CUBO A Mathematical Journal
Vol.16, Nº02, (49–52). June 2014

K-theory for the group C*-algebras of a residually finite discrete group with Kazhdan property T

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ABSTRACT

We compute the K-theory groups for the full and reduced group C^*-algebras of a residually finite, finitely generated discrete group with Kazhdan property T.

RESUMEN

Calculamos los grupos de la K-teoría para grupo de C^*-algebras de reducido y completo de un grupo discreto generado finitamente y residualmente finito con la propiedad T de Kazhdan.

Keywords and Phrases: Group C*-algebra, K-theory, discrete group, projection.

2010 AMS Mathematics Subject Classification: 46L05, 46L80, 19K14.

1 Introduction

In this paper, first of all, we compute the K-theory groups for the full group C^*-algebra of a residually finite, finitely generated discrete group with Kazhdan property T, such as $SL_n(\mathbb{Z})$ $(n \geq 3)$ the $n \times n$ special linear groups over the integers. The highly non-trivial and interesting problem to compute the K-theory groups has been considered by the author [6], but it was found to be not completed, and to be corrected as [7] (perhaps partly). This time we obtain a sort of solution for this problem to settle the issue, without using the results of [5] used in [6], but more precisely K_0 only, with a mysterious part left. We next compute the K-theory groups for the reduced group C^*-algebra of a residually finite, finitely generated discrete group with Kazhdan property T, by using the six-term exact sequence of K-groups and the results obtained in the full case.

2 The main result

Proposition 1. *Let Γ be a residually finite, finitely generated discrete group with Kazhdan property T and $C^*(\Gamma)$ its full group C^*-algebra. Then the K_0-group $K_0(C^*(\Gamma))$ has a direct summand isomorphic to the group generated by the infinite direct sum of copies of $\mathbb{Z}$ and one copy of $\mathbb{Z}$ corresponding to the unit.*

Proof. Since Γ is residully finite, then there is a (separable) family of finite dimensional irreducible representations π_λ of Γ such that the intersection of their kernels is trivial (see [4, p. 480]). Denote also by π_λ the corresponding finite dimensional irreducible representations of $C^*(\Gamma)$. Then $C^*(\Gamma)$ has a $*$-homomorphism (which can not be injective in general, see [2]) into the direct product C^*-algebra $\Pi_\lambda M_{n_\lambda}(\mathbb{C})$ of the $n_\lambda \times n_\lambda$ matrix algebras $M_{n_\lambda}(\mathbb{C})$ over $\mathbb{C}$, where $n_\lambda = \dim \pi_\lambda$, by the direct product representation $\Pi_\lambda \pi_\lambda$ of $C^*(\Gamma)$. The representation implies the K-theory homomorphism:

$$K_*(C^*(\Gamma)) \xrightarrow{(\Pi_\lambda \pi_\lambda)_*} K_*(\Pi_\lambda M_{n_\lambda}(\mathbb{C}))$$

for $* = 0, 1$, and $K_*(\Pi_\lambda M_{n_\lambda}(\mathbb{C})) \cong \Pi_\lambda K_*(M_{n_\lambda}(\mathbb{C}))$ with $K_0(M_{n_\lambda}(\mathbb{C})) \cong \mathbb{Z}$ and $K_1(M_{n_\lambda}(\mathbb{C})) \cong 0$. Note that since Γ is discrete, $C^*(\Gamma)$ has the unit and that the map $(\Pi_\lambda \pi_\lambda)_*$ is unital.

On the other hand, since Γ has Kazhdan propery T, then $C^*(\Gamma)$ has $M_{n_\lambda}(\mathbb{C})$ as a direct summand (see [9]). Hence $K_*(C^*(\Gamma))$ has $K_*(M_{n_\lambda}(\mathbb{C}))$ as a direct summand. Since $K_*(M_{n_\lambda}(\mathbb{C}))$ is mapped injectively under the induced map $(\Pi_\lambda \pi_\lambda)_*$, it follows that both $K_*(C^*(\Gamma))$ and the image of $K_*(C^*(\Gamma))$ contain the infinite direct sum $\oplus_\lambda K_*(M_{n_\lambda}(\mathbb{C}))$. Furthermore, all or nothing principle tells us that the image of $K_0(C^*(\Gamma))$ does not contain other classes corresponding to other non-trivial (infinite) projections in $\Pi_\lambda M_{n_\lambda}(\mathbb{C})$ except projections in the group generated by the direct sum $\oplus_\lambda \mathbb{Z}$ and $\mathbb{Z}$ of the unit class, because if it does contain, the principle implies that the image must be equal to $\Pi_\lambda K_0(M_{n_\lambda}(\mathbb{C}))$, so that $C^*(\Gamma)$ has $\Pi_\lambda M_{n_\lambda}(\mathbb{C})$ as a quotient, but the direct product is non-separable, while $C^*(\Gamma)$ is separable, a contradiction. Indeed, we can not find the difference among those extra infinite projections in $\Pi_\lambda M_{n_\lambda}(\mathbb{C})$. Hence the proof is completed. □

Remark. Unfortunately, we do not know about the mysterious kernel $\mathrm{Ker}(\Pi_\lambda \pi_\lambda)_*$ of the K-theory homomorphism in the K_0 and K_1-groups $K_*(C^*(\Gamma))$ which may not be trivial in general, so that we could not determine the K_0 and K_1-group.

Corollary 1. *For $n \geq 3$, the abelian group $K_0(C^*(SL_n(\mathbb{Z})))$ has a direct summand isomorphic to the group generated by an infinite direct sum of copies of $\mathbb{Z}$ and one copy of $\mathbb{Z}$.*

Proof. Note that $SL_n(\mathbb{Z})$ for $n \geq 3$ are residually finite, finitely generated groups with Kazhdan property T. Indeed, it is known that every finitely generated subgroup of $SL_n(\mathbb{C})$ is residually finite (see [1] and also [4]) and that $SL_n(\mathbb{Z})$ have Kazhdan property T (see [3, p. 34]). □

Theorem 1. *Let Γ be a non-amenable, residually finite, finitely generated discrete group with Kazhdan property T and $C^*_r(\Gamma)$ its reduced group C^*-algebra. Then*

$$K_0(C^*_r(\Gamma)) \cong \mathbb{Z} \oplus q_*[\mathrm{Ker}(\Pi_\lambda \pi_\lambda)_*]$$

and $K_1(C^*_r(\Gamma))$ is a quotient of $K_1(C^*(\Gamma))$, where this quotient and q_* are induced from the canonical quotient map $q : C^*(G) \to C^*_r(G)$.

Proof. Denote by $\mathfrak{I}_\Gamma$ the kernel of q. Then we have the following six-term diagram:

$$\begin{array}{ccccc} K_0(\mathfrak{I}_\Gamma) & \xrightarrow{i_*} & K_0(C^*(\Gamma)) & \xrightarrow{q_*} & K_0(C^*_r(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*_r(\Gamma)) & \xleftarrow{q_*} & K_1(C^*(\Gamma)) & \xleftarrow{i_*} & K_1(\mathfrak{I}_\Gamma) \end{array}$$

where i_* is induced by the inclusion $i : \mathfrak{I}_\Gamma \to C^*(\Gamma)$. Note that the infinite direct sum of $\mathbb{Z}$ in $K_0(C^*(\Gamma))$ is mapped to zero by q_* since Γ is non-amenable, so that $C^*_r(\Gamma)$ has no finite dimensional representation (a fact of the representation theory for Γ), and the other copy of $\mathbb{Z}$ in $K_0(C^*(\Gamma))$ is mapped injectively. It follows that $K_0(\mathfrak{I}_\Gamma)$ is isomorphic to the direct sum $\oplus\mathbb{Z}$. Since i_* on K_0 is injective, the index map from $K_1(C^*_r(\Gamma))$ is zero, so that q_* on $K_1(C^*(\Gamma))$ is surjective. Since the class of the unit in $K_0(C^*_r(\Gamma))$ is mapped to zero by the exactness of the diagram, it follows that $K_0(C^*_r(\Gamma)) \cong \mathbb{Z} \oplus q_*[\mathrm{Ker}(\Pi_\lambda \pi_\lambda)_*]$ and i_* on $K_1(\mathfrak{I}_\Gamma)$ is injective. □

Corollary 2. *For $n \geq 3$, we have*

$$K_0(C^*_r(SL_n(\mathbb{Z}))) \cong \mathbb{Z} \oplus q_*[\mathrm{Ker}(\Pi_\lambda \pi_\lambda)_*],$$

and $K_1(C^*_r(SL_n(\mathbb{Z})))$ is a quotient of $K_1(C^*(SL_n(\mathbb{Z})))$.

Remark. Note that $q_*[\mathrm{Ker}(\Pi_\lambda \pi_\lambda)_*]$ is not trivial. Because if it is zero, then $K_0(C^*_r(SL_n(\mathbb{Z}))) \cong \mathbb{Z}$, which implies that $C^*_r(SL_n(\mathbb{Z}))$ does not contain non-trivial projections. But $SL_n(\mathbb{Z})$ has torsion since it contains $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ as a subgroup, so that $C^*_r(SL_n(\mathbb{Z}))$ has non-trivial projections, a contradiction.

The Kadison-Kaplansky conjecture is that if Γ is a torsion free, discrete group, then $C^*_r(\Gamma)$ has no non-trivial projections. See [8] about the conjecture.

Received: March 2013. Revised: September 2013.

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