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# Numerical solution of singular and non singular integral equations

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## ABSTRACT

This paper is devoted to study the approximate solution of singular and non singular integral equations by means of Chebyshev polynomial and shifted Chebyshev polynomial. Some examples are presented to illustrate the mothed.

## RESUMEN

Este artículo se dedica al estudio de la solución aproximada de ecuaciones integrales singulares y no singulares por medio de polinomios de Chebyshev con o sin corrimiento. Se presentan algunos ejemplos para ilustrarlo.

**Keywords and Phrases:** Linear Hypersingular integral equations, nonlinear integral equations, Chebyshev polynomial, shifted Chebyshev polynomial, Approximate solution.

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# Introduction

Singular integral equations are usually difficult to solve analytically so it required to obtain the approximate solution [7,8]. Chebyshev polynomials are of great importance in many areas of mathematics particalarly approximation theory see ([1-3], [9-13] ). In this paper we analyze the numerical solution of singular and non singular integral equations by using Chebyshev polynomial and shifted Chebyshev polynomial.

This paper consists of two parts I $and$ II. In part I, we study the approximate solution of Hypersingular integral equations by means of Chebyshev polynomial. In part II, we study the approximate solution of nonlinear integral equations by means of shifted Chebyshev polynomial.

# Part I: Approximate solution of hypersingular integral equations

## 1. Formulation of the problem

Consider the following hypersingular integral equation:

$$\fint_{-1}^{1} \frac{k(x,t)}{(t-x)^2}\ \varphi(t)\ dt + \int_{-1}^{1} L(x,t)\ \varphi(t)\ dt = f(x)\ , \qquad -1 \le x \le 1 \tag{1.1}$$

where $k(x,t)$ , $L(x,t)$ $and$ $f(x)$ are given real-valued continuous functions defined on the set $[-1,1]\times[-1,1], [-1,1]\times[-1,1]$ $and$ $[-1,1]$ respectively and $\varphi(t)$ is unknown function satisfy the following condition $\varphi(\pm 1) = 0$.

The simplest hypersingular integral equation of the form (1.1) given by the following form:

$$\fint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\ dt = f(x)\ , \tag{1.2}$$

where the finite-part integral in (1.2) can be defined as the derivative of a Cauchy principle value integral as

$$\fint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\ dt = \frac{d}{dx}\int_{-1}^{1} \frac{\varphi(t)}{t-x}\ dt. \tag{1.3}$$

Thus equation (1.2) can be written in the following form

$$\frac{d}{dx} \int_{-1}^{1} \frac{\varphi(t)}{t-x} \, dt = f(x) \ . \tag{1.4}$$

Integrating both sides of equation (1.4) with respect to $x$ , we obtain

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x} \, dt = F(x) \ , \tag{1.5}$$

where

$$F(x) = \int f(x) \ dx \ . \tag{1.6}$$

The exact solution of equation (1.5) given by the following form:

$$\varphi(x) = -\frac{\sqrt{1-x^2}}{\pi^2} \int_{-1}^{1} \frac{F(t)}{\sqrt{1-t^2}(t-x)} \, dt \ . \tag{1.7}$$

The solution exists if and only if the function $F(x)$ satisfies the following condition :

$$\int_{-1}^{1} \frac{F(t)}{\sqrt{1-t^2}} \, dt = 0 \ . \tag{1.8}$$

## 2. The approximate solution

In this section we shall study the approximate solution of the hypersingular integral equation (1.1).

The unknown function $\varphi(x)$ satisfying the condition $\varphi(\pm 1) = 0$, can be represented by the following form:

$$\varphi(x) = \sqrt{1-x^2} \ \Phi(x), \qquad -1 \le x \le 1 \tag{2.1}$$

where $\Phi(x)$ is a well-behaved function of $x$ in the interval $x \in [-1, 1]$ . Let the unknown function $\Phi(x)$ be approximated by means of a polynomial of degree $n$ as the following :

$$\Phi(x) \approx \sum_{j=0}^{n} c_j \ x^j \ , \tag{2.2}$$

where $c_j$ $(j = 0, 1, 2, ..., n)$ are unknown constants.

Substituting from form (2.1) and (2.2) into equation (1.1) we obtain:

$$\sum_{j=0}^{n} c_j \left[ \fint_{-1}^{1} \frac{\sqrt{1-t^2}\; k(x,t)\; t^j}{(t-x)^2}\, dt + \int_{-1}^{1} \sqrt{1-t^2}\; L(x,t)\; t^j\, dt \right] = f(x)\ , \quad -1 \le x \le 1. \tag{2.3}$$

By using the following " Chebyshev approximations" to the kernels $k(x,t)$ and $L(x,t)$

$$\left.\begin{aligned} k(x,t) &\approx \textstyle\sum_{p=0}^{m} k_p(x)\; t^p\ , \\ L(x,t) &\approx \textstyle\sum_{q=0}^{s} L_q(x)\; t^q\ , \end{aligned}\right\} \tag{2.4}$$

with known functions $k_p(x)$ and $L_q(x)$ , then (2.3) takes

$$\sum_{j=0}^{n} c_j \left[ \sum_{p=0}^{m} k_p(x)\; u_{p+j}(x) + \sum_{q=0}^{s} L_q(x)\; \gamma_{q+j} \right] = f(x)\ , \qquad -1 \le x \le 1 \tag{2.5}$$

where

$$u_{p+j}(x) = \fint_{-1}^{1} \frac{\sqrt{1-t^2}\; t^{p+j}}{(t-x)^2}\, dt \tag{2.6}$$

and

$$\gamma_{q+j} = \int_{-1}^{1} \sqrt{1-t^2}\; t^{q+j}\, dt\ . \tag{2.7}$$

Using the zeros $x_k$ of the Chebyshev polynomial $T_{n+1}(x)$ into equation (2.5) we obtain the following system of linear equations with $(n+1)$ of the unknown constants $c_j$ $(j = 0, 1, 2, ..., n)$

$$\sum_{j=0}^{n} c_j\; \alpha_j(x_k) = f(x_k)\ , \tag{2.8}$$

where

$$x_k = \cos\left(\frac{(2k-1)}{2(n+1)}\pi\right)\ , \qquad k = 1, 2, 3, ..., n+1 \tag{2.9}$$

and

$$\alpha_j(x_k) = \sum_{p=0}^{m} k_p(x_k)\; u_{p+j}(x_k) + \sum_{q=0}^{s} L_q(x_k)\; \gamma_{q+j}\ . \tag{2.10}$$

By solving the system of linear equations (2.8) we obtain the unknown constants $c_j$ and substituting into (2.1) and (2.2) we obtain the approximate solution of equation (1.1).

# 3. Numerical examples

In this section we shall give two examples to illustrate the above results .

**Example 3.1**

Consider the following hypersingular integral equation

$$\fint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\, dt + \int_{-1}^{1} x\ t\ \varphi(t)\ dt = x + x^3\ , \qquad -1 \le x \le 1\ . \tag{3.1}$$

Equation (3.1) can be written in the following form:

$$\fint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\, dt = (1+\mu)\ x + x^3\ , \tag{3.2}$$

where

$$\mu = -\int_{-1}^{1} t\ \varphi(t)\ dt\ .$$

According to (1.3), (1.5), (1.6) and (1.7) it easy to show that the exact solution of equation (3.2) given by:

$$\varphi(x) = -\frac{1}{40\ \pi}\ \sqrt{1-x^2}\ [10\ x^3 + 27\ x\ ]\ . \tag{3.3}$$

Now, we study the approximate solution of equation (3.1). Since $k(x,t) = 1$ and $L(x,t) = x\ t$ , then we have

$$\left.\begin{array}{l} k_0(x) = 1 \quad , \quad k_1(x) = k_2(x) = ... = k_m(x) = 0 \\ \\ L_1(x) = x \quad , \quad L_0(x) = L_2(x) = ... = L_s(x) = 0\ . \end{array}\right\} \tag{3.4}$$

Substituting from (3.4) into (2.10) we obtain:

$$\alpha_j(x_k) = u_j\ (x_k) + x_k\ \gamma_{j+1}\ , \qquad (j = 0, 1, 2, ..., n) \tag{3.5}$$

By using the following formula

$$\fint_{-1}^{1} \frac{\sqrt{1-t^2}\ U_j(t)}{(t-x)^2}\ dt\ = -\pi\ (j+1)\ U_j(x)\ , \tag{3.7}$$

where $U_j(x)$ is Chebyshev polynomial of the second kind. From (2.6) and (3.7) it easy to show that

$$u_0(x) = -\pi\ , \qquad u_1(x) = -2\pi\ x, \qquad u_2(x) = -\pi\ (3x^2 - \frac{1}{2}), \qquad u_3(x) = -\pi\ (4x^3 - x)\ ,$$
$$u_4(x) = -\pi\ (5x^4 - \frac{3}{2}\ x^2 - \frac{1}{8})\ , \qquad u_5(x) = -\pi\ (6x^5 - 2x^3 - \frac{1}{4}x)\ ,\ ... \tag{3.8}$$

from (2.7) we have

$$\gamma_0 = \frac{\pi}{2}\ , \qquad \gamma_2 = \frac{\pi}{8}\ , \qquad \gamma_4 = \frac{\pi}{16}\ , \qquad \gamma_6 = \frac{5\pi}{128}\ , \qquad \gamma_8 = \frac{7\pi}{256}\ ,$$

$$\gamma_1 = 0\ , \qquad \gamma_3 = 0\ , \qquad \gamma_5 = 0\ , \qquad \gamma_7 = 0\ , \qquad \gamma_9 = 0\ . \tag{3.9}$$

Substituting from (3.8) and (3.9) into (2.10) and take $n = 5$ , we obtain the following system of linear equations:

$$-\pi\ [c_0 + \frac{15}{8}\ c_1\ x_k + c_2\ (3x_k^2 - \frac{1}{2}) + c_3\ (4x_k^3 - \frac{17}{16}x_k) + c_4\ (5x_k^4 - \frac{3}{2}x_k^2 - \frac{1}{8}) +$$
$$c_5\ (6x_k^5 - 2x_k^3 - \frac{37}{128}x_k)] = x_k + x_k^3\ , \qquad (k = 1, 2, ..., 6). \tag{3.10}$$

Solving system (3.10) by using the zeros $x_k$ of Chebyshev polynomial $T_{n+1}(x)$ , we obtain the values of the constants as follows:

$$\{\ c_0 = c_2 = c_4 = c_5 = 0\ \ ,\ \ c_1 = \frac{-27}{40\pi}\ \ \text{and}\ \ c_3 = \frac{-1}{4\pi}\ \}\ . \tag{3.11}$$

Substituting from (3.11) into (2.1) and (2.2) we obtain the approximate solution of equation (3.1) which is the same as the exact solution which given by (3.3).

**Example 3.2**

Consider the following hypersingular integral equation

$$\oint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\, dt + \int_{-1}^{1} (2x\ t + 4x^3\ t^3)\ \varphi(t)\ dt = 4x^3 - 2x + 1\ . \qquad -1 \le x \le 1 \tag{3.12}$$

Equation (3.12) can be written in the following form:

$$\oint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\, dt = 2(\mu_1 - 1)\ x + 4(\mu_2 + 1)x^3 + 1\ , \tag{3.13}$$

$$\mu_1 = -\int_{-1}^{1} t\ \varphi(t)\ dt \qquad \text{and} \qquad \mu_2 = -\int_{-1}^{1} t^3\ \varphi(t)\ dt\ . \tag{3.14}$$

It is easy to show that the exact solution of equation (3.13) given by:

$$\varphi(x) \ = -\frac{1}{\pi}\ \sqrt{1-x^2}\ \left[\ \frac{832}{825}\ x^3 - \frac{136}{275}\ x + 1\ \right]. \tag{3.15}$$

Similarly as in example 3.1 we study the approximate solution of equation (3.12). Since $k(x,t) = 1$ and $L(x,t) = 2x\ t + 4x^3\ t^3$ , then we have

$$\left.\begin{array}{c} k_0(x) = 1 \ \ , \ \ k_1(x) = k_2(x) = ... = k_m(x) = 0 \\ \\ L_1(x) = 2x \ \ , \ \ L_3(x) = 4x^3 \ \ , \ \ L_0(x) = L_2(x) = ... = L_s(x) = 0\ . \end{array}\right\} \tag{3.16}$$

Substituting from (3.16) into (2.10) we obtain:

$$\alpha_j(x_k) = u_j\ (x_k) + 2x_k\ \gamma_{j+1} + 4x_k^3\ \gamma_{j+3}\ . \qquad (j = 0, 1, 2, ..., n) \tag{3.17}$$

Substituting from (3.17) into (2.8) we obtain the following system of linear equations:

$$\sum_{j=0}^{n} c_j \ (u_j \ (x_k) + 2x_k \ \gamma_{j+1} + 4x_k^3 \ \gamma_{j+3} \ ) = 4x_k^2 - 2x_k + 1. \qquad (j = 0, 1, 2, ..., n) \qquad (3.18)$$

Substituting from (3.8) and (3.9) into (3.18) and take $n = 5$ , we obtain the following system of linear equations:

$$-\pi \ [c_0 + c_1 \ (\frac{7}{4}x_k - \frac{1}{4}x_k^3) + c_2 \ (3x_k^2 - \frac{1}{2}) + c_3 \ (\frac{123}{32}x_k^3 - \frac{9}{8}x_k) + c_4 \ (5x_k^4 - \frac{3}{2}x_k^2 - \frac{1}{8}) +$$
$$c_5 \ (6x_k^5 - \frac{135}{64}x_k^3 - \frac{21}{64}x_k)] = 4x_k^3 - 2x_k + 1 \ , \qquad (k = 1, 2, ..., 6). \quad (3.19)$$

Solving system (3.19) by using the zeros $x_k$ of Chebyshev polynomial $T_{n+1}(x)$ , we obtain the values of the constants as follows:

$$\{ \ c_2 = c_4 = c_5 = 0 \ \ , \ \ c_0 = -\frac{1}{\pi} \ \ , \ \ c_1 = \frac{-136}{275\pi} \ \ \text{and} \ \ c_3 = \frac{-832}{825\pi} \ \} \ . \qquad (3.20)$$

Substituting from (3.20) into (2.1) and (2.2) we obtain the approximate solution of equation (3.12) which is the same as the exact solution which is given by (3.15).

# Part II: Approximate solution of nonlinear integral equations

In this part we transform the integral equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Chebyshev coefficients.

## 1. Formulation of the problem

Consider the following nonlinear integral equation:

$$\phi(x) = f(x) + \lambda \int_0^1 \ K(x, t) \ [\phi(t)]^2 \ dt \ , \qquad (1.1)$$

where $f(x)$ , $K(x,t)$ are given functions, $\lambda$ is a real parameter and $\phi(x)$ is unknown function.

The unknown function $\phi(x)$ can be represented by truncated Chebyshev series as follows:

$$\phi(x) = {\sum_{j=0}^{N}}' a_j^* \; T_j^*(x) \;, \qquad 0 \le x \le 1 \tag{1.2}$$

where $T_j^*(x)$ denoted the shifted Chebyshev polynomial of the first kind, $a_j^*$ are the unknown Chebyshev coefficients, $\sum'$ is a sum whose first term is halved and $N$ is any positive integer.

Suppose that the solution $\phi(x)$ of equation (1.1) and $K(x,t)$ can be expressed as a truncated Chebyshev series. Then (1.2) can be written in the following form

$$\phi(x) = T^*(x) \; A^* \;, \tag{1.3}$$

where

$$T^*(x) = [\; T_0^*(x) \quad T_1^*(x) \quad \dots \quad T_N^*(x)] \;, \qquad A^* = [\; \frac{a_0^*}{2} \quad a_1^* \quad \dots \quad a_N^*]^T \;,$$

and the function $[\phi(t)]^2$ can be written in the following matrix form [1]

$$[\phi(t)]^2 = \overline{T^*(t)} \; B^* \;, \tag{1.4}$$

where

$$\overline{T^*(t)} = [T_0^*(x) \quad T_1^*(x) \quad \dots \quad T_{2N}^*(x)] \;, \quad B^* = [\; \frac{b_0^*}{2} \quad b_1^* \quad \dots \quad b_{2N}^*]^T \;,$$

and the elements $b_i^*$ consists of $a_i^*$ and $a_{-i}^* = a_i^*$ as follows:

$$b_i^* = \begin{cases} \frac{(a_{i/2}^*)^2}{2} + \sum_{r=1}^{N-i/2} (a_{\frac{i}{2}-r}^*)\,(a_{\frac{i}{2}+r}^*) & \text{for } \textit{even } i \\ \\ \sum_{r=1}^{N-\frac{i-1}{2}} (a_{\frac{i+1}{2}-r}^*)\,(a_{\frac{i-1}{2}+r}^*) & \text{for odd } i. \end{cases}$$

Now, $K(x,t)$ can be expanded by chebyshev series as follows:

$$K(x_i,t) = \sum_{r=0}^{N}{}'' \; k_r(x_i) \;\; T_r^*(t) \; ,$$

where $\sum''$ denotes a sum with first and last terms halved, $x_i$ are the chebyshev collocation points defined by

$$x_i = \frac{1}{2} \; [\; 1 + \cos \; (\frac{i\pi}{N})] \; , \qquad\qquad i = 0, 1, ..., N \tag{1.5}$$

and Chebyshev coefficients $k_r(x_i)$ are determined by the following relation:

$$k_r(x_i) = \frac{2}{N} \sum_{j=0}^{N}{}'' \; K(x_i, t_j) \; T_r^*(t_j) \; , \qquad t_i = \frac{1}{2} \; [\; 1 + \cos \; (\frac{j\pi}{N})]$$

which is given by [5 ].

Then the matrix representation of $K(x_i,t)$ given by

$$K(x_i,t) = K(x_i) \; T^*(t)^T \; . \tag{1.6}$$

where

$$K(x_i) = \; [\; \frac{k_0(x_i)}{2} \qquad k_1(x_i) \quad ... \quad k_{N-1}(x_i) \qquad \frac{k_N(x_i)}{2} \; ] \; .$$

## 2. Solution of nonlinear integral equation

Our aim in this section to find the Chebyshev coefficients of (1.2), that is the matrix $A^*$. By substituting from Chebyshev collocation points defined by (1.5) into equation (1.1) we obtain a matrix equation of the form

$$\Phi = F + \lambda \; I \; , \tag{2.1}$$

where $I(x)$ denotes the integral part of equation (1.1) and

$$\Phi = \begin{pmatrix} \phi(x_0) \\ \phi(x_1) \\ . \\ . \\ . \\ \phi(x_N) \end{pmatrix}, \quad F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ f(x_N) \end{pmatrix}, \quad I = \begin{pmatrix} I(x_0) \\ I(x_1) \\ . \\ . \\ . \\ I(x_N) \end{pmatrix}, \quad T^* = \begin{pmatrix} T^*(x_0) \\ T^*(x_1) \\ . \\ . \\ . \\ T^*(x_N) \end{pmatrix}.$$

When we substitute from Chebyshev collocation points (1.5) into (1.3), the matrix $\Phi$ becomes

$$\Phi = T^* \ A^* \ . \tag{2.2}$$

Substituting from (1.4) and (1.6) in $I(x_i)$ for $i = 0, 1, ..., N$ , $i = 0, 1, ..., 2N$ and using the following relation [6 ],

$$Z = \int_0^1 \ T^*(t)^T \ \overline{T^*(t)} \ dt \ = [\int_0^1 \ T_i^*(t) \ T_j^*(t) \ dt \ ] = \frac{1}{2} \ [z_{ij}] \ ,$$

where

$$z_{ij} = \begin{cases} \frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2} & \text{for } \ even \ i+j \\ \\ 0 & \text{for odd } i+j \ , \end{cases}$$

we obtain

$$I(x_i) = K(x_i) \ Z \ B^* \ . \tag{2.3}$$

Therefore, we obtain the matrix I in terms of Chebyshev coefficients matrix in the following form:

$$I = K \ Z \ B^* \ , \tag{2.4}$$

where

$$K = [\ k(x_0) \qquad k(x_1) \qquad ... \qquad k(x_N)]^T\ .$$

Now, by using the relation (2.2) and (2.4), the integral equation (1.1) transform to a matrix equation which is given by:

$$T^*\ A^* - \lambda\ K\ Z\ B^* = F\ . \tag{2.5}$$

The matrix equation (2.5) corresponds to a system of $(N+1)$ nonlinear algebraic equations with $(N+1)$ unknown Chebyshev coefficients. Thus the unknown coefficients $a_j^*$ can be computed from this equation and substituting from these coefficients into (1.2) we obtain the approximate solution.

## 3. Numerical examples

In this section we shall give two examples to illustrate the above results .

**Example 3.1**

Consider the following nonlinear integral equation

$$\phi(x) = x - \frac{1}{3} + \int_0^1\ [\phi(t)]^2\ dt\ . \tag{3.1}$$

From (3.1) we have

$$f(x) = x - \frac{1}{3}\ , \qquad K(x,t) = 1 \quad and \quad \lambda = 1\ .$$

For $N = 2$ , the Chebyshev collocation points on $[0,1]$ can be found from (1.5) as

$$x_0 = 1\ , \qquad x_1 = \frac{1}{2}\ , \qquad x_2 = 0$$

and the matrix equation corresponds to the integral equation (3.1) given by

$$T^*\ A^* -\ K\ Z\ B^* = F\ , \tag{3.2}$$

where

$$T^* = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad , \quad F = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \\ -\frac{1}{3} \end{pmatrix} \quad , \quad Z = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{15} \\ 0 & \frac{1}{3} & 0 & -\frac{1}{5} & 0 \\ -\frac{1}{3} & 0 & \frac{7}{15} & 0 & -\frac{19}{105} \end{pmatrix}$$

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad A^* = \begin{pmatrix} a_0^*/2 \\ a_1^* \\ a_2^* \end{pmatrix} \quad , \quad B^* = \begin{pmatrix} \frac{1}{2}(\frac{a_0^{*2}}{2} + a_1^{*2} + a_2^{*2}) \\ a_0^* \, a_1^* + a_1^* \, a_2^* \\ \frac{a_1^{*2}}{2} + a_0^* \, a_2^* \\ a_1^* \, a_2^* \\ \frac{a_2^{*2}}{2} \end{pmatrix} .$$

Substituting from these matrices into (3.2), we obtain a system of nonlinear algebraic equations, the solution of this given by:

$$(a_0^* = 1 \quad , \quad a_1^* = \frac{1}{2} \quad , \quad a_2^* = 0) \ .$$

Substituting from these values into (1.2) when $N = 2$ we obtain the approximate solution $\phi(x) = x$, which is the exact solution.

**Example 3.2**

Consider the following nonlinear integral equation

$$\phi(x) = x^2 - \frac{x}{6} - 1 + \int_0^1 x \ t \ [\phi(t)]^2 \ dt \ . \tag{3.3}$$

Similarly as in example 3.1 it is easy to show that the values of $a_j^*$ (for $N = 2$ , $j = 0, 1, 2$) given by

$$(a_0^* = -\frac{5}{4} \quad , \quad a_1^* = \frac{1}{2} \quad , \quad a_2^* = \frac{1}{8}\ ) \ ,$$

and the approximate solution of the integral equation (3.3) given by $\phi(x) = x^2 - 1$ which is the exact solution.

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