contact manifold have been major area of research and these submanifolds are divided into several types, mainly invariant, anti-invariant and semi-invariant.

The study of submanifolds of different contact manifolds is carried out from 1970 onwards by several authors, for example [9]-[12], while the study of submanifolds of (k,μ)-contact manifold have been done by Montano et al [13], Avjit Sarkar et al [1], Tripathi et al [16], Siddesha and Bagewadi [14] and others.

In [13], the authors have shown that invariant submanifolds of (k,μ)-contact manifold carries a (k,μ) structure and prove the totally geodesicity of invariant submanifolds when the second fundamental form is parallel. Later authors of [2] and [15] continued the work of above authors and they proved the totally geodesicity of recurrent, generalized recurrent of second fundamental form and semiparallel, pseudoparallel, Ricci-generalized pseudoparallel submanifolds.

Motivated by these studies of the above authors [2, 13, 16], in the present paper we find the necessary and sufficient conditions for the submanifolds to be invariant and anti-invariant. Also we study CR-submanifolds of (k,μ)-contact manifold and examine the integrability of the horizontal and vertical distributions involved in the definition of CR-submanifolds of (k,μ)-contact manifold. The paper is organized as follows: In section 2, we give a brief account of (k,μ)-contact manifolds and necessary details about submanifolds. In section 3, we show the existence of an invariant and anti-invariant submanifold, while, the section 4 deals with non-existence of an anti-invariant submanifold. Lastly in section 5, we consider CR-submanifolds of (k,μ)-contact manifold with distributions D and $D^\perp$, we find the conditions under which $D^\perp$ is integrable or totally geodesic.

2 Preliminaries

A contact manifold is a C^∞-$(2n+1)$ manifold $\tilde{M}^{2n+1}$ equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on $\tilde{M}^{2n+1}$. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on $\tilde{M}^{2n+1}$. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type (1,1) such that

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0, \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ g(X, \xi) = \eta(X), \tag{2.2}$$

for all vector fields X, Y on $\tilde{M}$. Then the structure (ϕ, ξ, η, g) on $\tilde{M}$ is called a contact metric structure and the manifold equipped with such a structure is called a contact metric manifold [3].

We now define a $(1,1)$ tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation, then h is symmetric and satisfies $h\phi = -\phi h$. Further, a q-dimensional distribution on a manifold M is defined as a mapping D on M which assigns to each point $p \in M$, a q-dimensional subspace D_p of T_pM.
The (k,μ)-nullity distribution of a contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ is a distribution

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) &= \{Z \in T_pM : \tilde{R}(X,Y)Z \\ &= k[\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y] + \mu[\tilde{g}(Y,Z)hX - \tilde{g}(X,Z)hY]\}, \end{aligned}$$

for all $X, Y \in T\tilde{M}$. Hence if the characteristic vector field ξ belongs to the (k,μ) nullity distribution, then we have

$$\tilde{R}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.3}$$

The contact metric manifold satisfying the relation (2.3) is called (k,μ) contact metric manifold [4]. It consists of both k-nullity distribution for $\mu = 0$ and Sasakian for $k = 1$. A (k,μ)-contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ satisfies

$$(\tilde{\nabla}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.4}$$

for all $X, Y \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g. From (2.4), we have

$$\tilde{\nabla}_X \xi = -\phi X - \phi hX, \tag{2.5}$$

for all $X, Y \in T\tilde{M}$. Again, if we put $\Omega(X,Y) = g(X, \phi Y)$, then Ω is a skew-symmetric $(0,2)$ tensor field [4]. Thus we have from (2.5)

$$\Omega(X + hX, Y) = (\tilde{\nabla}_X \eta)(Y). \tag{2.6}$$

Also from (2.4), it follows that

$$(\tilde{\nabla}_Z \Omega)(X,Y) = g(X, (\tilde{\nabla}_Z \phi)Y) = -g((\tilde{\nabla}_Z \phi)X, Y), \tag{2.7}$$

$$(\tilde{\nabla}_Z \Omega)(X,Y) = g(Z + hZ, Y)\eta(X) - \eta(Y)g(X, Z + hZ), \tag{2.8}$$

for any $X, Y \in T\tilde{M}$
Let M be a Riemannian submanifold of a (k,μ)-contact manifold $\tilde{M}$. Then the Gauss and Wein-

garten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.9}$$
$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.10}$$

for all $X, Y \in TM$ and each $N \in T^{\perp}M$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, σ is the second fundamental form of M and A is the shape operator with respect to the normal connection N. Then the shape operator A and the second fundamental form σ are related by

$$g(\sigma(X, Y), N) = g(A_N X, Y), \tag{2.11}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$. We denote by the same symbols g both metrics on $\tilde{M}$ and M.

Definition 1. *A submanifold* M *is said to be*
(i) totally geodesic in $\tilde{M}$ *if*

$$\sigma = 0 \text{ or equivalently } A_N = 0 \tag{2.12}$$

for each $N \in T^{\perp}M$.
(ii) Minimal in $\tilde{M}$ *if the curvature vector* H *satisfies*

$$H = \frac{Tr(\sigma)}{dim M} = 0 \tag{2.13}$$

and
(iii) totally umbilical if

$$\sigma(X, Y) = g(X, Y)H. \tag{2.14}$$

Put $\phi X = TX + NX$ for any tangent vector field X, where TX (resp. NX) denotes the tangential (resp. normal) component of ϕX. Similarly $\phi V = tV + nV$ for any normal vector field V with tV tangent and nV normal to M.
Then from straightforward calculation and using (2.4), (2.9) and (2.10), we obtain

Lemma 2.1. *Let* M *be a submanifold of a* (k, μ)*-contact manifold* $(\tilde{M}, \phi, \xi, \eta, g)$*, then*

$$(\nabla_X T)Y - t\sigma(X, Y) - A_{NY}X = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.15}$$
$$(\nabla_X N)Y + \sigma(X, TY) - n\sigma(X, Y) = 0, \tag{2.16}$$

for any vector fields $X, Y \in TM$.

3 Submanifolds of a (k, μ)-contact manifold

In this section, we define invariant and anti-invariant submanifolds of (k, μ)-contact manifold and prove the existence.

A submanifold M of a (k, μ)-contact manifold $\tilde{M}$ is said to be invariant (resp. anti-invariant) submanifold of $\tilde{M}$ if for each $x \in M$, $\phi(T_x M) \subset T_x M$ (resp. $\phi(T_x M) \subset T_x^{\perp} M$), here $T_x M$ and $T_x^{\perp} M$ are the tangent and normal bundles.
We first prove the following lemma:

Lemma 3.1. *For a submanifold M of a (k, μ)-contact manifold $\tilde{M}$, we have*

$$-\phi X - \phi hX = \nabla_X \xi + \sigma(X, \xi), \quad \xi \in TM, \tag{3.1}$$

$$-\phi X - \phi hX = -A_\xi X + \nabla_X^{\perp} \xi, \quad \xi \in T^{\perp} M \tag{3.2}$$

$$\eta(A_N X) = 0, \quad \xi \in T^{\perp} M \tag{3.3}$$

$$\eta(A_N X) = -g(\phi X + \phi hX, N), \quad \xi \in TM \tag{3.4}$$

for each $X \in TM$ and $N \in T^{\perp} M$.

Proof. From (2.5) and (2.9), we get (3.1). Also from (2.5) and (2.10), we obtain (3.2). Again, in view of (2.2), (3.3) is obvious. Now for $\xi \in TM$, and in view of (2.2), (2.5), (2.10) we get

$$\eta(A_N X) = g(\xi, A_N X) = -g(\xi, \tilde{\nabla}_X N) = g(\tilde{\nabla}_X \xi, N) = -g(\phi X + \phi hX, N).$$

This completes the proof of our lemma. □

Theorem 3.1. *Let M be a submanifold of a (k, μ)-contact manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then M is invariant if and only if $\sigma(X, \xi) = 0$, and M is anti-invariant if and only if $\nabla_X \xi = 0$.*

Since it is trivial from Lemma 3.2., we omit to prove our theorem.

Theorem 3.2. *If M is a totally umbilical submanifold of a (k, μ)-contact manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M, then*
(i) M is necessarily minimal and consequently totally geodesic and
(ii) M is an invariant submanifold of $\tilde{M}$ and $\nabla_X \xi \neq 0$.

Proof. Let M be a totally umbilical. Using (2.1), (2.2) and (3.1) in (2.14), we get

$$0 = \sigma(\xi, \xi) = g(\xi, \xi)H = H.$$

Hence in view of (2.13) and (2.14), we obtain (i). The second part follow from Theorem 3.1. and the above (i). □

Theorem 3.3. *A submanifold M of a (k, μ)-contact manifold $\tilde{M}$ with structure vector field ξ normal to M is anti-invariant in $\tilde{M}$ if and only if $A_\xi X = 0$. Consequently, if M is totally geodesic, then it is anti-invariant.*

Proof. Since ξ is normal to M, by virtue of (2.10) and (3.2) yields

$$g(-\phi X - \phi hX, Y) = g(A_\xi X, Y) = g(\sigma(X, Y), \xi), \ X, Y \in TM,$$

which provides the proof of our theorem. □

4 Non-existence of an anti-invariant distribution

This section is devoted to study of anti-invariant distribution.

A distribution D on a manifold M is said to be invariant under ϕ if $\phi D_x \subset D_x$ for each $x \in M$ and orthogonal complementary distribution $D^\perp$ on M is said to be anti-invariant under ϕ if $\phi D_x^\perp \subset D_x^\perp$ for each $x \in M$. Now we define semi-invariant submanifold as follows:
A submanifold M of a (k, μ)-contact manifold $\tilde{M}$ is said to be semi-invariant submanifold [6], if the following conditions are satisfied
(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D, $D^\perp$ are orthogonal distributions on M and $\{\xi\}$ is the 1-dimensional distribution spanned by ξ,
(ii) The distribution D is invariant by ϕ,
(iii) The distribution $D^\perp$ is anti-invariant under ϕ.
The distribution D(resp. $D^\perp$) is called the horizontal (resp. vertical) distribution. If both the distribution D and $D^\perp$ are non-zero then the semi-invariant submanifold is called a proper semi-invariant submanifold.
To prove the main result of this section first we prove the following lemmas:

Lemma 4.1. *For a submanifold* M *of a* (k, μ)*-contact manifold* $\tilde{M}$*, we have*

$$(\tilde{\nabla}_Z \Omega)(X, Y) = g(A_{\phi Y} X, Z) - \Omega(X, \nabla_Z Y) - \Omega(X, \sigma(Z, Y)) \tag{4.1}$$

for $Y \in D^\perp$, $X, Z \in TM$.

$$(\tilde{\nabla}_Z \Omega)(X, Y) = g(A_{\phi X} Y + A_{\phi Y} X, Z) \tag{4.2}$$

for all $X, Y \in D^\perp$, $Z \in TM$.

Proof. Let $Y \in D^\perp$, $Z \in TM$. Then, by virtue of (2.11) and the fact $\phi Y \in T^\perp M$, we get

$$(\tilde{\nabla}_Z \phi) Y = -A_{\phi Y} Z + \nabla_Z^\perp \phi Y - \phi(\tilde{\nabla}_Z Y). \tag{4.3}$$

Using this equation in (2.7), we can easily derive (4.1).
Next, in the special case of $X \in D^\perp$, since $\phi X \in T^\perp M$, (4.1) in view of (2.5) and (2.11) yields (4.2). □

Lemma 4.2. *Let* M *be a submanifold of a* (k, μ)*-contact manifold* $\tilde{M}$ *and* $D^\perp \perp \{\xi\}$*. Then we get*

$$(\tilde{\nabla}_Z \Omega)(X, X) = 0, \tag{4.4}$$

for $X \in D^{\perp}$ *and* $Z \in TM$*,and consequently*

$$A_{\phi X}X = 0, \tag{4.5}$$

for $X \in D^{\perp}$.

Proof. Since $D^{\perp} \perp \{\xi\}$, we have $\eta(X) = 0$ for any $X \in D^{\perp}$ and hence in view of (2.8), we get (4.4). Again (4.5) follows from (4.2) and (4.4). □

Theorem 4.1. *There does not exist any anti-invariant distribution* $D^{\perp}$ *on a submanifold* M *of a* (k, μ)*-contact manifold* $\tilde{M}$ *if* ξ *is tangent to* M *and* $D^{\perp} \perp \{\xi\}$.

Proof. Since $D^{\perp} \perp \{\xi\}$, we get $\eta(X) = 0$ for any $X \in D^{\perp}$. Thus, from (2.2), (3.4) and (4.5), we have

$$0 = \eta(A_{\phi X}X) = g(A_{\phi X}X, \xi) = -g(\phi X, \phi X + \phi hX) = -g(X, X + hX),$$

for any $X \in D^{\perp}$.
This implies

$$g(X, X) = 0.$$

Hence, X must be zero vector. Thus, if X is any arbitrary vector in $D^{\perp}$ then we have $X = 0$. Therefore, $D^{\perp} = 0$. This proves the theorem. □

Hence by virtue of Theorem 3.1, we have the following:

Corolary 1. *A* (k, μ)*-contact manifold does not admit any proper semi-invariant submanifold.*

5 CR-submanifolds of (k, μ)-contact manifold

In this section, we shall see the integrability conditions of the involved distributions D and $D^{\perp}$ in the definition of CR-submanifold M of a (k, μ)-contact manifold.

A submanifold M is said to be CR-submanifold in $\tilde{M}$ if there exist two orthogonal complementary distributions D and $D^{\perp}$ of TM such that $\xi \in TM$ and
(1) D is invariant by ϕ, i.e. $\phi(D_p) \subset D_p$, $\forall p \in M$,
(2) $D^{\perp}$ is anti-invariant by ϕ, i.e. $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, $\forall p \in M$.

Proposition 1. *Let* M *be a CR-submanifold of a* (k, μ)*-contact manifold* $\tilde{M}$*. Then,* D, $D^{\perp}$ *and* $D \oplus D^{\perp}$ *are* ξ*-parallel.*

Proof. For any $X \in D$ and $Y \in D^{\perp}$

$$\begin{aligned} g(\nabla_{\xi}X, \xi) &= \xi g(X, \xi) - g(X, \nabla_{\xi}\xi) = 0, \\ g(\nabla_{\xi}X, Y) &= \xi g(X, Y) - g(X, \nabla_{\xi}Y) = g(T^2X, \nabla_{\xi}Y) = -g(TX, T\nabla_{\xi}Y) = g(TX, \nabla_{\xi}TY) = 0, \end{aligned}$$

so $\nabla_\xi X \in D$, that is D is ξ-parallel.
Similarly, we can proceed for $D^\perp$. Finally, if D and $D^\perp$ are ξ-parallel, $D \oplus D^\perp$ also is. □

Lemma 5.1. *Let M be a submanifold of a (k, μ)-contact manifold. Then, $2g(X, TY) = \eta([X, Y])$ for all X, Y orthogonal to ξ.*

Proof. For a (k, μ)-contact manifold it holds that $d\eta = \Phi$. So $2g(X, TY) = 2\Phi(X, Y) = 2d\eta(X, Y) = \eta[X, Y]$. □

Lemma 5.2. *Let M be a CR-submanifold of a (k, μ)-contact manifold. Then, $D^\perp$ is integrable if and only if $d\Phi(X, Y, Z) = 0$, for any X tangent to M, $Y, Z \in D^\perp$.*

Proof. Consider X tangent to M and $Y, Z \in D^\perp$. Then,

$$\begin{aligned} 3d\Phi(X, Y, Z) &= X(\Phi(Y, Z)) + Y(\Phi(Z, X)) + Z(\Phi(X, Y)) \\ &\quad -\Phi([X, Y], Z) - \Phi([Z, X], Y) - \Phi([Y, Z], X) \\ &= -g([Y, Z], \phi X) = g(\phi[Y, Z], X), \end{aligned}$$

so $d\Phi(X, Y, Z) = 0$ if and only if $[Y, Z] \in KerT = D^\perp \oplus < \xi >$. This is equivalent to

$$[Y, Z] + \eta[Y, Z]\xi \in D^\perp,$$

but, using Lemma 5.5., $\eta([Y, Z]) = 2g(X, TY) = 0$. □

Now we can state the following theorem:

Theorem 5.1. *Let M be a CR-submanifold of a (k, μ)-contact manifold. Then, $D^\perp$ is always integrable.*

Proof. If M is a contact metric manifold, $d\Phi = d^2\eta = 0$ and so, the result follows from Lemma 5.6. □

Lemma 5.3. *Let M be a CR-submanifold of a (k, μ)-contact manifold. Then, $D^\perp \oplus < \xi >$ is integrable if and only if $d\Phi(X, Y, Z) = 0$, for any X tangent to M, $Y, Z \in D^\perp \oplus < \xi >$.*

Proof. Given $X \in TM$, $Y, Z \in D \oplus D^\perp$, we have $TY = TZ = 0$ and

$$3d\Phi(X, Y, Z) = -g([Y, Z], \phi X) = g(\phi[Y, Z], X).$$

So $[Y, Z]$ is normal if and only if $d\Phi(X, Y, Z) = 0$, for all X tangent to M. □

Again, from this lemma, we deduce:

Theorem 5.2. *Let M be a CR-submanifold of a (k, μ)-contact manifold. Then, $D^\perp \oplus < \xi >$ is always integrable.*

Finally, we characterize the integrability of $D\oplus <\xi>$

Theorem 5.3. *Let* M *be a CR-submanifold of a* (k,μ)*-contact manifold. Then,* $D\oplus <\xi>$ *is integrable if and only if* $\sigma(X,TY)-\sigma(Y,TX)=0$ *for all* $X,Y\in D\oplus <\xi>$*.*

Proof. Given $X,Y\in D\oplus <\xi>$, $[X,Y]$ belongs to $D\oplus <\xi>$ if and only if $N[X,Y]=0$.
Using (2.16),

$$\begin{aligned} N[X,Y] &= N\nabla_X Y - N\nabla_Y X \\ &= \nabla_X NY + \sigma(X,TY) - n\sigma(X,Y) - \nabla_Y NX - \sigma(Y,TX) + n\sigma(X,Y) \\ &= \sigma(X,TY) - \sigma(Y,TX), \end{aligned}$$

from which the proof follows. □

Theorem 5.4. *Let* M *be a CR-submanifold of a* (k,μ)*-contact manifold. Then,* M *is locally the product* $M_1\times M_2$*, where* M_1 *is a leaf of* $D\oplus <\xi>$ *and* M_2 *is a leaf of* $D^{\perp}$ *if and only if* $\sigma(X,TY)\in T\tilde{M}$*, for all* X *tangent to* M*,* $Y\in D^{\perp}$*.*

Proof. We shall prove that both $D\oplus <\xi>$ and $D^{\perp}$ are involutive and their leaves are totally geodesic immersed in M, so M is locally the product of these leaves.
For $Y\in D\oplus <\xi>$, $Z\in D^{\perp}$, by virtue of (2.4) and (2.9),

$$\begin{aligned} g(\nabla_X Y,Z) &= g(\tilde{\nabla}_X Y,Z) = g(\phi\tilde{\nabla}_X Y,\phi Z) \\ &= g(\tilde{\nabla}_X \phi Y + g(X+hX,Y)\xi - \eta(Y)(X+hX),NZ) \\ &= g(\tilde{\nabla}_X TY,NZ) \\ &= g(\sigma(X,TY),NZ). \end{aligned} \tag{5.1}$$

So if $X\in D\oplus <\xi>$, $\nabla_X Y\in D\oplus <\xi>$ if and only if $\sigma(X,TY)\in\tilde{D}$. Then $D\oplus <\xi>$ is involutive and its leaf is totally geodesic immersed in M.
Similarly, from (5.1), if $X\in D^{\perp}$, as $g(\nabla_X Z,Y)=-g(Z,\nabla_X Y)=-g(\sigma(X,TY),NZ)$, we have that $\nabla_X Y\in D\oplus <\xi>$ if and only if $\sigma(X,TY)\in\tilde{D}$. In this case, we obtain that $D\oplus <\xi>$ is also involutive and its leaf is totally geodesic immersed in M. □

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