

## Generalized Ulam - Hyers Stability of Derivations of a AQ - Functional Equation

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### ABSTRACT

In this paper, the author established the generalized Ulam - Hyers stability of Derivations of additive and quadratic (AQ)- functional equation

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y).$$

### RESUMEN

En este artículo el autor establece la estabilidad generalizada Ulam-Hyers de derivaciones de la ecuación (AQ)-funcional cuadrática y aditiva

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y).$$

**Keywords and Phrases:** Additive functional equations, Quadratic functional equation, Mixed type functional equation, Additive derivations, Quadratic derivations, Ulam - Hyers stability

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## 1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [33] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [13]. It was further generalized and excellent results obtained by number of authors [2, 10, 24, 30, 32].

Over the last six or seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 9, 14, 19, 22, 31].

C. Park [25] applied Gavruta's result to Banach modules over a  $C^*$ -algebra. Many authors have studied the structure of  $C^*$ -algebras for different types of functional equations in various settings one can refer [6, 8, 26, 28]. It seems that approximate derivations was first investigated by K.W. Jun and D.W. Park [16]. Recently, the stability of derivations have been investigated in [7, 11, 12, 20, 27, 29] and references therein. The stability of cubic derivations was first time introduced and investigated by M.E. Gordji et. al., [12]. With the help of [12], the stability of quadratic derivations was discussed by M. Arunkumar et. al., [3].

Very recently M. Arunkumar and J.M. Rassias [5], established the generalized Ulam - Hyers stability of an additive and quadratic (AQ)-mixed type functional equation

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y) \quad (1)$$

in Banach spaces. The solution and stability of several types of Mixed type additive and quadratic type functional equations were discussed in [4, 17, 18, 21, 23]

In this paper, the author first time established the generalized Ulam - Hyers stability of mixed derivations of a additive - quadratic (AQ)- functional equation (1).

Hereafter through out this paper, let us consider  $X$  and  $Y$  to be a normed Algebra and a Banach Algebra, respectively.

## 2 Stability Results: Additive Derivations

In this section, the authors investigate the generalized Ulam-Hyers stability of additive derivations of the AQ-functional equation (1).

**Definition 2.1.** A  $\mathbb{C}$ -linear mapping  $A : X \rightarrow X$  is called **Additive Derivation** on  $X$  if  $A$  satisfies

$$A(xy) = A(x)y + xA(y) \quad (1)$$

for all  $x, y \in X$ .

**Theorem 2.1.** Let  $j = \pm 1$ . Let  $f_a : X \rightarrow Y$  be an odd mapping for which there exist a function  $\alpha, \beta : X^2 \rightarrow [0, \infty)$  with the condition

$$\sum_{n=0}^{\infty} \frac{\alpha(2^{nj}x, 2^{nj}y)}{2^{nj}} \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha(2^{nj}x, 2^{nj}y)}{2^{nj}} = 0 \quad (2)$$

$$\sum_{n=0}^{\infty} \frac{\beta(2^{nj}x, 2^{nj}y)}{2^{2nj}} \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\beta(2^{nj}x, 2^{nj}y)}{2^{2nj}} = 0 \quad (3)$$

such that the functional inequalities

$$\|f_a(x+y) + f_a(x-y) - 2f_a(x) - f_a(y) - f_a(-y)\| \leq \alpha(x, y) \quad (4)$$

and

$$\|f_a(xy) - f_a(x)y - xf_a(y)\| \leq \beta(x, y) \quad (5)$$

for all  $x, y \in X$ . Then there exists a unique Additive Derivation mapping  $A : X \rightarrow Y$  satisfying the functional equation (1) and

$$\|f_a(x) - A(x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{kj}} \quad (6)$$

for all  $x \in X$ . The mapping  $A(x)$  is defined by

$$A(x) = \lim_{n \rightarrow \infty} \frac{f_a(2^{nj}x)}{2^{nj}} \quad (7)$$

for all  $x \in X$ .

*Proof.* Assume  $j = 1$ . Replacing  $y$  by  $x$  in (4) and using oddness of  $f$ , we get

$$\left\| f_a(x) - \frac{f_a(2x)}{2} \right\| \leq \frac{\alpha(x, x)}{2} \quad (8)$$

for all  $x \in X$ . Now replacing  $x$  by  $2x$  and dividing by 2 in (8), we get

$$\left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \leq \frac{\alpha(2x, 2x)}{2^2} \quad (9)$$

for all  $x \in X$ . From (8) and (9), we obtain

$$\begin{aligned} \left\| f_a(x) - \frac{f_a(2^2x)}{2^2} \right\| &\leq \left\| f_a(x) - \frac{f_a(2x)}{2} \right\| + \left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \\ &\leq \frac{1}{2} \left[ \alpha(x, x) + \frac{\alpha(2x, 2x)}{2} \right] \end{aligned} \quad (10)$$

for all  $x \in X$ . In general for any positive integer  $n$ , we get

$$\begin{aligned} \left\| f_a(x) - \frac{f_a(2^n x)}{2^n} \right\| &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^k x, 2^k x)}{2^k} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^k x, 2^k x)}{2^k} \end{aligned} \quad (11)$$

for all  $x \in X$ . In order to prove the convergence of the sequence

$$\left\{ \frac{f_a(2^n x)}{2^n} \right\},$$

replace  $x$  by  $2^m x$  and dividing by  $2^m$  in (11), for any  $m, n > 0$ , we deduce

$$\begin{aligned} \left\| \frac{f_a(2^m x)}{2^m} - \frac{f_a(2^{n+m} x)}{2^{n+m}} \right\| &= \frac{1}{2^m} \left\| f_a(2^m x) - \frac{f_a(2^n \cdot 2^m x)}{2^n} \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+m} x, 2^{k+m} x)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+m} x, 2^{k+m} x)}{2^{k+m}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

for all  $x \in X$ . Hence the sequence  $\left\{ \frac{f_a(2^n x)}{2^n} \right\}$  is Cauchy sequence. Since  $Y$  is complete, there exists a mapping  $A : X \rightarrow Y$  such that

$$A(x) = \lim_{n \rightarrow \infty} \frac{f_a(2^n x)}{2^n} \quad \forall x \in X.$$

Letting  $n \rightarrow \infty$  in (11) we see that (6) holds for all  $x \in X$ . To prove that  $A$  satisfies (1), replacing  $(x, y)$  by  $(2^n x, 2^n y)$  and dividing by  $2^n$  in (4), we obtain

$$\frac{1}{2^n} \left\| f_a(2^n x + 2^n y) + f_a(2^n x - 2^n y) - 2f_a(2^n x) + f_a(2^n y) + f_a(-2^n y) \right\| \leq \frac{1}{2^n} \alpha(2^n x, 2^n y)$$

for all  $x, y \in X$ . Letting  $n \rightarrow \infty$  in the above inequality and using the definition of  $A(x)$ , we see that

$$A(x + y) + A(x - y) = 2A(x) + A(y) + A(-y).$$

Hence  $A$  satisfies (1) for all  $x, y \in X$ . It follows from (5) that

$$\begin{aligned} &\|A(xy) - A(x)y - xA(y)\| \\ &= \frac{1}{2^{2n}} \|f_a(2^{2n}(xy)) - f_a(2^{2n}x)(2^{2n}y) - (2^{2n}x)f_a(2^{2n}y)\| \\ &\leq \frac{1}{2^{2n}} \beta(2^{2n}x, 2^{2n}y) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all  $x, y \in X$ . To prove that  $A$  is unique, let  $B(x)$  be another mapping satisfying (1) and (6), then

$$\begin{aligned} \|A(x) - B(x)\| &= \frac{1}{2^n} \|A(2^n x) - B(2^n x)\| \\ &\leq \frac{1}{2^n} \{ \|A(2^n x) - f_a(2^n x)\| + \|f_a(2^n x) - B(2^n x)\| \} \\ &\leq \sum_{k=0}^{\infty} \frac{\alpha(2^{k+n}x, 2^{k+n}x)}{2^{(k+n)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all  $x \in X$ . Hence  $A$  is unique. Thus the mapping  $A : X \rightarrow Y$  is a unique Additive Derivation mapping satisfying (6).

For  $j = -1$ , we can prove a similar stability result. This completes the proof of the theorem.  $\square$

The following Corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1).

**Corollary 2.2.** *Let  $f_a : X \rightarrow Y$  be a odd mapping and there exists real numbers  $\lambda$  and  $s$  such that*

$$\begin{aligned} &\|f_a(x+y) + f_a(x-y) - 2f_a(x) - f_a(y) - f_a(-y)\| \\ &\leq \begin{cases} \lambda, & \\ \lambda\{\|x\|^s + \|y\|^s\}, & s < 1 \text{ or } s > 1; \\ \lambda\|x\|^s\|y\|^s, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\ \lambda\{\|x\|^s\|y\|^s + \{\|x\|^{2s} + \|y\|^{2s}\}\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \end{aligned} \quad (12)$$

$$\begin{aligned} &\|f_a(xy) - f_a(x)y - xf_a(y)\| \\ &\leq \begin{cases} \lambda, & \\ \lambda\{\|x\|^s + \|y\|^s\}, & s < 1 \text{ or } s > 1; \\ \lambda\|x\|^s\|y\|^s, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\ \lambda\{\|x\|^s\|y\|^s + \{\|x\|^{2s} + \|y\|^{2s}\}\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \end{aligned} \quad (13)$$

for all  $x, y \in X$ . Then there exists a unique Additive Derivation function  $A : X \rightarrow Y$  such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} \lambda, & \\ \frac{2\lambda\|x\|^s}{|2 - 2^s|}, & \\ \lambda\|x\|^{2s} & \\ \frac{3\lambda\|x\|^{2s}}{|2 - 2^{2s}|} & \end{cases} \quad (14)$$

for all  $x \in X$ .

### 3 Stability Results: Quadratic Derivations

In this section, the author establish the generalized Ulam-Hyers stability of quadratic derivations of the AQ-functional equation (1).

**Definition 3.1. Quadratic Derivation.** A  $\mathbb{C}$ -linear mapping  $Q : X \rightarrow X$  is called **Quadratic Derivation** on  $X$  if  $Q$  satisfies

$$Q(xy) = Q(x)y^2 + x^2Q(y) \quad (1)$$

for all  $x, y \in X$ .

**Theorem 3.1.** Let  $j = \pm 1$ . Let  $f_q : X \rightarrow Y$  be a even mapping for which there exist a function  $\alpha, \beta : X^2 \rightarrow [0, \infty)$  with the condition

$$\sum_{n=0}^{\infty} \frac{\alpha(2^{nj}x, 2^{nj}y)}{2^{2nj}} \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha(2^{nj}x, 2^{nj}y)}{2^{2nj}} = 0 \quad (2)$$

$$\sum_{n=0}^{\infty} \frac{\beta(2^{nj}x, 2^{nj}y)}{2^{4nj}} \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\beta(2^{nj}x, 2^{nj}y)}{2^{4nj}} = 0 \quad (3)$$

such that the functional inequalities

$$\|f_q(x+y) + f_q(x-y) - 2f_q(x) - f_q(y) - f_q(-y)\| \leq \alpha(x, y) \quad (4)$$

and

$$\|f_q(xy) - x^2f_q(y) - f_q(x)y^2\| \leq \beta(x, y) \quad (5)$$

for all  $x, y \in X$ . Then there exists a unique Quadratic Derivation mapping  $Q : X \rightarrow Y$  satisfying the functional equation (1) and

$$\|f_q(x) - Q(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{2kj}} \quad (6)$$

for all  $x \in X$ . The mapping  $Q(x)$  is defined by

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f_q(2^{nj}x)}{2^{2nj}} \quad (7)$$

for all  $x \in X$ .

*Proof.* It follows from (5) that

$$\begin{aligned} & \|Q(xy) - x^2Q(y) - Q(x)y^2\| \\ &= \frac{1}{2^{4n}} \|f_q(2^n(xy)) - (2^n x)^2 f_q(2^n y) - f_q(2^n x)(2^n y)^2\| \\ &\leq \frac{1}{2^{4n}} \beta(2^n x, 2^n y) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all  $x, y \in X$ . The rest of the proof is similar tracing to that of Theorem 2.1. Thus the mapping  $Q : X \rightarrow Y$  is a unique Quadratic Derivation mapping satisfying (6).  $\square$

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1).

**Corollary 3.2.** *Let  $f_q : X \rightarrow Y$  be a even mapping and there exists real numbers  $\lambda$  and  $s$  such that*

$$\begin{aligned} & \|f_q(x+y) + f_q(x-y) - 2f_q(x) - f_q(y) - f_q(-y)\| \\ & \leq \begin{cases} \lambda, & \\ \lambda\{\|x\|^s + \|y\|^s\}, & s < 2 \text{ or } s > 2; \\ \lambda\|x\|^s\|y\|^s, & s < 1 \text{ or } s > 1; \\ \lambda\{\|x\|^s\|y\|^s + \{\|x\|^{2s} + \|y\|^{2s}\}\}, & s < 1 \text{ or } s > 1; \end{cases} \end{aligned} \quad (8)$$

$$\begin{aligned} & \|f_q(xy) - x^2f_q(y) - f_q(x)y^2\| \\ & \leq \begin{cases} \lambda, & \\ \lambda\{\|x\|^s + \|y\|^s\}, & s < 2 \text{ or } s > 2; \\ \lambda\|x\|^s\|y\|^s, & s < 1 \text{ or } s > 1; \\ \lambda\{\|x\|^s\|y\|^s + \{\|x\|^{2s} + \|y\|^{2s}\}\}, & s < 1 \text{ or } s > 1; \end{cases} \end{aligned} \quad (9)$$

for all  $x, y \in X$ . Then there exists a unique quadratic Deviation function  $Q : X \rightarrow Y$  such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{3}, \\ \frac{2\lambda\|x\|^s}{|4 - 2^s|}, \\ \frac{\lambda\|x\|^{2s}}{|4 - 2^{2s}|}, \\ \frac{3\lambda\|x\|^{2s}}{|4 - 2^{2s}|} \end{cases} \quad (10)$$

for all  $x \in X$ .

## 4 Stability Results: Mixed Derivations

In this section, the author present the generalized Ulam-Hyers stability of mixed derivations of the AQ-functional equation (1).

**Theorem 4.1.** *Let  $j = \pm 1$ . Let  $f : X \rightarrow Y$  be a odd mapping for which there exist a function  $\alpha, \beta : X^2 \rightarrow [0, \infty)$  with the conditions (2), (3), (2) and (3) such that the functional inequalities*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \alpha(x, y) \quad (1)$$

(5) and (5) for all  $x, y \in X$ . Then there exists a unique Additive Derivation mapping  $A : X \rightarrow Y$  and a unique Quadratic Derivation mapping  $Q : X \rightarrow Y$  satisfying the functional equation (1) and

$$\|f(x) - A(x) - Q(x)\| \leq \frac{1}{2} \left[ \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{kj}} \right) + \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{2kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{2kj}} \right) \right] \quad (2)$$

for all  $x \in X$ . The mapping  $A(x)$  and  $Q(x)$  are defined in (6) and (6) respectively for all  $x \in X$ .

*Proof.* Let  $f_o(x) = \frac{f_a(x) - f_a(-x)}{2}$  for all  $x \in X$ . Then  $f_o(0) = 0$  and  $f_o(-x) = -f_o(x)$  for all  $x \in X$ . Hence

$$\|f_o(x+y) + f_o(x-y) - 2f_o(x) - f_o(y) - f_o(-y)\| \leq \frac{\alpha(x, y)}{2} + \frac{\alpha(-x, -y)}{2} \quad (3)$$

By Theorem 2.1, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{kj}} \right) \quad (4)$$

for all  $x \in X$ . Also, let  $f_e(x) = \frac{f_q(x) + f_q(-x)}{2}$  for all  $x \in X$ . Then  $f_e(0) = 0$  and  $f_e(-x) = f_e(x)$  for all  $x \in X$ . Hence

$$\|f_e(x+y) + f_e(x-y) - 2f_e(x) - f_e(y) - f_e(-y)\| \leq \frac{\alpha(x, y)}{2} + \frac{\alpha(-x, -y)}{2} \quad (5)$$

By Theorem 3.1, we have

$$\|f_e(x) - Q(x)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{2kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{2kj}} \right) \quad (6)$$

for all  $x \in X$ . Define

$$f(x) = f_e(x) + f_o(x) \quad (7)$$

for all  $x \in X$ . From (4), (6) and (7), we arrive

$$\begin{aligned} \|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q(x)\| \\ &\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \\ &\leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{kj}} \right) \\ &\quad + \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \left( \frac{\alpha(2^{kj}x, 2^{kj}x)}{2^{2kj}} + \frac{\alpha(-2^{kj}x, -2^{kj}x)}{2^{2kj}} \right) \end{aligned}$$

for all  $x \in X$ . Hence the theorem is proved.  $\square$



Using Corollaries 2.2 and 3.2 we have the following Corollary concerning the stability of (1).

**Corollary 4.1.** *Let  $f : X \rightarrow Y$  be a mapping and there exists real numbers  $\lambda$  and  $s$  such that*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \begin{cases} \lambda, & s < 1 \text{ or } s > 1; \\ \lambda \{\|x\|^s + \|y\|^s\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\ \lambda \{\|x\|^s \|y\|^s + \{\|x\|^{2s} + \|y\|^{2s}\}\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \quad (8)$$

and (13), (9) for all  $x, y \in X$ . Then there exists a unique Additive Deviation function  $A : X \rightarrow Y$  and a unique quadratic Deviation function  $Q : X \rightarrow Y$  such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \lambda \left(1 + \frac{1}{3}\right), \\ 2\lambda \left(\frac{1}{|2-2^s|} + \frac{1}{|4-2^s|}\right) \|x\|^s, \\ \lambda \left(\frac{1}{|2-2^{2s}|} + \frac{1}{|4-2^{2s}|}\right) \|x\|^{2s}, \\ 3\lambda \left(\frac{1}{|2-2^{2s}|} + \frac{1}{|4-2^{2s}|}\right) \|x\|^{2s} \end{cases} \quad (9)$$

for all  $x \in X$ .

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