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Calderón's reproducing Formula For q-Bessel operator

Belgacem Selmi
Faculté des Sciences de Bizerte,
Département de Mathématiques,
7021 Zarzouna, Tunisie.
belgacem.selmi@fsb.rnu.tn

ABSTRACT

In this paper a Calderón-type reproducing formula for q-Bessel convolution is established using the theory of q-Bessel Fourier transform [13, 17], obtained in Quantum calculus.

RESUMEN

En este trabajo se prueba una fórmula de tipo Calderón para convolución q-Bessel, usando la teoría de q-Bessel transformada de Fourier [13, 17], obtenida en cálculo cuántico.

Keywords and Phrases: q-Calderon, q-Calculus, q-Bessel Convolution, q-Fourier Bessel transform, q-Measure.

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1 Introduction

Calderón's formula [1] involving convolutions related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform, in decomposition of certain spaces and in characterization of Besov spaces [6, 8, 10]. Calderón's reproducing formula was also established for Bessel operator [4, 5]. This work is a continuation of a last work [9], and we establish formula for q-Bessel convolution for both functions and measures witch generalize the above one.
In the classical case this formula is expressed for a suitable function f as follows:

$$f(x) = \int_0^{\infty} (g_t * h_t * f)(x) \frac{dt}{t}, \tag{1}$$

where $g, h \in L^2(\mathbb{R})$ and $g_t(x) = \frac{1}{t} g(\frac{x}{t})$, $h_t(x) = \frac{1}{t} h(\frac{x}{t})$, $t > 0$ satisfying

$$\int_0^{\infty} \hat{g}(xt)\hat{h}(xt)\frac{dx}{x} = 1, \quad \text{for all } t \in \mathbb{R} \setminus \{0\},$$

where $\hat{g}$ and $\hat{h}$ is the usual Fourier transform of g and h on $\mathbb{R}$.
If μ is a finite Borel measure on the real line $\mathbb{R}$, identity (1) has natural generalization as follow

$$f(x) = \int_0^{\infty} (f * \mu_t)(x) \frac{dt}{t}, \tag{2}$$

where μ_t is the dilated measure of μ under some restriction on μ, the L^p-norm of (2) has proved in [2]. A general form of (2) has been investigated in [3].

In this paper we study similar questions when in (1) and (2) the classical convolution $*$ is replaced by the q-Bessel convolution $*_{\alpha,q}$ on the half line generated by the q-Bessel operator defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[x^{2\alpha+1} D_q f\right](q^{-1}x). \tag{3}$$

In this paper we prove that, for φ and $\psi \in L^1_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$ satisfying

$$\int_0^{\infty} \mathcal{F}_{\alpha,q}(\varphi)(\xi)\mathcal{F}_{\alpha,q}(\psi)(\xi)\frac{d_q\xi}{\xi} = 1 \tag{4}$$

we have

$$f(x) = \int_0^{\infty} (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x)\frac{d_qt}{t}, \quad f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x)). \tag{5}$$

where $d_q\sigma(x) = \dfrac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)} x^{2\alpha+1} d_qx = b_{\alpha,q} x^{2\alpha+1} d_qx$, $\varphi_t(x) = \dfrac{1}{t^{2\alpha+2}}\varphi(\dfrac{x}{t})$.

In particular for $\varphi \in L^1_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$ such that

$$\int_0^{\infty} \left[\mathcal{F}_{\alpha,q}(\varphi)(\xi)\right]^2 \frac{d_q\xi}{\xi} = 1, \tag{6}$$

and for a suitable function f, put

$$f^{\varepsilon,\delta}(x) = \int_{\varepsilon}^{\delta} (f *_{\alpha,q} \varphi_t *_{\alpha,q} \varphi_t)(x) \frac{d_q t}{t} \tag{7}$$

then

$$\|f^{\varepsilon,\delta} - f\|_{2,\alpha,q} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0 \text{ and } \delta \to \infty. \tag{8}$$

In the case $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$ such that $\mathcal{F}_{\alpha,q} f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$ one has

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} f^{\varepsilon,\delta}(x) = f(x),\ x \in \mathbb{R}. \tag{9}$$

Then we prove that for $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, such that the q-integral

$$c_{\mu,\alpha,q} = \int_0^{\infty} \mathcal{F}_{\alpha,q}(\mu)(\lambda) \frac{d_q\lambda}{\lambda} \tag{10}$$

is finite. Then for all $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$, we have

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} f^{\varepsilon,\delta} = c_{\mu,\alpha,q} f. \tag{11}$$

where the limit is in $L^2_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$. And if $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ is such that the q-integral

$$\int_0^{\infty} |\mu([0,y])| \frac{d_q y}{y} \tag{12}$$

is finite, for all $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x))$

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} f^{\varepsilon,\delta} = c_{\mu,\alpha,q} f, \quad \text{in } L^2_{\alpha,q}(\mathbb{R}_{q,+}, d_q\sigma(x)). \tag{13}$$

The outline of this paper is as follows: In Section 2, basic properties of q-Bessel transform on $\mathbb{R}_q$ of functions and bounded measure and its underlying q-convolution structure are called and introduced here. In Section 3, we give the first main result of the paper, the q-Calderon's reproducing formula for functions. Section 4 is consecrate to establish the same result as in section 3 for finite measures.

2 Preliminaries

In this section we recall some basic result in harmonic analysis related to the q-Bessel Fourier transform. Standard reference here is Gasper & Rahman [7].

For $a, q \in \mathbb{C}$ the q-shifted factorial $(a;q)_k$ is defined as a product of k factors:

$$(a;q)_k = (1-a)(1-aq)...(1-aq^{k-1}), \quad k \in \mathbb{N}^*; \qquad (a;q)_0 = 1. \tag{14}$$

If $|q| < 1$ this definition remains meaningful for $k = +\infty$ as a convergent infinite product:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{15}$$

We also write $(a_1, ..., a_r; q)_k$ for the product of r q-shifted factorials:

$$(a_1, \cdots, a_r; q)_k = (a_1;q)_k...(a_r;q)_k \qquad (k \in \mathbb{N} \text{ or } k = \infty). \tag{16}$$

A q-hypergeometric series is a power series (for the moment still formal) in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1, ..., a_r$ and s complex lower parameters $b_1, ..., b_s$ as follows:

$${}_r\varphi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k (q;q)_k} [(-1)^k q^{\frac{k(k-1)}{2}}]^{1+s-r} x^k \quad (\text{for } r, s \in \mathbb{N}).$$

2.1 q-Exponential series

$$e_q(z) = {}_1\varphi_0(0; -; q, z) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \quad (|\, z \,| < 1) \tag{17}$$

$$E_q(z) = {}_0\varphi_0(-; -; q, -z) = \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}k(k-1)} z^k}{(q;q)_k} = (-z;q)_\infty \quad (z \in \mathbb{C}). \tag{18}$$

2.2 q-Derivative and q-Integral

The q-derivative of a function f given on a subset of $\mathbb{R}$ or $\mathbb{C}$ is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad (x \neq 0, q \neq 0), \tag{19}$$

where x and qx should be in the domain of f. By continuity we set $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The q-shift operators are

$$(\Lambda_q f)(x) = f(qx), \qquad (\Lambda_q^{-1} f)(x) = f(q^{-1}x). \tag{20}$$

For $a \in \mathbb{R} \setminus \{0\}$ and a function f given on $(0, a]$ or $[a, 0)$, we define the q-integral by

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{21}$$

provided the infinite sum converges absolutely (for instance if f is bounded). If $F(a)$ is given by the left-hand side of (21) then $D_q F = f$. The right-hand side of (21) is an infinite Riemann sum.

For a q-integral over $(0, \infty)$ we define

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{-\infty}^{+\infty} f(q^k) q^k. \tag{22}$$

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_q$, we have

$$\int_0^{\infty} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{\infty} f(x) d_q x, \quad \int_0^{a} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{aq^n} f(x) d_q x. \tag{23}$$

The q-integration by parts is given for suitable functions f and g by:

$$\int_a^b f(x) D_q g(x) d_q x = \Big[f(x) g(x) \Big]_a^b - \int_a^b D_q f(x) g(x) d_q x. \tag{24}$$

The q-Logarithm $\log_q$ is given by [19]

$$\log_q x = \int \frac{d_q x}{x} = \frac{1-q}{\log q} \log x. \tag{25}$$

For all $a, b \in q^{\mathbb{Z}}, a < b$

$$\log_q (b/a) = (1-q) \sum_{k: a \leq q^k \leq b} 1. \tag{26}$$

The improper integral is defined in the following way

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{-\infty}^{+\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}. \tag{27}$$

We remark that for $n \in \mathbb{Z}$, we have

$$\int_0^{\infty/q^n} f(x) d_q x = \int_0^{\infty} f(x) d_q x. \tag{28}$$

The following property holds for suitable function f

$$\int_0^{\infty} \int_0^{x} f(x, y) d_q y d_q x = \int_0^{\infty} \int_{qy}^{\infty} f(x, y) d_q x d_q y. \tag{29}$$

2.3 The q-gamma function

The q-gamma function is defined by [7, 16]

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad 0<q<1; z\neq 0,-1,-2,... \tag{30}$$

$$= \int_0^{(1-q)^{-1}} t^{z-1}E_q(-(1-q)qt)d_qt, \quad (\mathrm{Re}\, z>0) \tag{31}$$

moreover the q-duplication formula holds

$$\Gamma_q(2z)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2z-1}\Gamma_q^2(z)\Gamma_{q^2}(z+\frac{1}{2}). \tag{32}$$

2.4 Some q-functional spaces

We begin by putting

$$\mathbb{R}_{q,+} = \{+q^k, k\in\mathbb{Z}\}, \qquad \widetilde{\mathbb{R}}_{q,+} = \{+q^k, k\in\mathbb{Z}\}\cup\{0\} \tag{33}$$

and we denote by

- $L^p_{\alpha,q}(\mathbb{R}_{q,+})$, $\quad p\in[1,+\infty[$, (resp. $L^\infty_{\alpha,q}(\mathbb{R}_{q,+})$) the space of functions f such that,

$$\|f\|_{p,\alpha,q} = (\int_0^\infty |f(x)|^p d_q\sigma(x))^{\frac{1}{p}} < +\infty. \tag{34}$$

$$(\text{resp. } \|f\|_{\infty,q} = \text{ess}\sup_{x\in\mathbb{R}_{q,+}} |f(x)| < +\infty). \tag{35}$$

- $\mathcal{S}_{q,*}(\mathbb{R}_q)$ the q-analogue of Schwartz space of even functions defined on $\mathbb{R}_q$ such that $D^k_{q,x}f(x)$ is continuous in 0 for all $k\in\mathbb{N}$ and

$$N_{q,n,k}(f) = \sup_{x\in\mathbb{R}_q} |(1+x^2)^n D^k_{q,x}f(x)| < +\infty. \tag{36}$$

- The q-analogue of the tempered distributions is introduced in [12] as follow:

(i) A q-distribution T in $\mathbb{R}_q$ is said to be tempered if there exists $C_q>0$ and $k\in\mathbb{N}$ such that:

$$|\langle T,f\rangle| \leq C_q N_{q,n,k}(f); \qquad f\in\mathcal{S}_{q,*}(\mathbb{R}_q). \tag{37}$$

(ii) A linear form T: $S_{q,*}(\mathbb{R}_q) \longrightarrow \mathbb{C}$ is said continuous if there exist $C_q>0$ and $k\in\mathbb{N}$ such that:

$$|\langle T,f\rangle| \leq C_q N_{q,n,k}(f); \quad f\in\mathcal{S}_{q,*}(\mathbb{R}_q). \tag{38}$$

- $\mathcal{S}'_{q,*}(\mathbb{R}_q)$ the space of even q-tempered distributions in $\mathbb{R}_q$. That is the topological dual of $\mathcal{S}_{q,*}(\mathbb{R}_q)$.
- $\mathcal{D}_{q,*}(\mathbb{R}_q)$ the space of even functions infinitely q-differentiable on $\mathbb{R}_q$ with compact support in $\mathbb{R}_q$. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.
- $\mathcal{C}_{q,*,0}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ continuous on 0, infinitely q-differentiable and

$$\lim_{x\to\infty} f(x)=0, \quad \|f\|_{\mathcal{C}_{q,*,0}} = \sup_{x\in\mathbb{R}_q} |f(x)| < +\infty. \tag{39}$$

- $\mathcal{H}_{q,*}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ continuous on 0 with compact support such that

$$\|f\|_{\mathcal{H}_{q,*}} = \sup_{x\in\mathbb{R}_q} |f(x)| \quad < +\infty. \tag{40}$$

2.5 q-Bessel function

The following properties of the normalized q-Bessel function is given (see [13]) by

$$j_\alpha(x;q^2) \quad = \quad \Gamma_{q^2}(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}(\frac{x}{1+q})^{2k}. \tag{41}$$

This function is bounded and for every $x\in\mathbb{R}_q$ and $\alpha > -\frac{1}{2}$ we have

$$|j_\alpha(x;q^2)| \quad \leq \quad \frac{1}{(q;q^2)^2_\infty}, \tag{42}$$

$$\left(\frac{1}{x}D_q\right) j_\alpha(.;q^2)(x) = -\frac{(1-q)}{(1-q^{2\alpha+2})} j_{\alpha+1}(qx;q^2), \tag{43}$$

$$\left(\frac{1}{x}D_q\right)(x^{2\alpha}j_\alpha(x;q^2)) = \frac{1-q^{2\alpha}}{1-q}x^{2(\alpha-1)}j_{\alpha-1}(x;q^2), \tag{44}$$

$$|D_q j_\alpha(x;q^2)| \quad \leq \quad \frac{(1-q)}{(1-q^{2\alpha+2})}\frac{x}{(q;q^2)^2_\infty}. \tag{45}$$

We remark that for $\lambda\in\mathbb{C}$, the function $j_\alpha(\lambda x;q^2)$ is the unique solution of the q-differential system

$$\begin{cases} \Delta_{q,\alpha}U(x,q) = -\lambda^2 U(x,q), \\ \\ U(0,q)=1;\ D_{q,x}U(x,q)|_{x=0} = 0, \end{cases} \tag{46}$$

where $\Delta_{q,\alpha}$ is the q-Bessel operator defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[x^{2\alpha+1} D_q f\right](q^{-1}x) \quad (47)$$

$$= q^{2\alpha+1}\Delta_q f(x) + \frac{1-q^{2\alpha+1}}{(1-q)q^{-1}x} D_q f(q^{-1}x), \quad (48)$$

where

$$\Delta_q f(x) = \Lambda_q^{-1} D_q^2 f(x) = (D_q^2 f)(q^{-1}x), \quad (49)$$

and for $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}_{q,+}$,

$$\Delta_{q,x}^k j_\alpha(\lambda x; q^2) = (-1)^k \lambda^{2k} j_\alpha(\lambda x; q^2). \quad (50)$$

2.6 q-Bessel Translation operator

$T_{q,x}^\alpha, x \in \mathbb{R}_{q,+}$ is the q-generalized translation operator associated with the q-Bessel transform is introduced in [13] and rectified in [17], where it is defined by the use of Jackson's q-integral and the q-shifted factorial as

$$T_{q,x}^\alpha f(y) = \int_0^{+\infty} f(t) D_{\alpha,q}(x,y,t) t^{2\alpha+1} d_q t, \quad \alpha > -1 \quad (51)$$

with

$$D_{\alpha,q}(x,y,z) = c_{\alpha,q}^2 \int_0^{+\infty} j_\alpha(xt;q^2) j_\alpha(yt;q^2) j_\alpha(zt;q^2) t^{2\alpha+1} d_q t$$

where

$$c_{\alpha,q} = \frac{1}{1-q} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

In particular the following product formula holds

$$T_{q,x}^\alpha j_\alpha(y,q^2) = j_\alpha(x,q^2) j_\alpha(y,q^2).$$

It is shown in [18] that for $f \in L_{\alpha,q}^1(\mathbb{R}_{q,+})$, $T_{q,x}^\alpha f \in L_{\alpha,q}^1(\mathbb{R}_{q,+})$ and

$$||T_{q,x}^\alpha f||_{1,\alpha,q} = ||f||_{1,\alpha,q}.$$

2.7 The q-convolution and the q-Bessel Fourier transform

The q-Bessel Fourier transform $\mathcal{F}_{\alpha,q}$ and the q-Bessel convolution product are defined for suitable functions f, g as follows

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x; q^2) d_q\sigma(x),$$

$$f *_{\alpha,q} g(x) = \int_0^{+\infty} T^{\alpha}_{q,x} f(y) g(y) d_q\sigma(y).$$

The q-Bessel Fourier transform $\mathcal{F}_{\alpha,q}$ is a modified version of the q-analogue of the Hankel transform defined in [15].
It is shown in [13, 17, 14], that the q-Bessel Fourier transform $\mathcal{F}_{\alpha,q}$ satisfies the following properties:

Proposition 2.1. *If* $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$*, then* $\mathcal{F}_{\alpha,q}(f) \in \mathcal{C}_{q,*,0}(\mathbb{R}_{q,+})$ *and*

$$\|\mathcal{F}_{\alpha,q}(f)\|_{\mathcal{C}_{q,*,0}} \le B_{\alpha,q} \|f\|_{1,\alpha,q}.$$

where

$$B_{\alpha,q} = \frac{1}{(1-q)} \frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)}{(q^2;q^2)_\infty}.$$

Proposition 2.2. *Given two functions* f, $g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$*, then*

$$f *_{\alpha,q} g \in L^1_{\alpha,q}(\mathbb{R}_{q,+}),$$

and

$$\mathcal{F}_{\alpha,q}(f *_{\alpha,q} g) = \mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g).$$

Theorem 2.3. *(Inversion formula)*
1. If $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *such that* $\mathcal{F}_{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$*, then*
for all $x \in \mathbb{R}_{q,+}$*, we have*

$$f(x) = \int_0^{\infty} \mathcal{F}_{\alpha,q}(f)(y) j_\alpha(xy;q^2) d_q\sigma(y).$$

2. $\mathcal{F}_{\alpha,q}(f)$ *is an isomorphism of* $\mathcal{S}_{*,q}(\mathbb{R}_q)$ *and* $\mathcal{F}^2_{\alpha,q}(f) = Id$.

• Note that the inversion formula is valid for $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ without the additional condition $\mathcal{F}_{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$.
$\mathcal{F}_{\alpha,q}(f)$ can be extended to $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ and we have the following theorem:

Theorem 2.4. *(q-Plancherel theorem)*
$\mathcal{F}_{\alpha,q}(f)$ *is an isomorphism of* $L^2_{\alpha,q}(\mathbb{R}_{q,+})$*, we have* $\|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}$, *for* $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *and* $\mathcal{F}^{-1}_{\alpha,q}(f) = \mathcal{F}_{\alpha,q}(f)$.

Proposition 2.5.

(i) For $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$, $p \in [1,\infty[$, $g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *we have* $f *_{\alpha,q} g \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$ *and* $\|f *_{\alpha q} g\|_{p,\alpha,q} \le \|f\|_{p,\alpha,q}\|g\|_{1,\alpha,q}$.

(ii) $\displaystyle\int_0^{\infty} \mathcal{F}_{\alpha,q}(f)(\xi) g(\xi) d_q\sigma(\xi) = \int_0^{\infty} f(\xi)\mathcal{F}_{\alpha,q}(g)(\xi) d_q\sigma(\xi); \quad f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+}).$

(iii) $\mathcal{F}_{\alpha,q}(T^{\alpha}_{q,x}f)(\xi) = j_\alpha(\xi x; q^2)\mathcal{F}_{\alpha,q}(f)(\xi); \quad f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$.

Specially, we choose $q \in [0, q_0]$ where q_0 is the first zero of the function [17]: $q \mapsto {}_1\phi_1(0, q, q; q)$ under the condition $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$.

Definition 2.6. *[11, 9] A bounded complex even measure μ on $\mathbb{R}_q$ is a bounded linear functional μ on $\mathcal{H}_{q,*}(\mathbb{R}_q)$, i.e., for all f in $\mathcal{H}_{q,*}(\mathbb{R}_q)$, we have*

$$|\mu(f)| \leq C\|f\|_{\mathcal{H}_{q,*}}, \tag{52}$$

where $C > 0$ is a positive constant.
Denote the space of all such measure by $\mathcal{M}'(\mathbb{R}_{q,+})$.

Note that $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ can be identified with a function $\widetilde{\mu}$ on $\widetilde{\mathbb{R}}_{q,+}$ such that $\widetilde{\mu}$ restricted to $\mathbb{R}_{q,+}$ is $L^1_{\alpha,q}(\mathbb{R}_{q,+})$:

$$\mu(f) = \mu(\{0\})f(0) + \int_0^\infty \widetilde{\mu}(x)f(x)d_q(x), \quad (f \in \mathcal{H}_{q,*}(\mathbb{R}_q)).$$

For $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ denote $\|\mu\| = |\mu|(\mathbb{R}_{q,+})$ where $|\mu|$ is the absolute value of μ.

Definition 2.7. *The q-Bessel Fourier transform of a measure μ in $\mathcal{M}'(\mathbb{R}_{q,+})$ is defined for all $\varphi \in \mathcal{S}_{q,*}(\mathbb{R}_q)$ by*

$$\mathcal{F}_{\alpha,q}\mu(\lambda) = b_{\alpha,q}\int_0^{+\infty} j_\alpha(\lambda x; q^2)d\mu(x). \tag{53}$$

The q-Bessel convolution product of a measure $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ and a suitable function f on $\mathbb{R}_{q,+}$ is defined by

$$\mu *_{\alpha,q} f(x) = \int_0^\infty T^{\alpha}_{q,x}f(y)d\mu(y). \tag{54}$$

Proposition 2.8. (1) *The q-Bessel Fourier transform $\mathcal{F}_{\alpha,q}$ of a measure μ in $\mathcal{M}'(\mathbb{R}_{q,+})$ is the q-tempered distribution $\mathcal{F}_{\alpha,q}\mu$ given by:*

$$\langle \mathcal{F}_{\alpha,q}\mu, \varphi\rangle = \langle \mu, \mathcal{F}_{\alpha,q}\varphi\rangle = \int_0^{+\infty} \mathcal{F}_{\alpha,q}\varphi(\lambda)d_q\mu(\lambda). \tag{55}$$

(2) *For all $x, \lambda \in \mathbb{R}_{q,+}$ we have*

$$T^{\alpha}_{q,x}\mathcal{F}_{\alpha,q}\mu(\lambda) = b_{\alpha,q}\int_0^{+\infty} j_\alpha(xt; q^2)j_\alpha(\lambda t; q^2)d_q\mu(t). \tag{56}$$

(3) *For all $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, $\mathcal{F}_{\alpha,q}\mu$ is continuous on $\mathbb{R}_{q,+}$, and*

$$\lim_{\lambda\to\infty} \mathcal{F}_{\alpha,q}\mu(\lambda) = \mu(\{0\}). \tag{57}$$

$\mathcal{F}_{\alpha,q}$ maps one to one $\mathcal{M}'(\mathbb{R}_{q,+})$ into $\mathcal{C}_b(\mathbb{R}_{q,+})$, (the space of continuous and bounded functions on $\mathbb{R}_{q,+}$).

(4) *If* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ *and* $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$, $p = 1,2$ *then* $\mu *_{\alpha,q} f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$ *and*

$$\|\mu *_{\alpha,q} f\|_{p,\alpha,q} \le \|\mu\|\|f\|_{p,\alpha,q}. \tag{58}$$

(5) *For all* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ *and* $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$, $p = 1,2$ *we have*

$$\mathcal{F}_{\alpha,q}(\mu *_{\alpha,q} f) = \mathcal{F}_{\alpha,q}(\mu)\mathcal{F}_{\alpha,q}(f). \tag{59}$$

Definition 2.9. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ *and* $a > 0$. *We define the* q*-dilated measure* μ_a *of* μ *by*

$$\int_0^\infty \varphi(x)d_q\mu_a(x) = \int_0^\infty \varphi(ax)d_q\mu(x), \quad \varphi \in \mathcal{H}_{q,*}(\mathbb{R}_q). \tag{60}$$

Proposition 2.10. *(i) When* $\mu = f(x)x^{2\alpha+1}d_qx$, *with* $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *the measure* μ_a, $a > 0$, *is given by the function*

$$f_a(x) = \frac{1}{a^{2\alpha+2}}f(\frac{x}{a}), \quad x \ge 0. \tag{61}$$

(ii) Let $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *then*

$$\mathcal{F}_{\alpha,q}(\mu_a)(\lambda) = \mathcal{F}_{\alpha,q}(\mu)(a\lambda), \quad \text{for all} \quad \lambda \ge 0. \tag{62}$$

(iii) For $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ *and* $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+}), p = 1,2$ *we have*

$$\lim_{a\to 0} \mu_a *_{\alpha,q} f = \mu(\widetilde{\mathbb{R}}_{q,+})f. \tag{63}$$

where the limit is in $L^p_{\alpha,q}(\mathbb{R}_{q,+})$.

(iv) Let $g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *and* $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$, $1 < p < \infty$. *Then*

$$\lim_{a\to\infty} f *_{\alpha,q} g_a = 0 \tag{64}$$

where the limit is in $L^p_{\alpha,q}(\mathbb{R}_{q,+})$.

Proof. Statement of (i) and (ii) are obvious. A standard argument gives (iii). Let us verify (iv). If $f, g \in \mathcal{D}_{q,*}(\mathbb{R}_q)$ then by (58) and (61) we have

$$\begin{aligned}\|f *_{\alpha,q} g_a\|_{p,\alpha,q} &\le \|f\|_{1,\alpha,q}\|g_a\|_{p,\alpha,q}\\ &= a^{\frac{-2(\alpha+1)(p-1)}{p}}\|f\|_{1,\alpha,q}\|g\|_{p,\alpha,q} \to 0, \quad \text{as} \quad a \to \infty.\end{aligned}$$

For arbitrary $g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ and $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+})$ the result follows by density. □

Given a measure $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$. Denote

$$c_{\mu,\alpha,q} = \int_0^\infty \mathcal{F}_{\alpha,q}(\mu)(\lambda)\frac{d_q\lambda}{\lambda}. \tag{65}$$

3 q-Calderón's formula for functions

In this section, we establish the q-Calderón's reproducing identity for functions using the properties of q-Fourier Bessel transform $\mathcal{F}_{\alpha,q}$ and q-Bessel convolution $*_{\alpha,q}$.

Theorem 3.1. *Let* φ *and* $\psi \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *be such that following admissibility condition holds*

$$\int_0^\infty \mathcal{F}_{\alpha,q}(\varphi)(\xi)\mathcal{F}_{\alpha,q}(\psi)(\xi)\frac{d_q\xi}{\xi} = 1 \tag{66}$$

then for all $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *the following Calderón's reproducing identity holds:*

$$f(x) = \int_0^\infty (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x)\frac{d_q t}{t}. \tag{67}$$

Proof. Taking q-Bessel Fourier transform of the right-hand side of (67), we get

$$\begin{aligned}
\mathcal{F}_{\alpha,q}\left[\int_0^\infty (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x)\frac{d_q t}{t}\right](\xi) &= \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\xi)\mathcal{F}_{\alpha,q}(\varphi_t)(\xi)\mathcal{F}_{\alpha,q}(\psi_t)(\xi)\frac{d_q t}{t} \\
&= \mathcal{F}_{\alpha,q}(f)(\xi)\int_0^\infty \mathcal{F}_{\alpha,q}(\varphi_t)(\xi)\mathcal{F}_{\alpha,q}(\psi_t)(\xi)\frac{d_q t}{t} \\
&= \mathcal{F}_{\alpha,q}(f)(\xi)\int_0^\infty \mathcal{F}_{\alpha,q}(\varphi)(t\xi)\mathcal{F}_{\alpha,q}(\psi)(t\xi)\frac{d_q t}{t} \\
&= \mathcal{F}_{\alpha,q}(f)(\xi).
\end{aligned}$$

Now, by putting $t\xi = s$, we get

$$\int_0^\infty \mathcal{F}_{\alpha,q}(\varphi)(t\xi)\mathcal{F}_{\alpha,q}(\psi)(t\xi)\frac{d_q t}{t} = \int_0^\infty \mathcal{F}_{\alpha,q}(\varphi)(s)\mathcal{F}_{\alpha,q}(\psi)(s)\frac{d_q s}{s} = 1.$$

Hence, the result follows. □

The equality (67) can be interpreted in the following L^2-sense.

Theorem 3.2. *Suppose* $\varphi \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *and satisfies*

$$\int_0^\infty [\mathcal{F}_{\alpha,q}(\varphi)(\xi)]^2 \frac{d_q\xi}{\xi} = 1. \tag{68}$$

For $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \cap L^2_{\alpha,q}(\mathbb{R}_{q,+})$, *suppose that*

$$f^{\varepsilon,\delta}(x) = \int_\varepsilon^\delta (f *_{\alpha,q} \varphi_t *_{\alpha,q} \varphi_t)(x)\frac{d_q t}{t} \tag{69}$$

then

$$\|f^{\varepsilon,\delta} - f\|_{2,\alpha,q} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0 \text{ and } \delta \to \infty. \tag{70}$$

Proof. Taking q-Bessel Fourier transform of both sides of (69) and using Fubini's theorem, we get

$$\mathcal{F}_{\alpha,q}(f^{\varepsilon,\delta})(\xi) = \mathcal{F}_{\alpha,q}(f)(\xi)\int_{\varepsilon}^{\delta}[\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2\,\frac{d_qt}{t}$$

by Proposition 2.5, we have

$$\begin{aligned}\|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{2,\alpha,q} &\leq \|\varphi_t *_{\alpha,q} \varphi_t\|_{1,\alpha,q}\|f\|_{2,\alpha,q}\\ &\leq \|\varphi_t\|^2_{1,\alpha,q}\|f\|_{2,\alpha,q}.\end{aligned}$$

Now using above inequality, Minkowski's inequality and relation (29), we get

$$\begin{aligned}\|f^{\varepsilon,\delta}\|^2_{2,\alpha,q} &= \int_0^\infty |\int_\varepsilon^\delta(\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)\frac{d_qt}{t}|^2 d_q\sigma(x)\\ &\leq \int_\varepsilon^\delta\int_0^\infty |(\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)|^2 d_q\sigma(x)\frac{d_qt}{t}\\ &\leq \int_\varepsilon^\delta \|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{2,\alpha,q}\frac{d_qt}{t}\\ &\leq \|\varphi_t\|^2_{1,\alpha,q}\|f\|_{2,\alpha,q}\int_\varepsilon^\delta\frac{d_qt}{t}\\ &= \|\varphi_t\|^2_{1,\alpha,q}\|f\|_{2,\alpha,q}\log_q(\frac{\delta}{\varepsilon}).\end{aligned}$$

Hence, by Theorem 2.4, we get

$$\begin{aligned}\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}}\|f^{\varepsilon,\delta}-f\|^2_{2,\alpha,q} &= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}}\|\mathcal{F}_{\alpha,q}(f^{\varepsilon,\delta})-\mathcal{F}_{\alpha,q}(f)\|^2_{2,\alpha,q}\\ &= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}}\int_0^\infty|\mathcal{F}_{\alpha,q}(f)(\xi)\left(1-\int_\varepsilon^\delta[\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2\,\frac{d_qt}{t}\right)|^2 d_q\sigma(x) = 0.\end{aligned}$$

Since $|\mathcal{F}_{\alpha,q}(f)(\xi)\left(1-\int_\varepsilon^\delta[\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2\,\frac{d_qt}{t}\right)| \leq |\mathcal{F}_{\alpha,q}(f)(\xi)|$, therefore, by the dominated convergence theorem, the result follows. □

The reproducing identity (67) holds in the pointwise sense under different sets of nice conditions.

Theorem 3.3. *Suppose* f, $\mathcal{F}_{\alpha,q}f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$. *Let* $\varphi \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *and satisfies*

$$\int_0^\infty[\mathcal{F}_{\alpha,q}\varphi(t\xi)]^2\,\frac{d_qt}{t} = 1 \tag{71}$$

then

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} f^{\varepsilon,\delta}(x) = f(x), \tag{72}$$

where $f^{\varepsilon,\delta}$ *is given by (69).*

Proof. by Proposition 2.5, we have

$$\|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{1,\alpha,q} \leq \|\varphi_t\|^2_{1,\alpha,q}\|f\|_{1,\alpha,q}.$$

Now

$$\begin{aligned} \|f^{\varepsilon,\delta}\|_{1,\alpha,q} &= \int_0^\infty |\int_\varepsilon^\delta (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)\frac{d_q t}{t}| d_q\sigma(x) \\ &\leq \int_\varepsilon^\delta \int_0^\infty |(\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)| d_q\sigma(x)\frac{d_q t}{t} \\ &\leq \int_\varepsilon^\delta \|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{1,\alpha,q}\frac{d_q t}{t} \\ &\leq \|\varphi_t\|^2_{1,\alpha,q}\|f\|_{1,\alpha,q}\log_q(\frac{\delta}{\varepsilon}). \end{aligned}$$

Therefore, $f^{\varepsilon,\delta} \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$. Also using Fubini's theorem and taking q- Bessel Fourier transform of $f^{\varepsilon,\delta}$, we get

$$\begin{aligned} \mathcal{F}_{\alpha,q}f^{\varepsilon,\delta}(\xi) &= \int_0^\infty j_\alpha(x\xi;q^2)\left(\int_\varepsilon^\delta (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)\frac{d_q t}{t}\right) d_q\sigma(x) \\ &= \int_\varepsilon^\delta \int_0^\infty j_\alpha(x\xi;q^2)(\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x) d_q\sigma(x)\frac{d_q t}{t} \\ &= \int_\varepsilon^\delta \mathcal{F}_{\alpha,q}\varphi_t(\xi)\mathcal{F}_{\alpha,q}\varphi_t(\xi)\mathcal{F}_{\alpha,q}f(\xi)\frac{d_q t}{t} \\ &= \mathcal{F}_{\alpha,q}f(\xi)\int_\varepsilon^\delta [\mathcal{F}_{\alpha,q}\varphi(t\xi)]^2\,\frac{d_q t}{t}. \end{aligned}$$

Therefore by (71), $|\mathcal{F}_{\alpha,q}f^{\varepsilon,\delta}(\xi)| \leq |\mathcal{F}_{\alpha,q}f(\xi)|$. It follows that $\mathcal{F}_{\alpha,q}f^{\varepsilon,\delta} \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$. By inversion, we have

$$f(x) - f^{\varepsilon,\delta}(x) = \int_0^\infty j_\alpha(x\xi;q^2)\left[\mathcal{F}_q f(\xi) - \mathcal{F}_{\alpha,q}f^{\varepsilon,\delta}(\xi)\right] d_q\sigma(\xi). \tag{73}$$

Putting

$$\begin{aligned} g^{\varepsilon,\delta}(x,\xi) &= j_\alpha(x\xi;q^2)\left[\mathcal{F}_{\alpha,q}f(\xi) - \mathcal{F}_{\alpha,q}f^{\varepsilon,\delta}(\xi)\right] \\ &= j_\alpha(x\xi;q^2)\mathcal{F}_q f(\xi)\left[1 - \int_\varepsilon^\delta [\mathcal{F}_{\alpha,q}\varphi(t\xi)]^2\,\frac{d_q t}{t}\right], \end{aligned} \tag{74}$$

we get

$$\begin{aligned} f(x) - f^{\varepsilon,\delta}(x) &= \int_0^\infty j_\alpha(x\xi;q^2)\left[\mathcal{F}_{\alpha,q}f(\xi) - \mathcal{F}_{\alpha,q}f^{\varepsilon,\delta}(\xi)\right] d_q\sigma(\xi) \\ &= \int_0^\infty g^{\varepsilon,\delta}(x,\xi) d_q\sigma(\xi). \end{aligned}$$

Now using (71) and (74), we get

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} g^{\varepsilon,\delta}(x,\xi) = 0. \tag{75}$$

Since $|g^{\varepsilon,\delta}(x,\xi)| \leq \frac{1}{(q;q^2)^2_\infty}|\mathcal{F}_{\alpha,q}f(\xi)|$, the dominated convergence theorem yields the result.
□

4 q-Calderón's formula for finite measures

It is now possible to define analogues to (2) for the q-Bessel convolution $*_{\alpha,q}$ and investigate its convergence in the $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ q-norm. To this end we need some technical lemmas

Lemma 4.1. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *for* $0 < \varepsilon < \delta < \infty$ *define*

$$G_{\varepsilon,\delta}(x;q^2) = \frac{\mu([\frac{x}{\delta},\frac{x}{\varepsilon}])}{x^{2\alpha+2}}, \quad x > 0 \tag{76}$$

and

$$K_{\varepsilon,\delta}(\lambda;q^2) = \int_\varepsilon^\delta \mathcal{F}_{\alpha,q}(\mu)(qa\lambda)\frac{d_q a}{a},\ \lambda \geq 0. \tag{77}$$

Then $G_{\varepsilon,\delta} \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ *and*

$$\mathcal{F}_{\alpha,q}(G_{\varepsilon,\delta})(\lambda;q^2) = K_{\varepsilon,\delta}(\lambda;q^2) - \mu(\{0\})\log_q(\frac{\delta}{\varepsilon}), \tag{78}$$

where $\log_q$ *is given by (25).*

Proof. We have by (25) and (29),

$$\begin{aligned} \left|\int_0^\infty G_{\varepsilon,\delta}(x;q^2)x^{2\alpha+1}d_qx\right| &\leq \int_0^\infty \left(\int_{\frac{x}{\delta}}^{\frac{x}{\varepsilon}} d_q|\mu|(y)\right)\frac{d_qx}{x} \\ &= \int_0^\infty \left[\int_0^{\frac{x}{\varepsilon}} d_q|\mu|(y) - \int_0^{\frac{x}{\delta}} d_q|\mu|(y)\right]\frac{d_qx}{x} \\ &= \int_0^\infty \left[\int_{q\varepsilon y}^\infty \frac{d_qx}{x} - \int_{q\delta y}^\infty \frac{d_qx}{x}\right] d_q|\mu|(y) \\ &= \int_0^\infty \log_q(\frac{\varepsilon}{\delta}) d_q|\mu|(y) \\ &= |\mu|(\widetilde{\mathbb{R}}_{q,+})\log_q(\frac{\varepsilon}{\delta}) < \infty. \end{aligned}$$

Using again relation (29) and q-Fubini's theorem we obtain

$$\begin{aligned}
\mathcal{F}_{\alpha,q}(G_{\varepsilon,\delta})(\lambda) &= \int_0^{\infty}\int_{\frac{x}{\delta}}^{\frac{x}{\varepsilon}} d_q\mu(y) j_\alpha(\lambda x;q^2)\frac{d_qx}{x}\\
&= \int_0^{\infty}\int_{q\varepsilon y}^{q\delta y} j_\alpha(\lambda x;q^2)\frac{d_qx}{x}d_q\mu(y)\\
&= \int_0^{\infty}\int_{q\varepsilon}^{q\delta} j_\alpha(\lambda xy;q^2)\frac{d_qx}{x}d_q\mu(y)\\
&= \int_{q\varepsilon}^{q\delta}\int_0^{\infty} j_\alpha(\lambda xy;q^2)d_q\mu(y)\frac{d_qx}{x}\\
&= \int_{q\varepsilon}^{q\delta} \mathcal{F}_{\alpha,q}\mu(\lambda x)-\mu(\{0\})\frac{d_qx}{x}\\
&= \int_{\varepsilon}^{\delta} \mathcal{F}_{\alpha,q}\mu(q\lambda x)-\mu(\{0\})\frac{d_qx}{x}\\
&= K_{\varepsilon,\delta}(\lambda;q^2)-\mu(\{0\})\log_q(\frac{\delta}{\varepsilon}).
\end{aligned}$$

□

Lemma 4.2. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *then for* $f \in L^p_{\alpha,q}(\mathbb{R}_{q,+}), p = 1,2$ *and* $0 < \varepsilon < \delta < \infty$, *the function*

$$f^{\varepsilon,\delta}(x;q^2) = \int_{\varepsilon}^{\delta} f *_{\alpha,q} \mu_a(x;q^2)\frac{d_qa}{a} \tag{79}$$

belongs to $L^p_{\alpha,q}(\mathbb{R}_{q,+})$ *and has the form*

$$f^{\varepsilon,\delta}(x;q^2) = f *_{\alpha,q} G_{q\varepsilon,q\delta}(x;q^2) + \mu(\{0\})f(x)\log_q(\frac{\delta}{\varepsilon}). \tag{80}$$

where $G_{\varepsilon,\delta}$ *is given by (4.1).*

Proof. Applying q-Fubini's theorem we get

$$\begin{aligned}
f^{\varepsilon,\delta}(x) &= \int_{\varepsilon}^{\delta}\int_0^{\infty} T^{\alpha}_{q,x}f(ay)d_q\mu(y)\frac{d_qa}{a}\\
&= \int_0^{\infty}\int_{\varepsilon}^{\delta} T^{\alpha}_{q,ay}f(x)\frac{d_qa}{a}d_q\mu(y)\\
&= f(x)\mu(\{0\})\log_q(\frac{\delta}{\varepsilon}) + \int_{\widetilde{\mathbb{R}}_{q,+}}\int_{\varepsilon y}^{\delta y} T^{\alpha}_{q,x}f(a)\frac{d_qa}{a}d_q\mu(y)\\
&= f(x)\mu(\{0\})\log_q(\frac{\delta}{\varepsilon}) + \int_{\widetilde{\mathbb{R}}_{q,+}} T^{\alpha}_{q,x}f(a)(\int_{q\frac{a}{\delta}}^{q\frac{a}{\varepsilon}}\frac{d_qa}{a})d_q\mu(y)\\
&= f(x)\mu(\{0\})\log_q(\frac{\delta}{\varepsilon}) + f *_{\alpha,q} G_{q\varepsilon,q\delta}(x).
\end{aligned}$$

From this relation, inequality (58) and Lemma 4.1 we deduce that $f^{\varepsilon,\delta}$ belongs to $L^p_{\alpha,q}(\mathbb{R}_{q,+})$. □

Lemma 4.3. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *then for* $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, *we have*

$$\mathcal{F}_{\alpha,q}(f^{\varepsilon,\delta})(\lambda;q^2) = \mathcal{F}_q(f)(\lambda;q^2)K_{q^{\varepsilon},q^{\delta}}(\lambda;q^2), \tag{81}$$

where $K_{\varepsilon,\delta}$, *is the function defined in (67).*

Proof. This follows from (59), (67) and (80). □

Theorem 4.4. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *be such that the* q*-integral*

$$c_{\mu,\alpha,q} = \int_0^\infty \mathcal{F}_{\alpha,q}(\mu)(\lambda)\frac{d_q\lambda}{\lambda} \tag{82}$$

be finite. Then for all $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, *we have*

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \|f^{\varepsilon,\delta} - c_{\mu,\alpha,q}f\|_{2,\alpha,q} = 0. \tag{83}$$

Proof. By identity (81) and Theorem 2.4 we have

$$\begin{aligned} \|f^{\varepsilon,\delta} - c_{\mu,\alpha,q}f\|^2_{2,\alpha,q} &= \|\mathcal{F}_{\alpha,q}(f^{\varepsilon,\delta}) - c_{\mu,\alpha,q}\mathcal{F}_{\alpha,q}(f)\|^2_{2,\alpha,q} \\ &= \|\mathcal{F}_{\alpha,q}(f)[K_{\varepsilon,\delta} - c_{\mu,\alpha,q}]\|^2_{2,\alpha,q}. \end{aligned}$$

Or $\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} K_{\varepsilon,\delta}(\lambda) = c_{\mu,\alpha,q}$, for all $\lambda > 0$ the result follows from the dominate convergence theorem. □

Lemma 4.5. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$, *be such that the* q*-integral*

$$\int_0^\infty |\mu([0,y])|\frac{d_qy}{y} \tag{84}$$

be finite. Then the q*-integral* $c_{\mu,\alpha,q}$ *is finite and admits the representation*

$$c_{\mu,\alpha,q} = \int_0^\infty \mu([0,y])\frac{d_qy}{y}. \tag{85}$$

Proof. From (76) we have

$$G_{\varepsilon,\delta} = \frac{\mu([\frac{x}{\delta},\frac{x}{\varepsilon}])}{x^{2\alpha+2}} = G_\varepsilon - G_\delta, \tag{86}$$

where

$$G(y) = \frac{\mu([0,y])}{y^{2\alpha+2}} \tag{87}$$

and G_ε, G_δ the dilated function of G. Since $G \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, we deduce from (62) and (78) that

$$\begin{aligned}\mathcal{F}_{\alpha,q}G_{\varepsilon,\delta}(\lambda) &= \int_{\varepsilon\lambda}^{\delta\lambda} \mathcal{F}_{\alpha,q}\mu(a)\frac{d_q a}{a} - \mu(\{0\})\log_q(\frac{\delta}{\varepsilon}) \\ &= \mathcal{F}_{\alpha,q}G(\varepsilon\lambda) - \mathcal{F}_{\alpha,q}G(\delta\lambda),\end{aligned} \tag{88}$$

for all $\lambda > 0$. Or (84) implies necessarily $\mu(\{0\}) = 0$. Hence when $\varepsilon = 1$ and $\delta \to \infty$, a combination of (88) and (57) gives

$$\mathcal{F}_{\alpha,q}G(\lambda) = \int_\lambda^\infty \mathcal{F}_{\alpha,q}\mu(a)\frac{d_q a}{a}, \quad \text{for all} \quad \lambda > 0. \tag{89}$$

Now the result follows from Formula (84) by using the continuity of $\mathcal{F}_{\alpha,q}(\mu)$. □

Theorem 4.6. *Let* $\mu \in \mathcal{M}'(\mathbb{R}_{q,+})$ *such that*

$$\int_0^\infty |\mu([0,y])|\frac{d_q y}{y} \tag{90}$$

is finite and $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$. *Then*

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \|f^{\varepsilon,\delta} - c_{\mu,\alpha,q}f\|_{2,\alpha,q} = 0. \tag{91}$$

Proof. By (80) and (86) we have

$$f^{\varepsilon,\delta} = f *_{\alpha,q} G_\varepsilon - f *_{\alpha,q} G_\delta, \tag{92}$$

where G is as in (87). Equation (91) is now a consequence of Proposition 2.5. □

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