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Upper and Lower Solutions for $\phi-$Laplacian Third-order BVPs on the Half-Line

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ABSTRACT

In this paper, we investigate the existence of positive solution for a class of singular third-order boundary value problem associated with a ϕ-Laplacian operator and posed on the positive half-line:

$$\begin{cases} (\phi(-x''))'(t) + f(t, x(t)) = 0, \quad t > 0, \\ x(0) = \mu x'(0), \; x'(+\infty) = x''(+\infty) = 0 \end{cases}$$

where $\mu \geq 0$. By using the upper and lower solution approach and the fixed point theory, the existence of positive solutions is proved under a monotonic condition on f. The nonlinearity f may be singular at $x = 0$. An example of application is included to illustrate the main existence result.

RESUMEN

En este artículo investigamos la existencia de una solución positiva de una clase de problema singular de valores de frontera de tercer-orden asociado con el operador ϕ-Laplaciano y colocado sobre la semirecta real positiva:

$$\begin{cases} (\phi(-x''))'(t) + f(t, x(t)) = 0, \quad t > 0, \\ x(0) = \mu x'(0), \; x'(+\infty) = x''(+\infty) = 0 \end{cases}$$

donde $\mu \geq 0$. Usando la técnica de sub y súper soluciones y la teoría del punto fijo, se prueba la existencia de soluciones positivas bajo una condición de monotonicidad sobre f. La nolinealidad f puede ser singular en $x = 0$. Se incluye un ejemplo de aplicación para ilustrar el resultado principal de existencia.

Keywords and Phrases: Third order, half-line, $\phi-$Laplacian, singular problem, positive solution, fixed point, upper and lower solution.

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1 Introduction

This paper is concerned with the existence of positive solutions to the following third-order boundary value problem posed on the half-line and associated with a $\phi-$Laplacian operator:

$$\begin{cases} (\phi(-x''))'(t) + f(t, x(t)) = 0, \quad t > 0, \\ \\ x(0) = \mu x'(0),\ x'(+\infty) = x''(+\infty) = 0, \end{cases} \tag{1}$$

where $\mu \geq 0$ and $f = f(t, x) : \mathbb{R}^+ \times (0, +\infty) \longrightarrow \mathbb{R}^+$ is a continuous function which may have space singularity at $x = 0$ and $\mathbb{R}^+ = [0, +\infty)$. The map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, increasing homeomorphism such that $\phi(0) = 0$ (for instance the $p-$Laplacian $\varphi_p(s) = |s|^{p-1}s,\ \ p > 1$).

Boundary value problems (bvps for short) on the half-line appear in many applied problems relating to various phenomena in physics, biology, and combustion theory (see, e.g., [1] and references therein). In the last couple of years, the mathematical investigation of such problems, especially second-order boundary value problems have attracted several authors (see, e.g., [4], [5], [6], [7], [8], [9] and the references therein). However, only some of them were interested in higher-order differential equations on $[0, +\infty)$ (see [9], [11], [12]). The aim of this work to study a third-order differential equation with a ϕ-Laplacian derivative operator and posed on the positive half-line. Our approach is based on the upper and lower solution method adapted to this class of problems combined with the Schauder fixed point theorem. This papers essentially consists of three sections. Section 2 is devoted to some preliminaries facts and basic notions needed in this paper. A fixed point formulation is also provided in this section. In Section 3, we present our existence result of positive solutions when the nonlinearity f is monotonic with respect to x but may be singular at $x = 0$. The case f is not singular at $x = 0$ is also considered with less hypotheses. Our existence theorem is illustrated by means of an example of application. A function x is said to be a solution of problem (1) if

$$x \in X = \{x \mid x \in C^2((0, \infty), \mathbb{R})\} \text{ and } \phi(-x'') \in C^1((0, \infty), \mathbb{R})\} \tag{2}$$

and satisfies (1). In addition, x said to be a positive solution if $x(t) > 0$ for $t \in (0, +\infty)$.

2 Auxiliary Lemmas

A mapping defined on a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets. Let

$$C_l([0, \infty), \mathbb{R}) = \{x \in C([0, \infty), \mathbb{R}) \mid \lim_{t \to +\infty} x(t) \text{ exists}\}.$$

For $x \in C_l([0, \infty), \mathbb{R})$, define $\|x\|_l = \sup_{t \in \mathbb{R}^+} |x(t)|\cdot$ Then $(C_l, \|x\|_l)$ is a Banach space.

Lemma 2.1. *([3], p. 62) Let* $M \subseteq C_l(\mathbb{R}^+, \mathbb{R})$*. Then* M *is relatively compact in* $C_l(\mathbb{R}^+, \mathbb{R})$ *if the following three conditions hold:*
(a) M *is uniformly bounded in* $C_l(\mathbb{R}^+, \mathbb{R})$*.*
(b) The functions belonging to M *are almost equicontinuous on* $\mathbb{R}^+$*, i.e., equicontinuous on every compact interval of* $\mathbb{R}^+$*.*
(c) The functions from M *are equiconvergent, that is, for every* $\varepsilon > 0$*, there exists* $T(\varepsilon) > 0$ *such that* $|x(t) - x(+\infty)| < \varepsilon$ *for any* $t \geq T(\varepsilon)$ *and* $x \in M$*.*

Note that the space

$$E = \{x \in C([0, \infty), \mathbb{R}) \mid \lim_{t \to +\infty} \frac{x(t)}{1+t} \text{ exists}\}. \tag{3}$$

is also a Banach space with the norm $\|x\| = \sup\limits_{t \in \mathbb{R}^+} \frac{|x(t)|}{1+t}$. From Lemma 2.1, we easily deduce

Lemma 2.2. *Let* $M \subseteq E$*. Then* M *is relatively compact in* E *if the following conditions hold:*
(a) M *is bounded in* E*,*
(b) the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ *are locally equicontinuous on* $[0, +\infty)$*,*
(c) the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ *are equiconvergent at* $+\infty$*.*

Definition 2.3. *Let* $\alpha, \beta \in X$*. Then* α *is called a lower solution of (1) if* α *satisfies*

$$\begin{cases} (\phi(-\alpha''(t)))' + f(t, \alpha(t)) \geq 0, \quad t > 0 \\ \\ \alpha(0) \leq \mu\alpha'(0), \quad \lim\limits_{t \to +\infty} \alpha'(t) \leq 0, \ \lim\limits_{t \to +\infty} \alpha''(t) \geq 0. \end{cases}$$

β is called an upper solution of (1) if the above inequalities are reversed. Let

$$G(t, s) = \begin{cases} s + \mu, & 0 \leq s \leq t < +\infty \\ t + \mu, & 0 \leq t \leq s < +\infty, \end{cases}$$

be the Green function of the linear problem $-x'' = x(0) - \mu x'(0) = x'(+\infty) = 0$. The following lemmas are straightforward; the proofs are omitted.

Lemma 2.4. *Assume that* $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies* $\int_0^{+\infty} \delta(s)ds < +\infty$ *and let* $x(t) = \int_0^{+\infty} G(t, s)\delta(s)ds$*. Then*

$$\begin{cases} x''(t) + \delta(t) = 0, \quad t > 0, \\ \\ x(0) = \mu x'(0), \quad \lim\limits_{t \to +\infty} x'(t) = 0. \end{cases} \tag{4}$$

Lemma 2.5. *Assume that* $\delta \in C(\mathbb{R}^+, \mathbb{R}^+) \cap L^1(r, +\infty)$ *for all* $r > 0$ *and*

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right) ds < +\infty.$$

If $x(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty}\delta(\tau)d\tau\right)ds$, *then* $x \in X$ *and*

$$\begin{cases} (\phi(-x''(t)))' + \delta(t) = 0, \quad t > 0, \\ \\ x(0) = \mu x'(0),\ x'(+\infty) = x''(+\infty) = 0. \end{cases} \tag{5}$$

Consider the positive cone

$$S = \{x \in C^1[0,+\infty) \mid\ x(t) \geq 0, \text{ concave on } [0,+\infty),\ \lim_{t\to+\infty} x'(t) = 0\}. \tag{6}$$

In addition to the null function, S contains, e.g., $\ln(1+t)$, so S is a nonempty subset; moreover S has the following properties:

Lemma 2.6. *Let $x \in S\setminus\{0\}$. Then there exists a positive constant λ_x such that*

(a) for all $\theta > 1$, $\quad x(t) \geq \lambda_x \frac{1}{\theta}, \quad \forall\, t \in [1/\theta, \theta]$,

(b) If

$$\rho(t) = \begin{cases} t, & t \in [0,1] \\ 1, & t \geq 1 \end{cases} \tag{7}$$

then

$$x(t) \geq \lambda_x \rho(t),\ \forall\, t \in [0,+\infty).$$

Proof.

(a) Notice that every $x \in S$ is nondecreasing and thus by L'Hopital's rule $\lim\limits_{t\to+\infty} \frac{x(t)}{1+t} = 0$; as a consequence, the function $\frac{x(t)}{1+t}$ achieves its maximum at some point $t_m \in [0,+\infty)$; let $\lambda_x = \frac{x(t_m)}{1+t_m} = \|x\| > 0$. By concavity of x, we have for $t \in [1/\theta, \theta]$

$$\begin{aligned} x(t) &\geq \min_{t\in[\frac{1}{\theta},\theta]} x(t) = x(\tfrac{1}{\theta}) = x(\tfrac{\theta-1+\theta t_m}{\theta+\theta t_m}\tfrac{1}{\theta-1+\theta t_m} + \tfrac{1}{\theta+\theta t_m}t_m) \\ &\geq \tfrac{\theta-1+\theta t_m}{\theta+\theta t_m}x(\tfrac{1}{\theta-1+\theta t_m}) + \tfrac{1}{\theta+\theta t_m}x(t_m) \\ &\geq \tfrac{1}{\theta+\theta t_m}x(t_m) = \tfrac{1}{\theta}\tfrac{x(t_m)}{1+t_m} = \lambda_x\tfrac{1}{\theta}, \end{aligned}$$

whence the first part of the lemma.

(b) Fix $t_0 \in [0,+\infty)$ and distinguish between four cases.

(i) If $t_0 = 0$, then $x(0) \geq 0 = \lambda_x\rho(0)$.

(ii) If $t_0 \in (0,1)$, then $\frac{1}{t_0} \in (1,+\infty)$. From Part (a), $x(s) \geq \lambda_x t_0, \forall\, s \in [t_0, \frac{1}{t_0}]$. In particular for $s = t_0$, $x(t_0) \geq \lambda_x t_0 = \lambda_x\rho(t_0)$.

(iii) If $t_0 = 1$, let $\{t_n\}_n$ be a real sequence such that $0 < t_n < 1$ and $t_n \to 1$, as $n \to +\infty$. By (ii), we know that $x(t_n) \geq \lambda_x t_n$, $\forall n \geq 1$. Then

$$x(1) = \lim_{n\to+\infty} x(t_n) \geq \lambda_x \lim_{n\to+\infty} t_n = \lambda_x = \lambda_x \rho(1).$$

(iv) Finally, let $t_0 \in (1, +\infty)$, since x is nondecreasing, then $x(t_0) \geq x(1) \geq \lambda_x = \lambda_x \rho(t_0)$, ending the proof of the lemma. □

3 Main Existence Results

First we list some assumptions:

$(\mathcal{H}_1)$ $f \in C(\mathbb{R}^+ \times (0, +\infty), \mathbb{R}^+)$ and $f(t, x)$ is a nonincreasing relatively to the second argument.

$(\mathcal{H}_2)$ For every $\lambda > 0$,

$$\int_0^{+\infty} f(\tau, \lambda\rho(\tau))d\tau < +\infty, \quad \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda\rho(\tau))d\tau\right) ds < +\infty.$$

$(\mathcal{H}_3)$ There exists a function $a \in S \setminus \{0\}$ such that for $t \geq 0$

$$\begin{cases} b(t) := \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, a(\tau))d\tau\right) ds & \geq & a(t), \\ \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, b(\tau))d\tau\right) ds & \geq & a(t). \end{cases}$$

For $x \in S \setminus \{0\}$, define a fixed point operator T by

$$Tx(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, x(\tau))d\tau\right) ds.$$

We have

Lemma 3.1. *Assume $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ holds. Then the operator T maps $S \setminus \{0\}$ into $X \cap S$. In addition*

$$\begin{cases} (\phi(-(Tx)''))'(t) + f(t, x(t)) = 0, \quad t > 0, \\ \\ (Tx)(0) = \mu(Tx)'(0),\ (Tx)'(+\infty) = (Tx)''(+\infty) = 0. \end{cases} \tag{8}$$

Proof.

(a) For $\lambda > 0$, let

$$F_\lambda(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda\rho(\tau))d\tau\right) ds,$$

then

$$\lim_{t\to+\infty} \frac{F_\lambda(t)}{1+t} = 0.$$

Indeed, using the convergence of the second integral in $(\mathcal{H}_2)$, we get

$$\lim_{t\to+\infty} F'_\lambda(t) = \lim_{t\to+\infty} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau,\lambda\rho(\tau))d\tau\right) ds = 0,$$

then F_λ is nondecreasing and

$$\lim_{t\to+\infty} \frac{F_\lambda(t)}{1+t} = \begin{cases} 0, & \text{if} \quad \lim_{t\to+\infty} F_\lambda(t) < \infty, \\ \lim_{t\to+\infty} F'_\lambda(t) = 0, & \text{if} \quad \lim_{t\to+\infty} F_\lambda(t) = \infty. \end{cases}$$

(b) Given $x \in S\setminus\{0\}$, by Lemma 2.6, there exists $\lambda_x > 0$ such that $x(t) \geq \lambda_x \rho(t), t \in \mathbb{R}^+$. By $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and Part (a), we have

$$\begin{aligned} \frac{Tx(t)}{1+t} &= \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau,x(\tau))d\tau\right)ds}{1+t} \\ &\leq \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau,\lambda_x\rho(\tau))d\tau\right)ds}{1+t} \\ &= \frac{F_{\lambda_x}(t)}{1+t}. \end{aligned}$$

Hence $\lim_{t\to+\infty} \frac{Tx(t)}{1+t} = 0$. Then $Tx \in E$ and even $Tx \in X \cap S$. Indeed $Tx(t) \geq 0$,

$$(Tx)'(t) = \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau,x(\tau))d\tau\right) ds \Longrightarrow \lim_{t\to+\infty} (Tx)'(t) = 0,$$

$$(Tx)''(t) = -\phi^{-1}\left(\int_t^{+\infty} f(\tau,x(\tau))d\tau\right) \leq 0,$$

and thus (8) is satisfied. □

Now we state and prove our main existence result:

Theorem 3.2. *Assume that Assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold. Then the boundary value problem (1) has at least one positive solution* $x \in X$ *which satisfies* $x(t) \geq \lambda_0\rho(t)$ *for some* $\lambda_0 > 0$.

Proof. The proof is be split into three steps.

Step 1. We first determine appropriate upper and lower solution for the bvp (1). Since $a \in S\setminus\{0\}$ and $b(t) = Ta(t)$, then by $(\mathcal{H}_3)$ and Lemma 3.1, we have $b, Tb \in S\setminus\{0\}$. Moreover T being nonincreasing relatively to x, we have

$$a \leq b \Rightarrow a \leq Tb \leq Ta = b. \tag{9}$$

Therefore, for all $t > 0$

$$\begin{cases} (\phi(-(Tb)''))'(t) + f(t,Tb(t)) \geq (\phi(-(Tb)''))'(t) + f(t,b(t)) = 0 \\ (Tb)(0) = \mu(Tb)'(0),\ (Tb)'(+\infty) = 0,\ (Tb)''(+\infty) = 0 \end{cases} \tag{10}$$

and

$$\begin{cases} (\phi(-(Ta)''))'(t) + f(t, Ta(t)) \leq (\phi(-(Ta)''))'(t) + f(t, a(t)) = 0, \\ \\ (Ta)(0) = \mu(Ta)'(0),\ (Ta)'(+\infty) = 0,\ (Ta)''(+\infty) = 0. \end{cases} \tag{11}$$

The functions $\alpha(t) = Tb(t)$ and $\beta(t) = Ta(t)$ are lower and upper solution of the bvp (1), respectively with $\alpha \leq \beta$.

Step 2. We claim that the following regular modified boundary value problem

$$\begin{cases} (\phi(-x''))'(t) + f^*(t, x(t)) = 0, \quad t > 0, \\ \\ x(0) = \mu x'(0),\ x'(+\infty) = x''(+\infty) = 0 \end{cases} \tag{12}$$

has a positive solution, where

$$f^*(t,x) = \begin{cases} f(t,\alpha), & x < \alpha(t), \\ f(t,x), & \alpha(t) \leq x \leq \beta(t), \\ f(t,\beta), & x > \beta(t), \end{cases} \tag{13}$$

To see this, consider the operator $A : E \to E$ defined by

$$Ax(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x(\tau))d\tau\right) ds.$$

It is clear that a fixed point of the operator A is a solution of the boundary value problem (12). Since $\alpha \in S \setminus \{0\}$, by lemma 2.6 (b), there exists a positive constant λ_α such that $\alpha(t) \geq \lambda_\alpha \rho(t)$, $\forall\, t \in \mathbb{R}^+$. Moreover $f(t,x)$ being nonincreasing in x, we have

$$f^*(t,x) \leq f(t,\alpha(t)) \leq f(t,\lambda_\alpha\rho(t)) \tag{14}$$

for all positive t.

(a) $A(E) \subseteq E$. For $x \in E$ and $t \in \mathbb{R}^+$, we have, using (14)

$$\begin{aligned} \frac{Ax(t)}{1+t} &= \frac{\int_0^{+\infty} G(t,s)\phi^{-1}(\int_s^{+\infty} f^*(\tau,x(\tau))d\tau)ds}{1+t} \\ &\leq \frac{\int_0^{+\infty} G(t,s)\phi^{-1}(\int_s^{+\infty} f(\tau,\lambda_\alpha\rho(\tau))d\tau)ds}{1+t} \\ &= \frac{F_{\lambda_\alpha}(t)}{1+t}, \end{aligned}$$

hence $\lim\limits_{t\to+\infty} \frac{Ax(t)}{1+t} = 0$ and $A(E) \subseteq E$.

(b) A is continuous. Let some sequence $\{x_n\}_{n\geq1} \subseteq E$ be such that $\lim\limits_{n\to+\infty} x_n = x_0 \in E$. Then we have

$$\begin{aligned}
&\|Ax_n - Ax_0\| \\
= \;& \sup_{t\in\mathbb{R}^+} \tfrac{|Ax_n(t)-Ax_0(t)|}{1+t} \\
= \;& \sup_{t\in\mathbb{R}^+} \int_0^{+\infty} \tfrac{G(t,s)}{1+t} |\phi^{-1}(\int_s^{+\infty} f^*(\tau, x_n(\tau))d\tau) - \phi^{-1}(\int_s^{+\infty} f^*(\tau, x_0(\tau))d\tau)|ds \\
\leq \;& \max(1,\mu) \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} f^*(\tau, x_n(\tau))d\tau) - \phi^{-1}(\int_s^{+\infty} f^*(\tau, x_0(\tau))d\tau)|ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x_n(\tau))d\tau\right) - \phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x_0(\tau))d\tau\right)\right| \\
\leq \;& 2\phi^{-1}\left(\int_0^{+\infty} f(\tau, \lambda_\alpha \rho(\tau)\right) d\tau,
\end{aligned}$$

then the continuity of f^*, ϕ^{-1}, $(\mathcal{H}_2)$ and the Lebesgue dominated convergence theorem, we deduce $\|Ax_n - Ax_0\| \longrightarrow 0$, as $n \longrightarrow +\infty$

(c) $A(E)$ is relatively compact. Indeed

(i) $A(E)$ is uniformly bounded. For $x \in E$, we have

$$\begin{aligned}
\|Ax\| &= \sup_{t\in\mathbb{R}^+} \tfrac{|Ax(t)|}{1+t} \\
&\leq \sup_{t\in\mathbb{R}^+} \int_0^{+\infty} \tfrac{G(t,s)}{1+t} \phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x(\tau))d\tau\right) ds \\
&\leq \max(1,\mu) \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x(\tau))d\tau\right) ds \\
&\leq \max(1,\mu) \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda_\alpha \rho(\tau))d\tau\right) ds \\
&< +\infty.
\end{aligned}$$

(ii) $\{\frac{A(E)}{1+t}\}$ is almost equicontinuous. For a given $T > 0$, $x \in E$, and $t, t' \in [0,T]$ $(t > t')$, we have

$$\begin{aligned}
&\left|\tfrac{Ax(t)}{1+t} - \tfrac{Ax(t')}{1+t'}\right| \\
\leq \;& \int_0^{+\infty} \left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right| \phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x(\tau))d\tau\right) ds \\
\leq \;& \int_0^{T} \left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right| \phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda_\alpha \rho(\tau))d\tau\right) ds \\
& + \left|\tfrac{t+\mu}{1+t} - \tfrac{t'+\mu}{1+t'}\right| \int_T^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda_\alpha \rho(\tau))d\tau\right) ds,
\end{aligned}$$

then by $(\mathcal{H}_2)$, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $\left|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}\right| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$. Hence $\{\frac{A(E)}{1+t}\}$ are almost equicontiuous.

(iii) $\{\frac{A(E)}{1+t}\}$ is equiconvergent at $+\infty$. Since $\lim\limits_{t\to+\infty} \frac{Ax(t)}{1+t} = 0$, then by $(\mathcal{H}_2)$ we have

$$\begin{aligned}
\lim_{t\to+\infty} \sup_{x\in E} \left|\frac{Ax(t)}{1+t} - \lim_{t\to+\infty} \frac{Ax(t)}{1+t}\right| &= \lim_{t\to+\infty} \sup_{x\in E} \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f^*(\tau, x(\tau))d\tau\right)ds}{1+t} \\
&\leq \lim_{t\to+\infty} \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda_\alpha \rho(\tau))d\tau\right)ds}{1+t} \\
&= \lim_{t\to+\infty} \frac{F_{\lambda_\alpha}(t)}{1+t} = 0.
\end{aligned}$$

Lemma 2.2 guarantees that $A(E)$ is relatively compact. Finally by the Schauder fixed point theorem (see, e.g., [2]), the operator A has at least one fixed point $x \in E$, which is further in X by Lemma 3.1, solution of the bvp (12).

Step 3. Next we will prove that the boundary value problem (1) has at least one positive solution. For this, we only need to check that $\alpha(t) \leq x(t) \leq \beta(t)$, $\forall t \in \mathbb{R}^+$. Since x is a solution of the bvp (12)

$$x(0) = \mu x'(0), \ \lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} x''(t) = 0 \tag{15}$$

In addition, $f(t, x)$ is nonincreasing in x

$$f(t, \beta(t)) \leq f^*(t, x) \leq f(t, \alpha(t)), \ \forall\, t \in \mathbb{R}^+. \tag{16}$$

It follows from (9) and $(\mathcal{H}_3)$ that

$$f(t, b(t)) \leq f^*(t, x) \leq f(t, a(t)), \ \forall\, t \in \mathbb{R}^+. \tag{17}$$

Since $a \in S \setminus \{0\}$, by Lemma 3.1

$$(\phi(-\beta''(t)))' = (\phi(-Ta)''(t)))' = -f(t, a(t)), \ \forall\, t \in \mathbb{R}^+.$$

These, together with Lemma 3.1 (9), (15)-(17) yield

$$\begin{cases} (\phi(-\beta''(t)))' - (\phi(-x''(t)))' = -f(t, a(t)) + f^*(t, x(t)) \leq 0, \ t \in \mathbb{R}^+ \\ \\ (\beta - x)(0) = \mu(\beta - x)'(0), \ (\beta - x)'(+\infty) = 0, \ (\beta - x)''(+\infty) = 0 \end{cases} \tag{18}$$

This implies that the function z defined by $z(t) = (\phi(-\beta''(t))) - (\phi(-x''(t)))$ is a nonincreasing function in $\mathbb{R}^+$. Moreover $z(+\infty) = 0$ implies $z(t) \geq 0$, $\forall\, t \geq 0$ and then $(\beta - x)''(t) \leq 0, \forall\, t \in \mathbb{R}^+$ which means that $(\beta - x)'$ is nonincreasing in $\mathbb{R}^+$. Now $(\beta - x)'(+\infty) = 0$ then $(\beta - x)'(t) \geq 0$, $\forall t \in \mathbb{R}^+$ and so $\beta - x$ is nondecreasing on $\mathbb{R}^+$. Finally the boundary condition $(\beta - x)(0) = \mu(\beta - x)'(0) \geq 0$ implies that $x(t) \leq \beta(t)$, for all $t \in \mathbb{R}^+$. In a similar way, we can prove that $x(t) \geq \alpha(t)$, for all $t \in \mathbb{R}^+$. Therefore, x is a solution of the bvp (1). In addition, there existence of a positive constant $\lambda_0 = \lambda_\alpha$ such that $x(t) \geq \alpha(t) \geq \lambda_0 \rho(t)$, $\forall\, t \in \mathbb{R}^+$. The proof of Theorem 3.2 is completed.

□

However, when $f(t,x)$ is nonsingular at $x = 0$, i.e. $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function, then for all $x \geq 0$, $f(t,x) \leq f(t,0)$. In this case, we have

Theorem 3.3. *Assume that assumption* $(\mathcal{H}_1)$ *holds and*

$(\mathcal{H}_2)'$ $\quad 0 < \int_0^{+\infty} f(\tau,0)d\tau < +\infty$ *and* $\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau,0)d\tau\right) ds < +\infty$. *Then the bvp (1) has at least one positive solution* $x \in X$ *such that* $x(t) \geq \lambda_0 \rho(t)$ *for some* $\lambda_0 > 0$.

The proof is similar to that of Theorem 3.2. We only check that $T(S) \subset S \cap X$ and if we take $a(t) = 0, \forall\, t \geq 0$, then condition $(\mathcal{H}_3)$ holds. Finally the condition $(\mathcal{H}_2)'$ implies that $\beta = Ta = b, \alpha = Tb$ belong to $S \setminus \{0\}$.

Example 3.4. *Consider the singular boundary value problem*

$$\begin{cases} (\phi(-x''(t)))' + e^{-t}m(t)g(x(t)) = 0, \\ x(0) = \mu x'(0), \ \lim\limits_{t\to+\infty} x'(t) = \lim\limits_{t\to+\infty} x''(t) = 0, \end{cases} \tag{19}$$

where $0 \leq \mu \leq \frac{8}{3}$, $\phi(x) = x^{\frac{1}{3}}$, $f(t,x) = e^{-t}m(t)g(x)$,

$$g(x) = \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 1, & x \geq 1, \end{cases}$$

and

$$m(t) = \begin{cases} t^3, & t \in [0,1] \\ \frac{1}{t^2}, & t \geq 1, \end{cases}$$

Then, we have

$(\mathcal{H}_1)$ $f \in C((0,+\infty) \times \mathbb{R}^+, \mathbb{R}^+)$ *and* $f(t,x)$ *is a nonincreasing with respect to* x *for every positive* t.

$(\mathcal{H}_2)$ *For all* $\lambda > 0$,

$$\int_0^{+\infty} f(\tau, \lambda\rho(\tau))d\tau \leq \max\{1, \frac{1}{\lambda}\} < +\infty,$$

and

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} f(\tau, \lambda\rho(\tau))d\tau\right) ds < +\infty.$$

$(\mathcal{H}_3)$ *Let* $a_0(t) = 1$*, then* $a_0 \in S$ *and if we put*

$$a(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) g(a_0(\tau)) d\tau\right) ds,$$

then

$$a(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) d\tau\right) ds.$$

Moreover for all $t \in \mathbb{R}^+$

$$\begin{aligned} a(t) &= \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) d\tau\right) \\ &\leq \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} d\tau\right) \\ &\leq \int_0^{+\infty} (s + 8/3)\phi^{-1}(e^{-s}) ds \\ &\leq 1 = a_0(t). \end{aligned}$$

Hence

$$\begin{aligned} b(t) &= \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) g(a(\tau)) d\tau\right) ds \\ &\geq \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) g(a_0(\tau)) d\tau\right) ds \\ &= a(t). \end{aligned}$$

Finally, since $g \geq 1$*, we have*

$$\begin{aligned} & \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} f(\tau, b(\tau)) d\tau\right) ds \\ = & \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) g(b(\tau)) d\tau\right) ds \\ \geq & \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} e^{-\tau} m(\tau) d\tau\right) ds \\ = & \; a(t). \end{aligned}$$

Then all conditions of Theorem 3.2 are fulfilled which guarantees that the bvp (19) has at least one positive solution.

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