CUBO A Mathematical Journal
Vol.15, Nº01, (113–117). March 2013

A Girsanov formula associated to a big order pseudo-differential operator

Rémi Léandre
Université de Franche -Comté,
Laboratoire de Mathématiques,
25030 Besançon, France.
remi.leandre@univ-fcomte.fr

ABSTRACT

We give a quasi-invariance formula involved with a semi-group generated by a big order elliptic pseudo-differential operator.

RESUMEN

Entregamos una fórmula de cuasi-invarianza relacionada con un semigrupo generado por un operador seudo-diferencial elíptico de orden superior.

Keywords and Phrases: Pseudo-differential operators. Girsanov formula.

2010 AMS Mathematics Subject Classification: 35K41, 35S05, 60G20.

1 Introduction

Dedicated to professor doctor N'Guérékata for his birthday

There are two basic tools in the theory of stochastic processes ([2], [6], [17]):

-) Itô formulas.

-)Quasi-invariance formulas of Girsanov type.

Roughly speaking, a semi group $\exp[tL]$ governed by a generator L whose domain is continuously densely imbedded in the space of bounded continuous functions $C_b(\mathbb{R}^d)$ endowed with the uniform norm is represented by a stochastic process

$$\exp[tL]f(x) = E[f(x_t(x))] \tag{1}$$

if and only if the generator satisfies the maximum principle: $Lf(x) \geq 0$ if the function f reaches his maximum in x. Such semi-groups are called Markov semi-groups.

There are much more semi-groups than Markov semi-groups.

Itô formula was extended for more general partial differential equations in [7], [8], [9], [10], [11], [15]. For an approach to Itô formula to generalized Wiener chaos, we refer to [13], [14].

Girsanov formula was extended in the framework of white noise analysis to bilaplacians in [13] and [14].

We refer to [16] to a review.

The object of this paper is to extend the Girsanov formula to very general semi-groups generated by general pseudo-differential operators.

2 Statement and proof of the main theorem

Let $(x, \xi) \to a(x, \xi)$ a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$. According the terminology of [3], [4], [5] it is called a symbol. We suppose that if $|\xi| \leq C$ the symbol is smooth with bounded derivatives at each order. If $|\xi| > C$, we suppose that there exist a strictly positive integer m such that

$$\sup_{x \in \mathbb{R}^d} |D_x^k D_\xi^{k'} a(x, \xi)| \leq C|\xi|^{2m-k'} \tag{2}$$

We suppose that the symbol is elliptic:

$$\inf_{x \in \mathbb{R}^d} |a(x, \xi)| \geq C|\xi|^{2m} \tag{3}$$

We put by standard theory on pseudo-differential operators ([3], [4], [5])

$$\hat{L}f(x) = \int_{\mathbb{R}^d} a(x, \xi)\hat{f}(\xi)d\xi \tag{4}$$

where $\xi \to \hat{f}(\xi)$ is the Fourier transform of $x \to f(x)$.He can be extended continuously on the space of smooth functions with bounded derivatives at each order. We suppose because later we consider Girsanov type formulas that $L1 = 0$.

Hypothesis: We suppose that $-L$ is positive essentially self-adjoint on $L^2(\mathbb{R}^d)$.

L generates a contraction semigroup $\exp[tL]$ on $L^2(\mathbb{R}^d)$. By elliptic theory,

$$\exp[tL]f(x) = \int_{\mathbb{R}^d} f(y)\mu_t(x, dy) \tag{5}$$

where $\mu_t(x, dy)$ is a measure on $\mathbb{R}^d$ (But not a **probability** measure).

We consider an operator L^1 on $L^2(\mathbb{R}^d)$ and we suppose that it is a pseudodifferential operator of order strictly smaller than $2m - 1$ of the type (2) and (4). He can be extended continuously on the space of smooth functions with bounded derivatives at each order. We suppose because later we consider Girsanov type formulas that $L^1 1 = 0$. We consider the pseudo-differential operator densely defined on $L^2(\mathbb{R}^d \times \mathbb{R})$

$$-L^{tot} = -L - L^1 \frac{\partial}{\partial u} + (-1)^m \frac{\partial^{2m}}{\partial u^{2m}} \tag{6}$$

By elliptic theory, it generates a semi group $\exp[tL^{tot}]$on $L^2(\mathbb{R}^d \times \mathbb{R})$ (But not a contraction semi-group due to the perturbation term $L^1 \frac{\partial}{\partial u}$ in the total operator L^{tot}). The main remark is that if f depends only on u $L^1 \frac{\partial}{\partial u} f = 0$! By elliptic theory

$$\exp[tL^{tot}]f(x, u) = \int_{\mathbb{R}^d \times \mathbb{R}} f(y, v)\mu_t^{tot}(x, u, dy, dv) \tag{7}$$

where μ^{tot} is a measure on $\mathbb{R}^d \times \mathbb{R}$ (But not a **probability** measure).

We consider the operator densely defined on $L^2(\mathbb{R}^d)$

$$-L^{per} = -L - L^1 \tag{8}$$

By elliptic theory, it generates a semi-group on $L^2(\mathbb{R}^d)$ (But not a contraction semi-group due to the perturbation term L^1). By elliptic theory, it generates a semi-group on $L^2(\mathbb{R}^d)$ (but not a contraction semi-group due to the perturbation term L^1). By elliptic theory,

$$\exp[tL^{per}]f(x) = \int_{\mathbb{R}^d} f(y)\mu_t^{per}(x, dy) \tag{9}$$

where $\mu_t^{per}(x, dy)$ is a measure on $\mathbb{R}^d$ (but not a **probability** measure).

We get

Theorem 2.1. *(Girsanov): We have if f is continuous with compact support and if we consider the Doleans-Dade exponential* $\exp[u + (-1)^m t] = g(u, t)$

$$\exp[tL^{per}]f(x) = \exp[tL^{tot}][f(.)g(., t)](x, 0) \tag{10}$$

Proof: Let us begin by doing formal computations. $\frac{\partial}{\partial u}$ commute with L^{tot}. Therefore

$$L^{tot}\exp[tL^{tot}][f(.)g(.,t)](x,u) = L\exp[tL^{tot}][f(.)g(.,t)](x,u)+$$
$$L^1\exp[tL^{tot}][f(.)\frac{\partial}{\partial v}g(.,t)](x,u) + \exp[tL^{tot}][f(.)(-1)^{m+1}\frac{\partial^{2m}}{\partial v^{2m}}g(.,t)](x,u) =$$
$$A_1 + A_2 + A_3 \quad (11)$$

The term A_3 is boring. This explain that we introduce $\exp[(-1)^m t]$ in the Doleans-Dade exponential in order to remove it. Namely we consider **linear** semi-groups such that

$$\exp[tL^{tot}][f(.)g(.,t)](x,u) = \exp[tL^{tot}][f(.)\exp[.]](x,0)\exp[(-1)^m t] \quad (12)$$

Therefore A_3 disappears and

$$\frac{\partial}{\partial t}\exp[tL^{tot}][f(.)g(.,t)](x,0) = L^{per}\exp[tL^{tot}][f(.)g(.,t)](x,0) \quad (13)$$

The only problem in this formal comutation is that $u \to \exp[u]$ is not bounded!. But if f is with compact support continuous

$$|\exp[tL^{tot}][f(.)\exp[.]](x,0)| \leq \int_{\mathbb{R}^d\times\mathbb{R}}|f(y)|\exp[v]|\mu_t^{tot}|(x,u,dy,dv)$$
$$\leq (\int_{\mathbb{R}^d}|f(y)|^2|\mu_t|(x,dy))^{1/2}(\int_{\mathbb{R}}\exp[2u]|\nu_t|(0,dv))^{1/2} \quad (14)$$

In (14), $\nu_t(u,dv)$ represents the semi group associated to $L_{2m} = (-1)^{m+1}\frac{\partial^{2m}}{\partial u^{2m}}$. By [1], this semi-group has an heat-kernel bounded by $Ct^{-1/4m}G_{2m,a}(\frac{|u-v|}{t^{1/4m}})$ $(a > 0)$ where

$$G_{2m,a}(u) = \exp[-au^{2m/2m-1}] \quad (15)$$

This inequality justifies the formal considerations above!

◇

Received: November 2012. Revised: February 2013.

References

[1] Davies B.: Unifomly elliptic operators with measurable coefficients. J. Funct. Ana. 132, 141–169. (1995).

[2] Dellacherie C. Meyer P.A.: Probabilités et potentiel. Vol II. Hermann, Paris (1978).

[3] Dieudonné J.: Eléments d'analyse VII. Gauthiers-Villars, Paris (1977).

[4] H́ormander L.:The analysis of linear partial operators III, Springer, Berlin (1984).

[5] H́ormander L.:The analysis of linear partial operators IV. Springer, Berlin (1984).

[6] Ikeda N. Watanabe S.: Stochastic differential equations and diffusion processes, 2nd edition, North-Holland (1989).

[7] Léandre R.: Itô-Stratonovitch for a four order operator on a torus. In: Nagy P. et al eds. Non-Euclidean geometry and its applications. Acta. Physica Debrecina 42, 133–138 (2008).

[8] Léandre R.: Itô-Stratonovitch for the Schroedinger equation associated to a big order operator on a torus. In: Zaslavsky G. and al eds. Fractional Order differentiation. Physica Scripta 136, 014028 (2009).

[9] Léandre R.: Itô-Stratonovitch formula for the wave equation on a torus. In: El Tawil M.A. (eds), Computations of stochastic systems, 68–75. Trans. Comp. Sciences. VII. L.N.C.S. 5890 (2010).

[10] Léandre R.: Itô formula for an integro differential operator without a stochastic process. In: Ruzhansky M., Wirth J. (eds). ISAAC 2009, 225–231, World Scientific (2010).

[11] Léandre R.: Stochastic analysis without probability: study of some basic tools. Journal Pseudo Differential Operators and Applications 1, 389–410 (2010).

[12] Léandre R.: Long time behaviour on a path group of the heat semi-group associated to a bilaplacian. In: Garrido A. eds. Symmetry measures on complex networks. Symmetry 3, 72–83, 2011.

[13] Léandre R.: A path-integral approach to the Cameron-Martin-Maruyama-Girsanov formula associated to a Bilaplacian. In: Ti-Jun Xiao and al eds. Integral and differential systems in function spaces. Journal of function spaces and applications, Article 458738, 2012.

[14] Léandre R.: A generalized Fock space associated to a Bilaplacian. In: Victor Jin (eds). 2011 World Congress Engineering Technology, 68-72, C.D. I.E.E.E., (2011).

[15] Léandre R.: An Itô formula for an accretive operator. In : A. Garrido Eds: Axioms: feature papers. Axioms 1, 4-8 (2012).

[16] Léandre R.: Stochastic analysis for a non-markovian generator: an introduction.

[17] Rogers L.C.G., Williams D.: Stochastic differential equations and martingales, vol 2, Itô Calculus, Wiley, New-York (1987).