CUBO A Mathematical Journal
Vol.15, Nº 03, (123–132). October 2013

K-theory for the group C*-algebras of nilpotent discrete groups

Takahiro Sudo
Department of Mathematical Sciences,
Faculty of Science, University of the Ryukyus,
Senbaru 1, Nishihara, Okinawa 903-0213, Japan.
sudo@math.u-ryukyu.ac.jp

ABSTRACT

We study the K-theory groups for the group C*-algebras of nilpotent discrete groups, mainly, without torsion. We determine the K-theory class generators for the K-theory groups by using generalized Bott projections.

RESUMEN

Estudiamos los grupos de la K-teoría para el grupo de álgebras C* de grupos discretos nilpotentes principalmente sin torsión. Determinamos los generadores de la clase de K-teoría para los grupos de la K-teoría usando proyecciones generalizadas de Bott.

Keywords and Phrases: group C*-algebra, K-theory, nilpotent discrete group, Bott projection.

2010 AMS Mathematics Subject Classification: 46L05, 46L80, 19K14.

1 Introduction

The K-theory groups for the group C^*-algebra of the discrete Heisenberg nilpotent group are computed in the paper [1] of Anderson and Paschke by determining the K-thoery class generators for the K-theory groups by using the Bott projection on the two dimensional torus. The K-theory groups for the group C^*-algebras of the generalized discrete Heisenberg nilpotent groups are computed in the paper [3] of the author by determining the K-theoy class generators for the K-theory groups by using generalized Bott projections on the higher dimensional torus defined in [3].

In this paper, based on those results in the typical case of two-step, nilpotent discrete groups, we study the K-theory groups for the group C^*-algebras of general, nilpotent discrete groups, mainly, without torsion, and it is found out that we can determine the K-theory class generators for the K-theory groups by using the generalized Bott projections. Moreover, several consequences of this main result are also obtained.

Notation. We denote by $C(X)$ the C^*-algebra of all continuous, complex-valued functions on a compact Hausdorff space X. Denote by $C^*(G)$ the (full or reduced) group C^*-algebra of a nilpotent, discrete group G (that is amenable). Note that $C^*(G)$ is generated by unitaries that correspond to generators of G. Denote by $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ the K_0-group and the K_1-group of a C^*-algebra $\mathfrak{A}$ respectively (see [2]).

2 Finitely generated nilpotent discrete case

Recall that a k-dimensional noncommutative torus denoted by $\mathbb{T}^k_\theta$ is the universal C^*-algebra generated by k unitaries U_j $(1 \leq j \leq k)$ with the relations $U_iU_j = e^{2\pi i\theta_{ij}}U_jU_i$ for $i \neq j$ and $\theta_{ij} \in \mathbb{R}$ and $\theta = (\theta_{ij}) \in M_k(\mathbb{R})$ a $k \times k$ skew adjoint matrix over the field $\mathbb{R}$ of real numbers with $\theta_{ii} = 0$ and $\theta_{ji} = -\theta_{ij}$ $(i \neq j)$.

Lemma 2.1. *Let G be a finitely generated, two-step nilpotent discrete group without torsion and with Z its center and $C^*(G)$ be the group C^*-algebra of G.*

Then $C^*(G)$ can be viewed as a continuous field C^*-algebra over the dual group $Z^\wedge$ of Z with fibers given by noncommutative tori $\mathbb{T}^{n-k}_{\theta_\lambda}$ with the relations by θ_λ varing over elements $\lambda \in Z^\wedge$, where $Z^\wedge$ is an ordinary torus $\mathbb{T}^k$ by Pontrjagin duality theorem with $1 \leq k = \mathrm{rank}(Z)$ the rank of Z, and $n = \mathrm{rank}(G)$.

Proof. This is certainly known and may follow from the same way as done by [1] in the case of G the discrete Heisenberg group of rank 3.

Indeed, note that since G/Z is commutative, the commutator subgroup $[G, G]$ of G is contained in Z. As a fact of the unitaty representation theory for G, that is identified with the representation theory of $C^*(G)$, any element λ in $Z^\wedge$ induces an irreducible induced representation π_λ of G and of $C^*(G)$ and any element of $[G, G]$ is mapped to a complex number in the one-torus $\mathbb{T}$, so that the image of $C^*(G)$ under π_λ is a noncommutive torus $\mathbb{T}^{n-k}_{\theta_\lambda}$ with θ_λ associated to λ. Since elements $\lambda \in Z^\wedge = \mathbb{T}^k$ vary continuously on $Z^\wedge$, the norms of $\pi_\lambda([u_i, u_j])$ for u_i, u_j unitary generators of $C^*(G)$ corresponding to generators of G also vary continuously, to make a continuous field C^*-algebra over $Z^\wedge$ with fibers noncommutative tori $\mathbb{T}^{n-k}_{\theta_\lambda}$. □

As the main result we obtain

Theorem 1. *Let* G *be a finitely generated, nilpotent discrete group without torsion and* $C^*(G)$ *be the group* C^**-algebra of* G.

Then the K-theory class generators in the K_0*-group* $K_0(C^*(G))$ *are given by the class of the identity of* $C^*(G)$ *and the classes of generalized Bott projections combinatoricly corresponding to abelian subalgebras of* $C^*(G)$ *that correspond to even subsets of mutually commuting generators, even numbered, in the set of generators of* G.

Moreover, the K-theory class generators in the K_1*-group* $K_1(C^*(G))$ *are given by the class of unitary generators of* $C^*(G)$ *that correspond to each of generators of* G*, or correspond both to generators of* G *and to the generalized Bott projections, each of which is obtained combinatoricly from both the generalized Bott projection and each of generators of* G *which is not involved in the generalized Bott projection.*

The statement above can be understood precisely by helpful examples and remark below the following proof.

Proof. Recall that under the assumption on G, the group G is isomorphic to a successive semi-direct product of $\mathbb{Z}$ the group of integers:

$$G \cong \mathbb{Z} \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$$

crossed by $\mathbb{Z}$ rank$(G) - 1$ times, with rank(G) the rank of G. Then

$$C^*(G) \cong C^*(\mathbb{Z}) \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$$

a successive crossed product C^*-algebra by $\mathbb{Z}$, and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform.

Set rank$(G) = n$. Let $U = \{g_1, \cdots, g_n\}$ be the set of generators of G. Note that since G is discrete, the generators of G can be identified with corresponding unitary generators of $C^*(G)$ (via the left regular, or universal representation on the corresponding Hilbert space since G is amenable). Suppose that V is an even subset of U with some mutually commuting generators of G. Denote by $C^*(V)$ the C^*-algebra generated by elements of V. Then $C^*(V)$ is an abelian subalgebra of $C^*(G)$ and is isomorphic to $C(\mathbb{T}^{|V|})$ the C^*-algebra of all continuous, complex-valued functions on the $|V|$-dimensional torus $\mathbb{T}^{|V|}$, where $|V|$ is the cardinality of V. We assign such even subset V each to the generalized Bott projection P_V in $M_2(C(\mathbb{T}^{|V|}))$ the 2×2 matrix algebra over $C(\mathbb{T}^{|V|})$, involving all elements of V. See the remark below for the definition of P_V.

It follows that $K_0(C(\mathbb{T}^{|V|}))$ can be embedded in $K_0(C^*(G))$ canonically. Therefore, the K_0-group class $[P_V]$ can be viewed in $K_0(C^*(G))$. It also follows that if $V \neq V'$ even subsets in U, then $[P_V] \neq [P_{V'}]$, i.e., P_V is not equivalent to $P_{V'}$. Indeed, if P_V is equivalent to $P_{V'}$, then we can deduce a contradition, by observing that the coordinates of $\mathbb{T}^{|V|}$ corresponding to V are different from those of $\mathbb{T}^{|V'|}$ of V'.

If G is commutative, then $G \cong \mathbb{Z}^n$, and $C^*(G) \cong C(\mathbb{T}^n)$ by the Fourier transform, and it is shown by [3] that the K_0-group classes of generalized Bott projections on the even dimensional tori $\mathbb{T}^{2k}$ $(2 \leq 2k \leq n)$ combinatoricly in $\mathbb{T}^n$ and the class of the identity generate all classes in $K_0(C(\mathbb{T}^n))$.

By the lemma above, if G is a finitely generated, two-step nilpotent discrete group without torsion, then $C^*(G)$ can be viewed as a continuous field C^*-algebra over the dual group $Z^\wedge$ of the center Z of G with fibers given by noncommutative tori, that are successive crossed product C^*-algebras by $\mathbb{Z}$, generated by unitaries corresponding to generators of G not in Z, where their relations vary over $Z^\wedge$. It is also shown by [3] that even in this case, the same holds as in the commutative case.

Indeed, the class of the identity and the classes of generalized Bott projections in $M_2(C^*(G))$ generate all classes in $K_0(C^*(G))$, because it is noticed in [3] that the classes of the genearalized Rieffel projections defined in [3] and the class of the identity generate all classes in the K_0-group of a fiber, a noncommutative torus, and the classes of the genearalized Rieffel projections can not contribute to a class of $K_0(C^*(G))$ since those projections are not continuous over $Z^\wedge$. Therefore, a projection for a class of $K_0(C^*(G))$ can not involve the generalized Rieffel projections in fibers.

We now consider the general case by induction. Suppose that the theorem on K_0 is true when $\mathrm{rank}(G) \leq n$. Let $\mathrm{rank}(G) = n + 1$. Let $[p] \in K_0(C^*(G))$ for a projection p in a matrix algebra over $C^*(G)$.

If p is generated by k unitaries corresponding to k generators of G with $k \leq n$, then p is contained in the group C^*-algebra $C^*(H)$ of a nilpotent subgroup H of G generated by V the set of the k generators of G, that is $C^*(H) = C^*(V) \subset C^*(G)$. By induction hypothesis, the class $[p]$ is spanned by the class of the identity and the classes of generalized Bott projections in $M_2(C^*(H))$.

We now assume that the projection p involves all elements of U. We also may assume that G is not two-step nilpotent. Therefore, the quotient group G/Z is not commutative and nilpotent. There is a quotient map q from $C^*(G)$ to $C^*(G/Z)$ and is extended to their matrix algebras. Then $q(p)$ is a projection that involves all generators of G/Z. But by induction, and since G/Z is non commutative, the K_0-group classes of $K_0(C^*(G/Z))$ can not involve all generators of G/Z. This is a contradiction. Hence, there is no such projection p. In fact, this reduction can be continued until that p is contained in an abelian subalgebra of $C^*(G)$ that is generated by unitaries corresponding to a set of mutually commuting generators of G

The K_1-group case for $C^*(G)$ is treated similarly as in the K_0-group case above. Indeed, when G is commutative, it is shown by [3] that the K_1-group $K_1(C^*(G))$ can be generated by the classes represented by either unitary generators of $C^*(G)$ corresponding to generators of G or the unitaires that combinatoricly correspond to both generalized Bott projections and each of unitary generators of $C^*(G)$ corresponding to generators of G. See the remark below for the definition of the unitaries. Moreover, even in the case of G two-step nilpotent, the same holds for $K_1(C^*(G))$. And the general case can be proved by the same way as in the proof for that case of $K_0(C^*(G))$. In fact, the construction of generators of $K_1(C^*(G))$ can be made by bijectively correponding to the generators of $K_0(C^*(G))$ constructed, in a suitable and combinatoric way (see the examples below). □

Remark. Recall from [3] (or [1] originally) that the Bott projection P in $M_2(C(\mathbb{T}^2))$ is defined as a projection-valued function from $\mathbb{T}^2$ to $M_2(\mathbb{C})$:

$$P(w,z) = \mathrm{Ad}(U(w,z))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}), \quad (w,z) \in \mathbb{T}^2,$$

where $U(w,z) = Y(t,z)^*$ with $w = e^{2\pi i t} \in \mathbb{T}$ for $t \in [0,1]$ and

$$Y(t,z) = \exp\left(\frac{i\pi t}{2}K(z)\right)\exp\left(\frac{i\pi t}{2}S\right)$$
$$K(z) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad S = K(1).$$

Moreover, the generalized Bott projection Q_k in $M_2(C(\mathbb{T}^{2k}))$ is defined in [3] by a projection-valued function from $\mathbb{T}^{2k}$ to $M_2(\mathbb{C})$:

$$Q_k(z_1,\cdots,z_{2k}) = \mathrm{Ad}(U_1(z_1,z_2))\mathrm{Ad}(U_2(z_3,z_4))\cdots\mathrm{Ad}(U_k(z_{2k-1},z_{2k}))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $U_j(\cdot,\cdot) = U(\cdot,\cdot)$ for $1 \le j \le k$. Furthermore, the unitary V_k in $M_2(C(\mathbb{T}^{2k+1}))$ obtained from the generalized Bott projection Q_k and a unitary generator u of $C^*(G)$ corresponding to a generator of G is defined in [3] by

$$V_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (u-1)\otimes Q_k \in M_2(C(\mathbb{T}^{2k+1})).$$

Example 2.2. If $G = \mathbb{Z}^n$, then $C^*(G) \cong C(\mathbb{T}^n)$ by the Fourier transform, and $K_*(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ for $* = 0,1$ ([4]). Note that the generators of $K_0(C(\mathbb{T}^n))$ are given by the class of the identity and the classes of generalized Bott projections defined as above and the generators of $K_1(C(\mathbb{T}^n))$ are given by the classes of unitary generators of $C^*(\mathbb{Z}^n)$ corresponding to generators of $\mathbb{Z}^n$ and the classes of the unitaries associated to both generalized Bott projections and the unitary generators of $C^*(\mathbb{Z}^n)$ defined as above (see [3]).

More precisely, when $n = 4$, the generators of $K_0(C^*(\mathbb{Z}^4)) \cong \mathbb{Z}^8$ is given by the following classes:

$$[1], [P_{12}], [P_{13}], [P_{14}],$$
$$[P_{23}], [P_{24}], [P_{34}], [Q_{1234}],$$

where 1 is the identity of $C^*(\mathbb{Z}^4)$ and each P_{ij} over $\mathbb{T}^4$ is identified with the Bott projection over $\mathbb{T}^2$ that corresponds to i, j coordinates in $\mathbb{T}^4$, and Q_{1234} is the generalized Bott projection over $\mathbb{T}^4$. Also, the generators of $K_1(C^*(\mathbb{Z}^4)) \cong \mathbb{Z}^8$ is given by the following classes:

$$[u_1], [u_2], [u_3], [u_4],$$
$$[V_{123}], [V_{124}], [V_{134}], [V_{234}],$$

where each u_j is the unitary generator of $C^*(\mathbb{Z}^4)$ corresponding to generators of $\mathbb{Z}^4$ and each unitary V_{ijk} in $M_2(C^*(\mathbb{Z}^4))$ is obtained by P_{ij} and u_k. Note that V_{ijk} may be obtained from either P_{jk} and u_i, or P_{ik} and u_j.

Example 2.3. Let G be the discrete Heisenberg group of rank 3:

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Then $Z = \mathbb{Z}$ and $G/Z \cong \mathbb{Z}^2$. Also, $C^*(G)$ is viewed as a continuous filed C^*-algebra over $\mathbb{T} = Z^\wedge$ with fibers noncommutative 2-tori $\mathbb{T}^2_{\theta_\lambda}$. It is computed by [1] (and also [3]) that

$$K_0(C^*(G)) \cong \mathbb{Z}^3, \quad K_1(C^*(G)) \cong \mathbb{Z}^3,$$

and the generators of $K_0(C^*(G))$ is given by the class of the identity of $C^*(G)$ and two classes of the Bott projections over $\mathbb{T}^2$, where their domains are different in the sense as one $\mathbb{T}^2 = \mathbb{T} \times Z^\wedge$ with the first factor $\mathbb{T}$ corresponding to one of two generators of the fibers and the other $\mathbb{T}^2 = \mathbb{T} \times Z^\wedge$ with the first factor $\mathbb{T}$ corresponding to the other of two generators of the fibers, and the generators of $K_1(C^*(G))$ is given by two classes of unitary generators of $C^*(G)$ corresponding both to generators of G and to one of two Bott projections and the class of the unitary of $M_2(C^*(G))$ obtained from both the chosen Bott projection and the rest of unitary generators of $C^*(G)$ corresponding to generators of G. Namely,

$$K_0(C^*(G)) \cong \langle [1], [P_{13}], [P_{23}] \rangle,$$
$$K_1(C^*(G)) \cong \langle [u_1], [u_3], [V_{123}] \rangle,$$

where the equations mean that the left hand sides are generated by the classes in the brackets in the right hand sides, and the third coordinate in $\mathbb{T}^3$ corresponds to $Z^\wedge$ and the unitary V_{123} is obtained from the Bott projection P_{13} and u_2. Note that the above set of generators of $K_1(C^*(G))$ may be replaced with $\{[u_2], [u_3], [V'_{123}]\}$, where V'_{123} is obtained from the Bott projection P_{23} and u_1.

Example 2.4. Let $G \times G$ be the direct product of G the discrete Heisenberg nilpotent group of rank 3. Then $C^*(G \times G) \cong C^*(G) \otimes C^*(G)$ the tensor product of $C^*(G)$. Since $K_j(C^*(G))$ $(j = 0, 1)$ are torsion free, the Künneth theorem in K-theory for C^*-algebras (see [2]) implies that

$$\begin{aligned} K_0(C^*(G \times G)) &\cong [K_0(C^*(G)) \otimes K_0(C^*(G))] \oplus [K_1(C^*(G)) \otimes K_1(C^*(G))] \\ &\cong [\mathbb{Z}^3 \otimes \mathbb{Z}^3] \oplus [\mathbb{Z}^3 \otimes \mathbb{Z}^3] \cong \mathbb{Z}^{18}, \\ K_1(C^*(G \times G)) &\cong [K_0(C^*(G)) \otimes K_1(C^*(G))] \oplus [K_1(C^*(G)) \otimes K_0(C^*(G))] \\ &\cong [\mathbb{Z}^3 \otimes \mathbb{Z}^3] \oplus [\mathbb{Z}^3 \otimes \mathbb{Z}^3] \cong \mathbb{Z}^{18}. \end{aligned}$$

Our theorem tells us that

$$\begin{aligned} K_0(C^*(G \times G)) \cong \langle &[1], [P_{13}], [P_{23}], [P_{46}], [P_{56}], \\ &[P_{14}], [P_{15}], [P_{16}], [P_{24}], [P_{25}], [P_{26}], [P_{34}], [P_{35}], [P_{36}], \\ &[Q_{1346}], [Q_{1356}], [Q_{2346}], [Q_{2356}]\rangle, \end{aligned}$$

where the subindices $1, 2, 3$ correspond to the unitary generators u_j of $C^*(G) \otimes \mathbb{C}$ and the subindices $4, 5, 6$ correspond to the unitary generators u_j of $\mathbb{C} \otimes C^*(G)$ and both subindices 3 and 6 corresponds to the center Z of G. Also,

$$\begin{aligned} K_1(C^*(G \times G)) \cong \langle &[u_1], [u_3], [V_{123}], [u_4], [u_6], [V_{456}], \\ &[V(P_{14}, u_2)], [V(P_{15}, u_2)], [V(P_{16}, u_2)], \\ &[V(P_{24}, u_3)], [V(P_{25}, u_3)], [V(P_{26}, u_3)], \\ &[V(P_{34}, u_5)], [V(P_{35}, u_6)], [V(P_{36}, u_4)], \\ &[V(Q_{1346}, u_2)], [V(Q_{1356}, u_2)], [V(Q_{2346}, u_1)]\rangle, \end{aligned}$$

where each $V(P_{ij}, u_k)$ means the unitary obtained from P_{ij} and u_k and each $V(Q_{ijkl}, u_m)$ means the unitary obtained from Q_{ijkl} and u_m. Note that the unions of subindeices such as $(1, 2, 4)$ of $(14, 2)$ and $(1, 2, 5)$ of $(15, 2)$ are taken only once among combinations of (i, j, k) with $i < j < k$. Also, the choice of adding u_m to either P_{ij} or Q_{ijkl} may be different to make the same set of unions of subindices, and the set of generators of $K_0(C^*(G \times G))$ corresponds to the set of generators of $K_1(C^*(G \times G))$ bijectively.

Corollary 2.5. *If* G *is a finitely generated, discrete nilpotent group without torsion, then*

$$K_0(C^*(G)) \cong K_1(C^*(G)).$$

Example 2.6. The isomorphism in the corollary above does not hold if G has torsion. Indeed, if $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ $(n \geq 2)$ a cyclic group, then $C^*(G) \cong \mathbb{C}^n$, so that $K_0(C^*(G)) \cong \mathbb{Z}^n$ but $K_1(C^*(G)) \cong 0$.

Corollary 2.7. *If* G *is a finitely generated, discrete nilpotent group without torsion, then the K-theory groups* $K_0(C^*(G))$ *and* $K_1(C^*(G))$ *are torsion free.*

Proof. This follows from the construction of the generators of $K_0(C^*(G))$ and $K_1(C^*(G))$ obtained in the theorem above. □

Remark. Possibly, in the last corollary, the group G may have torsion.

Example 2.8. We consider a version of the discrete Heisenberg nilpotent group with torsion (see [1] or [3] for the discrete Heisenberg nilpotent group). Let $G = \mathbb{Z}_2^2 \rtimes_\alpha \mathbb{Z}_2$ be a semi-direct product of the product group $\mathbb{Z}_2^2$ of the cycic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ by an action of $\mathbb{Z}_2$ defined by $\alpha_t(b + tc, c)$ for $b, c, t \in \mathbb{Z}_2$. Then the group C^*-algebra $C^*(G)$ is isomorphic to the crossed product $C(\mathbb{Z}_2^\wedge \times \mathbb{Z}_2^\wedge) \rtimes_{\alpha^\wedge} \mathbb{Z}_2$ via the Fourier transform, where the dual action $\alpha^\wedge$ of $\mathbb{Z}_2$ on the product space of the dual group $\mathbb{Z}_2^\wedge \cong \mathbb{Z}_2$ is defined by $\alpha_t^\wedge(z, w) = (z, z^t w)$ for $z, w \in \mathbb{Z}_2^\wedge$ via the duality

$$\alpha_t^\wedge(\varphi_{z,w})(b, c) = \varphi_{z,w}(b + tc, c) = z^{b+tc}w^c = \varphi_{z,z^t w}(b, c)$$

where $\varphi_{z,w} \in \mathbb{Z}_2^\wedge \times \mathbb{Z}_2^\wedge$ indentified with (z, w) (cf. [5]). We then obtain the following decomposition:

$$\begin{aligned} & C(\mathbb{Z}_2^\wedge \times \mathbb{Z}_2^\wedge) \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \\ & \cong [\mathbb{C} \otimes (\mathbb{C}^2 \rtimes_{\alpha^\wedge} \mathbb{Z}_2)] \oplus [\mathbb{C} \otimes (\mathbb{C}^2 \rtimes_{\alpha^\wedge} \mathbb{Z}_2)] \\ & \cong [\mathbb{C}^2 \otimes C^*(\mathbb{Z}^2)] \oplus [M_2(\mathbb{C})] \\ & \cong [\mathbb{C}^2 \otimes \mathbb{C}^2] \oplus M_2(\mathbb{C}) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C}) \end{aligned}$$

where the action $\alpha^\wedge$ on $\mathbb{C}^2$ in the first direct summand is trivial and that in the second is the shift. Therefore,

$$K_0(C^*(G)) \cong \mathbb{Z}^5, \quad \text{but} \quad K_1(C^*(G)) \cong 0.$$

Hence the K-theory groups are torsion free.

In this case, the direct sum factor $\mathbb{Z}^4$ in $\mathbb{Z}^5 = K_0(C^*(G))$ comes from $\mathbb{C}^4$ in $C^*(G)$ which is a maximal abelian subalgebra of $C^*(G)$ but the other direct sum factor $\mathbb{Z}$ in $\mathbb{Z}^5 = K_0(C^*(G))$ comes from $M_2(\mathbb{C})$ in $C^*(G)$ which is a noncommutative subalgebra of $C^*(G)$. Therefore, the case with torsion is certainly different from the torsion free case considered above, but the nilpotent case with torsion is just the same as the abelian case with torsion as in the example above, in the K-theory level.

Corollary 2.9. *If* G *is a finitely generated, discrete nilpotent group without torsion, then both* $K_0(C^*(G))$ *and* $K_1(C^*(G))$ *are isomorphic to a finitely generated, free abelian group, i.e.,* $\mathbb{Z}^m$ *for some positive integer* m.

3 Infinitely generated case

We assume that G is a countable discrete group.

Theorem 2. *Let G be an infinitely generated, nilpotent discrete group without torsion. Then both $K_0(C^*(G))$ and $K_1(C^*(G))$ of the group C^*-algebra $C^*(G)$ are isomorphic to an inductive limit of finitely generated free abelian groups*:

$$K_0(C^*(G)) \cong K_1(C^*(G)) \cong \varinjlim \mathbb{Z}^{m_n},$$

for some positive integers m_n with $m_n < m_{n+1}$, where the connecting maps $\mathbb{Z}^{m_n} \to \mathbb{Z}^{m_{n+1}}$ are injective.

Therefore,

$$K_0(C^*(G)) \cong K_1(C^*(G)) \cong \oplus^\infty \mathbb{Z},$$

which is the infinite direct sum of $\mathbb{Z}$, as a group.

Proof. Let $U = \{g_1, g_2, \cdots\}$ be an infinite set of generators of G and set $U_n = \{g_1, g_2, \ldots, g_n\}$, where g_{n+1} is not generated by $g_1, \cdots, g_n$. Let $C^*(U_n)$ denote the C^*-algebra generated by the elements of $C^*(G)$ that correspond to the elements of U_n. Then $C^*(U_n)$ is a C^*-subalgebra of $C^*(G)$. There is the canonical inclusion i_n from $C^*(U_n) \to C^*(U_{n+1})$. It follows that $C^*(G)$ is an inductive limit of the C^*-subalgebras $C^*(U_n)$ under the inclusions i_n. By continuity of K-theory, we have

$$K_j(C^*(G)) \cong \varinjlim K_j(C^*(U_n))$$

for $j = 0, 1$. By the theorem in the previous section, we see that both $K_j(C^*(U_n))$ for $j = 0, 1$ are isomorphic to $\mathbb{Z}^{m_n}$ for some positive integer m_n and $m_n \leq m_{n+1}$ and also that there is a canonical inclusion from $K_j(C^*(U_n)) \cong \mathbb{Z}^{m_n}$ to $K_j(C^*(U_{n+1})) \cong \mathbb{Z}^{m_{n+1}}$.

We need to check that $m_n \neq m_{n+1}$ for each n. Note that the group H_n generated by elements of U_n can be written as a successive semi-direct product by $\mathbb{Z}$:

$$H_n \cong \mathbb{Z} \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$$

crossed by $\mathbb{Z}$ $n-1$ times. Then $H_{n+1} \cong H_n \rtimes \mathbb{Z}$. It follows that the action of $\mathbb{Z}$ on H_n can not be non-trivial on every generator of H_n. Because, if non-trivial, H_{n+1} is not nilpotent (but solvable). Indeed, then there is no center in H_{n+1}, a contradiction to the nilpotentness of H_{n+1}. Therefore, there is a generator of H_n such that the action of $\mathbb{Z}$ is trivial on it. Therefore, we can construct a new Bott projection from these commuting elements of H_{n+1}, and not from H_n. It follows that $m_n < m_{n+1}$. □

Example 3.1. If G is an infinitely generated abelian discrete group, then $C^*(G)$ is isomorphic to an inductive limit of $C(\mathbb{T}^n)$ with the canonical inclusion from $C(\mathbb{T}^n)$ to $C(\mathbb{T}^{n+1})$. Then

$$K_j(C^*(G)) \cong \varinjlim K_j(C(\mathbb{T}^n)) \cong \varinjlim \mathbb{Z}^{2^{n-1}} \cong \oplus^\infty \mathbb{Z}$$

for $j = 0, 1$.

If G is an inductive limit of the product groups $\Pi^n \mathbb{Z}_2$ with the canonical inclusion from $\Pi^n \mathbb{Z}_2$ to $\Pi^{n+1}\mathbb{Z}_2$, then G is commutative and infinitely generated and has torsion. Then

$$\begin{aligned} C^*(G) &\cong \varinjlim C^*(\Pi^n \mathbb{Z}_2) \cong \varinjlim \otimes^n C^*(\mathbb{Z}^2) \\ &\cong \varinjlim \otimes^n \mathbb{C}^2 \cong \varinjlim \mathbb{C}^{2^n} \cong \oplus^\infty \mathbb{C} \end{aligned}$$

where the last side means the infinite direct sum of $\mathbb{C}$, so that

$$K_j(C^*(G)) \cong K_j(\oplus^\infty \mathbb{C}) \cong \oplus^\infty K_j(\mathbb{C}) \cong \begin{cases} \oplus^\infty \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{if } j = 1. \end{cases}$$

Received: March 2012. Accepted: September 2013.

References

[1] J. Anderson and W. Paschke, *The rotation algebra*, Houston J. Math. **15** (1989), 1-26.

[2] B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[3] T. Sudo, *K-theory of continuous fields of quantum tori*, Nihonkai Math. J. **15** (2004), 141-152.

[4] N. E. Wegge-Olsen, *K-theory and* C^**-algebras*, Oxford Univ. Press (1993).

[5] D. P. Williams, *Crossed products of* C^**-algebras*, SURV 134, Amer. Math. Soc. (2007).