

Characterizations for certain analytic functions by series expansions with Hadamard gaps

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ABSTRACT

In this paper we characterize $Q_{K,\omega}(p, q)$ functions by lacunary series under mild conditions posed on the weight functions K and ω , where $Q_{K,\omega}(p, q)$ is a space of analytic functions defined in the unit disk generalizing the well known analytic Besov-type space.

RESUMEN

En este artículo caracterizamos las funciones $Q_{K,\omega}(p, q)$ por series lacunarias bajo condiciones medianas impuestas en las funciones de peso K y ω , donde $Q_{K,\omega}(p, q)$ es un espacio de funciones analíticas definidas en el disco unitario generalizando el conocido espacio analítico del tipo Besov.

Keywords and Phrases: $Q_{K,\omega}(p, q)$ -type spaces, Lacunary series.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} , let $H(\mathbb{D})$ denote the class of functions analytic in the unit disc \mathbb{D} , while $dA(z)$ denotes the Lebesgue area measure on the plane, normalized so that $A(\mathbb{D}) = 1$.

Let the Green's function of \mathbb{D} be defined as $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$. For $0 < r < 1$, let $\mathbb{D}(a, r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with center $a \in \mathbb{D}$ and radius r .

Definition 1.1. [17] Let $K : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and nondecreasing function, $0 < p < \infty, -2 < q < \infty$ and for given reasonable function $\omega : (0, 1] \rightarrow (0, \infty)$, an analytic function f in \mathbb{D} is said to belong to the space $Q_{K, \omega}(p, q)$ if

$$\|f\|_{Q_{K, \omega}(p, q)} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z, a)) dA(z) < \infty.$$

In the past few decades both Taylor and Fourier series expansions for various classes of analytic function spaces where the studies are done by the help of Hadamard gap class (see [1, 3, 10, 16] and others).

It is well known that a lacunary series belongs to BMOA if and only if it is in the Hardy space H^2 , (see [2]) for example. Very recently, in [6, 7, 14] there are some characterizations for some classes of meromorphic functions by the coefficients of certain lacunary series expansions in the unit disk. On the other hand there are some studies of the same problem in Clifford analysis (see [4, 5, 12, 13]).

We assume throughout the paper that

$$\int_0^1 (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} r dr < \infty.$$

An important tool in the study of $Q_{K, \omega}(p, q)$ space is the auxiliary functions ϕ_K and ψ_ω defined by

$$\phi_K(s) = \sup_{0 < t < 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty,$$

and

$$\psi_\omega(s) = \sup_{0 < t < 1} \frac{\omega(st)}{\omega(t)}, \quad 0 < s < 1.$$

The following conditions have played a crucial roles in the study of $Q_{K, \omega}(p, q)$ space:

$$\int_1^\infty \phi_K(s) \frac{ds}{s^2} < \infty. \quad (1)$$

(See [9, 17]) for example.

$$\int_1^{\frac{1}{s}} \psi_\omega(s) \frac{ds}{s^2} < \infty. \quad (2)$$

The function theory of $Q_{K,\omega}(p, q)$ obviously depends on the properties of K and ω . Given the weight functions ω_1, ω_2, K_1 and K_2 . We write $\omega_1 \lesssim \omega_2, K_1 \lesssim K_2$ if there exists a constant $C > 0$, independent of t , such that $\omega_1(t) \leq C\omega_2(t), K_1(t) \leq CK_2(t)$ for all t . The notation $\omega_1 \gtrsim \omega_2, K_1 \gtrsim K_2$ is used in a similar fashion. When $\omega_1 \lesssim \omega_2 \lesssim \omega_1$, we write $\omega_1 \approx \omega_2$. Also for $K_1 \lesssim K_2 \lesssim K_1$, we write $K_1 \approx K_2$.

2 Auxiliary Lemmas

In what follows we say $f \lesssim g$ (for two functions f and g) if there is a constant C such that $f \leq Cg$. We say $f \approx g$ (that is, f is comparable with g) whenever $g \lesssim f \lesssim g$. In this section we prove several result about the weight function that are needed for subsequent sections and are of some independent interest.

Lemma 2.1. [20] *If K satisfies condition (1), then the function*

$$K^*(t) = t \int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

has the following properties :

(A) K^* is nondecreasing on $(0, \infty)$.

(B) $K^*(t)/t$ is nondecreasing on $(0, \infty)$.

(C) $K^*(t) \geq K(t)$ for all $t \in (0, \infty)$.

(D) $K^* \lesssim K$ on $(0, 1]$.

If $K(t) = K(1)$ for $t \geq 1$, then we also have

(E) $K^*(t) = K^*(1) = K(1)$ for $t \geq 1$, so $K^* \approx K$ on $(0, \infty)$.

Lemma 2.2. [20] *If K satisfies condition (1), then we can find another non-negative weight function given by*

$$K^*(t) = t \int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

such that $Q_{K,\omega}(p, q) = Q_{K^,\omega}(p, q)$ and that the new function K^* has the following properties :*

(A) K^* is nondecreasing on $(0, \infty)$.

(B) $K^*(t)/t$ is nondecreasing on $(0, \infty)$.

(c) $K^*(t)$ satisfies condition (1)

(d) $K^*(2t) \approx K^*(t)$ on $(0, \infty)$.

(e) $K^*(t) \approx K(t)$ on $(0, 1]$.

(f) K^* is differentiable on $(0, \infty)$.

(g) K^* is concave on $(0, \infty)$.

(h) $K^*(t) = K^*(1)$ for $t \geq 1$,

Lemma 2.3. [8] If ω satisfies condition (2), then the function

$$\omega^*(t) = t \int_t^1 \frac{\omega(s)}{s^2} ds \quad (\text{where, } 0 < t < 1),$$

has the following properties :

(A) ω^* is nondecreasing on $(0, 1)$.

(B) $\omega^*(t)/t$ is nondecreasing on $(0, 1)$.

(C) $\omega^*(t) \geq \omega(t)$ for all $t \in (0, 1)$.

(D) $\omega^* \lesssim \omega$ on $(0, 1)$.

If $\omega(t) = \omega(1)$ for $t \geq 1$, then we also have

(E) $\omega^*(t) = \omega^*(1) = \omega(1)$ for $t \geq 1$, so $\omega^* \approx \omega$ on $(0, 1)$.

Lemma 2.4. Let $\alpha \geq 1$ and $\beta > 0$. If K satisfies (1) and ω satisfies (2), then

$$\int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \approx (\frac{1}{\alpha})^\beta \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})}$$

Proof. Let

$$I = \int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr.$$

By the change variables we have

$$I = \int_0^1 e^{-\alpha t} t^{\beta-1} \frac{K(t)}{\omega^p(t)} dt.$$

We write $I = I_1 + I_2$ where

$$I_1 = \int_{\frac{1}{\alpha}}^1 e^{-\alpha t} t^{\beta-1} \frac{K(t)}{\omega^p(t)} dt,$$

and

$$I_2 = \int_0^{\frac{1}{\alpha}} e^{-\alpha t} t^{\beta-1} \frac{K(t)}{\omega^p(t)} dt.$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} I_1 &\leq \frac{K(\frac{1}{\alpha})}{(\frac{1}{\alpha})} \times \frac{(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \int_{\frac{1}{\alpha}}^1 e^{-\alpha t} t^{\beta-1} dt, \\ &= \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \int_{\frac{1}{\alpha}}^1 e^{-\alpha t} t^{\beta-1} dt. \end{aligned}$$

Making the change of variables $s = \alpha t$, we have

$$\begin{aligned} I_1 &\leq \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \int_1^{\alpha} e^{-s} s^{\beta-1} \left(\frac{1}{\alpha}\right)^{\beta-1} \left(\frac{1}{\alpha}\right) ds \\ &= \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^{\beta} \int_1^{\alpha} e^{-s} s^{\beta-1} ds. \end{aligned}$$

Then

$$I_1 \leq C(\beta) \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^{\beta}.$$

Since $K(t)$ and $\omega(t)$ are non-decreasing, then by making the change of variables $s = \alpha t$, we obtain

$$I_2 \leq \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \int_0^{\frac{1}{\alpha}} e^{-\alpha t} t^{\beta-1} dt,$$

and

$$I_2 \leq \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^{\beta} \int_0^1 e^{-s} s^{\beta-1} ds.$$

Then

$$I_2 \leq C(\beta) \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^{\beta}.$$

Combining this with what was proved in the previous paragraph, we have

$$\int_0^1 r^{\alpha-1} \left(\log \frac{1}{r}\right)^{\beta-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \leq C(\beta) \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^{\beta},$$

where $C(\beta)$ is constant which only depends on (β) .

On the other hand, recall that $K(t)$ and $\omega(t)$ are non-decreasing. Then,

$$I \geq \int_{\frac{1}{\alpha}}^1 e^{-\alpha t} t^{\beta-1} \frac{K(t)}{\omega^p(t)} dt \geq \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \int_{\frac{1}{\alpha}}^1 e^{-\alpha t} t^{\beta-1} dt.$$

Making the change of variables $s = \alpha t$, we have

$$I \geq C(\beta) \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^\beta.$$

Then

$$\int_0^1 r^{\alpha-1} (\log \frac{1}{r})^{\beta-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \approx C(\beta) \frac{K(\frac{1}{\alpha})}{\omega^p(\frac{1}{\alpha})} \left(\frac{1}{\alpha}\right)^\beta.$$

This completes the proof of the lemma.

Lemma 2.5. Let $0 < \gamma \leq 1$ and $\eta(r) = \sum_{n=0}^{\infty} 2^{n\gamma} r^{2^n}$, $0 \leq r < 1$, then

$$\eta(r) \leq 2\Gamma(\gamma) (\log \frac{1}{r})^{-\gamma}.$$

Then above result can be found in [15].

Lemma 2.6. [19]

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta = \begin{cases} \log \frac{1}{|a|}, & 0 < r \leq |a|, \\ \log \frac{1}{r}, & |a| < r \leq 1. \end{cases}$$

By Jensen's formula (see [11, 18]), we can directly obtain the above result.

Theorem 2.7. Let K satisfy condition (1) and ω satisfy condition (2). Suppose that

$$f(x) = \sum_{n=1}^{\infty} a_n x^n,$$

with $a_n \geq 0$. For $\alpha > 0$, $p > 0$, we have that

$$\int_0^1 (1-x)^{\alpha-1} (f(x))^p \frac{K(\log \frac{1}{x})}{\omega^p(\log \frac{1}{x})} dx \approx \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})} \quad (3)$$

where $t_n = \sum_{k \in I_n} a_k$, $n \in \mathbb{N}$, $I_n = \{k : 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$.

Proof. Let $r_n = 1 - 2^{-n}$, $n = 1, 2, \dots$, then $r_n^{2^n-1} \geq \frac{1}{e}$. By Lemma 2.3, we have that

$$\begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n \right)^p (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \geq \int_{\frac{1}{2}}^1 \left(\sum_{n=0}^{\infty} t_n r^{2^{n+1}-1} \right)^p (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \gtrsim \sum_{n=0}^{\infty} t_n^p (r_{n+1}^{2^{n+1}-1})^p \int_{r_{n+1}}^{r_{n+2}} (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r_{n+1}})}{\omega^p(\log \frac{1}{r_{n+1}})} dr \\ & \gtrsim \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})}. \end{aligned}$$

The last inequality holds because of $(\log \frac{1}{r}) \geq 1 - r$.

Conversely, we first suppose that $p > 1$. Let $\gamma = \min\{1, \alpha/p\}$ and $\eta(r) = \sum_{k=0}^{\infty} 2^{k\gamma} r^{2k}$, $0 \leq r < 1$. Then by Jensen's inequality (see [11, 18]), we have

$$\left(\sum_{n=0}^{\infty} a_n r^n\right)^p \leq \left(\sum_{n=0}^{\infty} t_n r^{2^n}\right)^p \lesssim (\eta(r))^{p-1} \sum_{n=0}^{\infty} 2^{n\gamma(1-p)} r^{2^n} |t_n|^p.$$

From Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5, it follows that

$$\begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n\right)^p (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq \int_0^1 (2\Gamma(\gamma))^{p-1} (\log \frac{1}{r})^{-\gamma(p-1)} \left(\sum_{n=0}^{\infty} 2^{n\gamma(1-p)} r^{2^n} |t_n|^p (1-r)^{\alpha-1}\right) \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq 2^{n\gamma(1-p)} \sum_{n=0}^{\infty} |t_n|^p \int_0^1 r^{2^n-1} (\log \frac{1}{r})^{-\gamma(p-1)+\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq 2^{n\gamma(1-p)} \sum_{n=0}^{\infty} |t_n|^p \left(\frac{1}{2^n}\right)^{-\gamma(p-1)+\alpha} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} \\ & \leq \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})}. \end{aligned} \tag{4}$$

Denote $r(1-r)^{\alpha-1} \leq (\log \frac{1}{r})^{\alpha-1}$, $0 \leq r < 1$. Secondly suppose that $p = 1$, by Lemma 2.4, we obtain that

$$\begin{aligned} & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n\right) (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \int_0^1 \left(\sum_{n=0}^{\infty} t_n r^{2^n}\right) (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq \sum_{n=0}^{\infty} t_n \int_0^1 (r^{2^n-1}) r(1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq \sum_{n=0}^{\infty} t_n \int_0^1 (r^{2^n-1}) (\log \frac{1}{r})^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq \sum_{n=0}^{\infty} 2^{-n\alpha} t_n \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})}. \end{aligned} \tag{5}$$

Finally, if $p < 1$, we have

$$\begin{aligned}
 & \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^n \right)^p (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\
 & \int_0^1 \left(\sum_{n=0}^{\infty} t_n r^{2^n} \right)^p (1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\
 & \leq \sum_{n=0}^{\infty} t_n^p \int_0^1 \left(r^{p2^n-1} \right) r(1-r)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\
 & \leq \sum_{n=0}^{\infty} t_n^p \int_0^1 \left(r^{p2^n-1} \right) \left(\log \frac{1}{r} \right)^{\alpha-1} \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\
 & \leq \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})}
 \end{aligned} \tag{6}$$

From (4), (5) and (6), we obtain the desired result and the proof is therefore established.

3 Main results

We prove that an analytic function f on the unit disk \mathbb{D} with Hadamard gaps, that is, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ satisfying $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$, belongs to the space $Q_{K,\omega}(p, q)$ with mild conditions on the weight functions K and ω if and only if

$$\sum_{k=0}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty,$$

where $0 < p < \infty$ and $-1 < q < \infty$. Now, we give the following result:

Theorem 3.1. *Let K satisfy condition (1) and ω satisfy condition (2). Suppose that*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

with $a_n \geq 0$. For $\alpha > 0$, $p > 0$, we have

$$\int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(\log \frac{1}{|z|}) dA(z) \approx \sum_{n=1}^{\infty} n^{p-q-1} |a_n|^p \frac{K(\frac{1}{n})}{\omega^p(\frac{1}{n})}, \tag{7}$$

where $t_n = \sum_{k \in I_n} a_k$, $n \in \mathbb{N}$, $I_n = \{k : 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$.

Proof. We write

$$I(f) = \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(\log \frac{1}{|z|}) dA(z).$$

Integrating in polar coordinates we gets

$$\begin{aligned} I(f) &= \int_0^1 \int_0^{2\pi} \left(\sum_{n=1}^{\infty} n |a_n| r^{n-1} \right)^p (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} r dr d\theta \\ &\lesssim \int_0^1 \left(\sum_{n=1}^{\infty} n |a_n| r^{n-1} \right)^p (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr, \end{aligned}$$

by theorem 2.7, we obtain

$$\begin{aligned} I(f) &\lesssim \sum_{n=0}^{\infty} 2^{-n(q+1)} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})} \\ &\approx \sum_{n=1}^{\infty} n^{p-q-1} |a_n|^p \frac{K(\frac{1}{n})}{\omega^p(\frac{1}{n})}. \end{aligned}$$

The proof is therefore established.

Theorem 3.2. *Let $0 < p < \infty$, $-1 < q < \infty$. Let K satisfy condition (1) and ω satisfy condition (2). Suppose that*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k},$$

has Hadamard gaps, then f belongs to $Q_{K,\omega}(p, q)$ if and only if

$$\sum_{k=1}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty. \tag{8}$$

Proof. Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is a lacunary series in $Q_{K,\omega}(p, q)$. Without loss generality, we assume that $n_k \geq 1$. If f has Hadamard gaps, then $M_p(r, f') \approx M_2(r, f')$ (see [21]), where

$$M_p^p(r, f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta.$$

since $f \in Q_{K,\omega}(p, q)$, by theorem 2.7, we obtain

$$\begin{aligned} \infty &> \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(\log \frac{1}{|z|}) dA(z) \\ &= \int_0^1 M_p^p(r, f') (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} r dr, \\ &\gtrsim \int_0^1 \left(\sum_{k=1}^{\infty} n_k^2 |a_k|^2 r^{2(n_k-1)} \right)^{\frac{p}{2}} (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} r dr, \\ &\gtrsim \sum_{k=1}^{\infty} 2^{-k(q+1)} t_k^{\frac{p}{2}} \frac{K(\frac{1}{2^k})}{\omega^p(\frac{1}{2^k})}, \end{aligned}$$

where $t_k = \sum_{n_j \in I_k} n_j^2 |a_j|^2$. The Taylor series of f has at most $[\log_{\lambda} 2] + 1$ terms $a_j z^{n_j}$ when $n_j \in I_k$ for $k \geq 1$. By Hölder's inequality, we note that $t_k^{\frac{p}{2}} \gtrsim \sum_{n_j \in I_k} n_j^p |a_j|^p$. Then

$$\sum_{k=0}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} \lesssim \sum_{k=0}^{\infty} 2^{-k(q+1)} \left(\sum_{n_j \in I_k} n_j^p |a_j|^p \right) \frac{K(\frac{1}{2^k})}{\omega^p(\frac{1}{2^k})} < \infty.$$

Next suppose that condition (8) holds, we write $z = re^{i\theta}$ in polar form and observe that

$$|f(z)| = \sum_{k=1}^{\infty} |a_k| r^{n_k}.$$

Since K and ω satisfies conditions (1) and (2) respectively, K is concave. Then, by Jensen's inequality, Lemma 2.2 and Lemma 2.6, we deduce that

$$\int_0^{2\pi} K(g(re^{i\theta}, a)) d\theta \leq K\left(\int_0^{2\pi} (g(re^{i\theta}, a)) d\theta\right) \leq K\left(\log \frac{1}{r}\right).$$

Hence,

$$\begin{aligned} \|f\|_{\mathbb{K}, \omega}^p &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K\left(\log \frac{1}{|z|}\right) dA(z) \\ &\leq \sup_{a \in \mathbb{D}} \int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(1-r)} \\ &\quad \times \left\{ \frac{1}{2\pi} \int_0^{2\pi} K(g(re^{i\theta}, a)) d\theta \right\} r dr \\ &\leq \sup_{a \in \mathbb{D}} \int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(1-r)} K\left(\log \frac{1}{r}\right) r dr \end{aligned}$$

by Theorem 2.7, we obtain

$$\begin{aligned} \|f\|_{Q_{\mathbb{K}, \omega}(p, q)} &\lesssim \sum_{n=0}^{\infty} 2^{-n(q+1)} t_n^p \frac{K(\frac{1}{2^n})}{\omega^p(\frac{1}{2^n})} \\ &\lesssim \sum_{k=1}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty. \end{aligned}$$

The proof is therefore established.

Theorem 3.3. Let $0 < p < \infty$, $-1 < q < \infty$. Let K satisfy condition (1) and ω satisfy condition (2). Suppose that

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k},$$

has Hadamard gaps, $\omega > 0$, then f belongs to $Q_{\mathbb{K}, \omega}(p, q)$ if and only if $f \in Q_{\mathbb{K}, \omega, 0}(p, q)$.

Proof. Since sufficiency is obvious because of $Q_{\mathbb{K}, \omega, 0}(p, q) \subset Q_{\mathbb{K}, \omega}(p, q)$.

Now we will prove necessity of Theorem 3.3. Suppose that the lacunary series f belongs to $Q_{\mathbb{K}, \omega}(p, q)$. We must show that $I(a) \rightarrow 0$ as $|a| \rightarrow 1^-$, where

$$I(a) = \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K\left(\log \frac{1}{|z|}\right) dA(z).$$

From the proof of Theorem 3.2, we know that $f \in Q_{K,\omega}(p, q)$ implies that

$$\int_0^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(\log \frac{1}{r})} K(\log \frac{1}{r}) r dr < \infty.$$

For any $\epsilon < 0$, there is a $\delta \in (0, 1)$ such that

$$\int_{\delta}^1 \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr < \epsilon.$$

We may as well assume that $\lim_{|a| \rightarrow 1^-} K(\log \frac{1}{|a|}) = 0$. If K satisfies the condition (1). Then we choose a such that $1 > |a| > \delta$. By Lemma 2.6, Theorem 2.7 and Theorem 3.2, we have

$$\begin{aligned} & \int_0^{\delta} \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(1-r)} \int_0^{2\pi} K(g(z, a)) r dr d\theta \\ & \leq \int_0^{\delta} \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(1-r)} K(\log \frac{1}{|a|}) dr \\ & = K(\log \frac{1}{|a|}) \int_0^{\delta} \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p \frac{(1-r)^q}{\omega^p(1-r)} dr \\ & \leq \frac{K(\log \frac{1}{|a|})}{K(\log \frac{1}{\delta})} \int_0^{\delta} \left(\sum_{k=1}^{\infty} n_k |a_k| r^{(n_k-1)} \right)^p (1-r)^q \frac{K(\log \frac{1}{r})}{\omega^p(\log \frac{1}{r})} dr \\ & \leq \frac{K(\log \frac{1}{|a|})}{K(\log \frac{1}{\delta})} \sum_{k=1}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})}. \end{aligned}$$

Since ϵ is arbitrary, we conclude that $I(a) \rightarrow 0$ as $|a| \rightarrow 1^-$. So $f \in Q_{K,\omega,0}(p, q)$ and the proof is complete.

Carefully checking the proof of Theorems 3.2 and 3.3, we also obtain the following sufficient condition for $f \in Q_{K,\omega,0}(p, q)$ and hence in $f \in Q_{K,\omega,0}(p, q)$ in terms of Taylor coefficients.

Theorem 3.4. *Let $0 < p < \infty$ and $-1 < q < \infty$. Let K satisfy condition (1) and ω satisfy condition (2). Suppose that*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k},$$

has Hadamard gaps, then f belongs to $Q_{K,\omega,0}(p, q)$ if and only if

$$\sum_{k=0}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty. \tag{9}$$

4 Conclusion

From Theorems 3.2, 3.3 and 3.4, we can give the following theorem;

Theorem 4.1. Let $0 < p < \infty$, $-1 < q < \infty$. Let K satisfy condition (1) and ω satisfy condition (2). Suppose that

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in H(\mathbb{D})$$

has Hadamard gaps, then the following statements are equivalent:

- (i) $f \in Q_{K,\omega}(p, q)$;
- (ii) $f \in Q_{K,\omega,0}(p, q)$;
- (iii) $\sum_{k=0}^{\infty} n_k^{p-q-1} |a_k|^p \frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty$.

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