CUBO A Mathematical Journal
Vol.14, N$^{\underline{o}}$02, (01–13). June 2012

The ϵ−Optimality Conditions for Multiple Objective Fractional Programming Problems for Generalized (ρ, η)−Invexity of Higher Order

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ABSTRACT

Motivated by the recent investigations in literature, a general framework for a class of (ρ, η)−invex n-set functions of higher order is introduced, and then some results on the ϵ−optimality conditions for multiple objective fractional subset programming are explored. The obtained results are general in nature, while generalize and unify results on generalized invexity as well as on generalized invexity of higher order to the context of multiple fractional programming.

RESUMEN

Motivado por investigaciones recientes en la literatura, se introduce un marco general para una clase de funciones (ρ, η)−invex n-set de orden superior y se exploran algunos resultados sobre condiciones de épsilon-optimalidad para objetivos múltiples fraccionales de subconjuntos de programación. Los resultados obtenidos son de naturaleza general, dado que generalizan y unifican resultados sobre invexity generalizada e invexity generalizada de orden superior en el contexto de la programación múltiple fraccionaria.

Keywords and Phrases: Generalized invexity of higher order, multiple objective fractional subset programming, ϵ−efficient solution, semi-parametric sufficient ϵ−optimality conditions.

2010 AMS Mathematics Subject Classification: 49J40, 90C25.

1 Introduction

Recently, Kim et al. [9] investigated some results based on $\epsilon-$optimality conditions for multiple objective fractional optimization problems. They used both approaches of parametric as well as non-parametric sufficient conditions to achieving an equivalence between them. Motivated by these developments, we examine some $\epsilon-$optimality conditions for multiple objective fractional programming problems based on a generalized $(\rho, \eta)-$invexity for higher order [1,7,8] of n-set functions, more specifically, results on parametric and semi-parametric sufficient $\epsilon-$efficiency conditions for multiobjective fractional subset programming. More recently, Mishra et al. [11] published some results on optimality conditions for multiple objective fractional subset programming with invex and related non-convex n-set functions (also studied by Verma [13,15] and Zalmai [16]) to the case of parametric and semi-parametric sufficient efficiency conditions for a multiobjective fractional subset programming problem. Jeyakumar et al. [5,6] and Kim et al. [9] investigated some results on $\epsilon-$optimality conditions for multiobjective fractional programming problems. We present using the generalized invexity of higher order for differentiable functions, the following multiple objective fractional subset programming problem:

(P)

$$Minimize\,(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, ..., \frac{F_p(S)}{G_p(S)})$$

subject to

$$H_j(S) \leq 0 \text{ for } j \in \{1, ..., m\}, S \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, ..., m\}\},$$

where X is an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space), $F_i, G_i,\ i \in \{1, ..., p\}$ and H_j for $j \in \{1, ..., m\}$ are real-valued functions defined on X and $G_i(S) > 0$ for each $i \in \{1, ..., p\}$ and for all $S \in X$.

Next, we observe that problem (P) is equivalent to the parametric multiobjective non-fractional programming problem:

(Pλ)

$$Minimize\,(F_1(S) - \lambda_1 G_1(S), ..., F_p(S) - \lambda_p G_p(S)),$$

where $\lambda_i, i = 1, 2, ..., p$ are parameters, and $S \in Q$.

Mishra et al. [11] investigated several parametric and semi-parametric sufficient conditions for the multiobjective fractional subset programming problems based on generalized invexity assumptions. Moreover, these results are also applicable to other classes of problems with multiple, fractional, and conventional objective functions.

Furthermore, among other results, the obtained results generalize the recent results on generalized invexity to the case of the generalized invexity of higher order $m \geq 1$ relating to the case of semi-parametric sufficient $\epsilon-$efficiency conditions for the multiobjective fractional subset programming problems. For more details, we refer the reader [1–17].

2 Generalized Invexities of Higher Order

In this section, we develop some concepts and notations for the problem on hand. Let X be an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space). Let $\langle \cdot, \cdot \rangle$ the inner product, and let $\eta : X \times X \to \mathfrak{R}^n$ be a vector-valued function. Suppose that $\bigtriangledown f(x)$denotes the gradient of f at x defined by

$$\bigtriangledown f(x) = (\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_n}),$$

where $f : X \to \mathfrak{R}^n$ is real-valued function on X.

Next, we recall the notions of the generalized invexity. Let $S, S^* \in X$, let the function $F : X \to \mathfrak{R}^n$ with components F_i for $i \in \{1, ..., n\}$, be differentiable at S^*.

Definition 2.1. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$F(S) - F(S^*) \geq \langle \bigtriangledown F(S^*), \eta(S, S^*) \rangle + \rho \|S - S^*\|^m,$$

where $m \geq 1$ is an integer.

Definition 2.2. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$strictly-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$F(S) - F(S^*) > \langle \bigtriangledown F(S^*), \eta(S, S^*) \rangle + \rho \|S - S^*\|^m,$$

where $m \geq 1$ is an integer.

Definition 2.3. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$quasi-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$F(S) \leq F(S^*) \Rightarrow \langle \bigtriangledown F(S^*), \eta(S, S^*) \rangle + \rho \|S - S^*\|^m \leq 0,$$

where $m \geq 1$ is an integer.

Definition 2.4. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$prestrictly-quasi-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each

$S^* \in X$, and $\rho > 0$,

$$F(S) < F(S^*) \Rightarrow \langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m \leq 0,$$

where $m \geq 1$ is an integer.

Definition 2.5. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$pseudo-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$\langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m \geq 0 \Rightarrow F(S) \geq F(S^*),$$

where $m \geq 1$ is an integer.

Definition 2.6. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$strictly-pseudo-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$\langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m \geq 0 \Rightarrow F(S) > F(S^*),$$

where $m \geq 1$ is an integer.

Definition 2.7. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$prestrictly-pseudo-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$\langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m > 0 \Rightarrow F(S) \geq F(S^*),$$

where $m \geq 1$ is an integer.

Definition 2.8. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$strictly-quasi-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$F(S) \leq F(S^*) \Rightarrow \langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m < 0,$$

where $m \geq 1$ is an integer.

Definition 2.9. A differentiable function $F : X \to \mathfrak{R}^n$ is said to be $(\rho, \eta)-$prestrictly-quasi-invex of higher order at S^* if there exists a vector-valued function $\eta : X \times X \to \mathfrak{R}^n$ such that for each $S^* \in X$, and $\rho > 0$,

$$F(S) < F(S^*) \Rightarrow \langle \bigtriangledown F(S^*), \eta(S, S^*)\rangle + \rho\|S - S^*\|^m \leq 0,$$

where $m \geq 1$ is an integer.

Now we introduce the generalized $\epsilon-$solvability conditions for (P) and (Pλ) problems as follows: S^* is a generalized $\epsilon-$efficient solution to (P) if there does not exist an $S \in Q$ such that

$$\frac{F_i(S)}{G_i(S)} \leq \frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*) \, \forall \, i = 1, .., p,$$

$$\frac{F_j(S)}{G_j(S)} < \frac{F_j(S^*)}{G_j(S^*)} - \epsilon_j(S^*) \text{ for some } j \in \{1, .., p\},$$

where ϵ_i,$\epsilon_j : \mathfrak{R}^n \to \mathfrak{R}$ are with $\epsilon(S^*) = (\epsilon_1(S^*), ..., \epsilon_p(S^*))$, $\epsilon_i \geq 0$ for i=1,...,p.
For $\epsilon = \epsilon(S^*)$, (P) reduces to Kim et al. [9], and for $\epsilon = 0$, it reduces to the case that $S^* \in Q$ is an efficient solution to (P) if there exists no $S \in Q$ such that

$$\frac{F_i(S)}{G_i(S)} \leq \frac{F_i(S^*)}{G_i(S^*)} \, \forall \, i = 1, ..., p.$$

To this context, based on Mishra et al. [11], we consider the following auxiliary problem:
(Pλ)

$$minimize_{S \in Q}(F_1(S) - \lambda_1 G_1(S), ..., F_p(S) - \lambda_p G_p(S)),$$

where λ_i for $i \in \{1, ..., p\}$ are parameters.

Next, we introduce the generalized $\epsilon-$solvability conditions for (Pλ) problem as follows: S^* is a generalized $\epsilon-$efficient solution to (Pλ) if there does not exist an $S \in Q$ such that

$$F_i(S) - \lambda_i G_i(S) \leq F_i(S^*) - \lambda_i G_i(S^*) - \bar{\epsilon_i}(S^*) \, \forall \, i = 1, .., p,$$

$$F_j(S) - \lambda_j G_j(S) < F_j(S^*) - \lambda_j G_j(S^*) - \bar{\epsilon_j}(S^*) \text{ for some } j \in \{1, .., p\},$$

where $\lambda_i = \frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i$, $\bar{\epsilon_i}(S^*) = \epsilon_i(S^*) G_i(S^*)$, and $\bar{\epsilon_j}(S^*) = \epsilon_j(S^*) G_j(S^*)$, where ϵ_i,$\epsilon_j : \mathfrak{R}^n \to \mathfrak{R}$ are with $\epsilon(S^*) = (\epsilon_1(S^*), ..., \epsilon_p(S^*))$, $\epsilon_i \geq 0$ for i=1,...,p.
For $\epsilon = \epsilon(S^*)$, (P) reduces to Kim et al. [9], and for $\epsilon = 0$, it reduces to the case that is an efficient solution to (P) if there exists no $S \in \Xi$ such that

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, ..., \frac{F_p(S)}{G_p(S)}\right) \leq \left(\frac{F_1(S^*)}{G_1(S^*)}, \frac{F_2(S^*)}{G_2(S^*)}, ..., \frac{F_p(S^*)}{G_p(S^*)}\right).$$

Lemma 2.1. [9] Let $S^* \in Q = \{x \in X : H_j(x) \leq 0 \text{ for } i = 1, ..., m\}$, where $H_j : X \to \mathfrak{R}$ is a real-valued function on X. Then the following statements are mutually equivalent:

(i) S^* is a generalized $\epsilon(S^*)-$efficient solution to (P).

(ii) S^* is a generalized $\epsilon^*(S^*)-$solution to (Pλ), where

$$\lambda = (\frac{F_1(S^*)}{G_1(S^*)} - \epsilon_1(S^*), ..., \frac{F_p(S^*)}{G_p(S^*)} - \epsilon_p(S^*))$$

and $\epsilon^*(S^*) = (\epsilon_1(S^*)G_1(S^*), ..., \epsilon_p(S^*)G_p(S^*))$.

Lemma 2.2. [15] Let $S^* \in Q$=$\{x \in X : H_j(x) \leq 0 \text{ for } i = 1, ..., m\}$, where $H_j : X \to \mathfrak{R}$ is a real-valued function on X. Then the following statements are mutually equivalent:

(i) S^* is a generalized $\epsilon(S^*)-$efficient solution to (P).

(ii) There exists $S \in Q$ such that

$$\Sigma_{i=1}^{p}[F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S)] \geq 0.$$

Lemma 2.3. [15] Let $S^* \in Q = \{x \in X : H_j(x) \leq 0 \text{ for } i = 1, ..., m\}$, where $H_j : X \to \mathfrak{R}$ is a real-valued function on X. Then the following statements are mutually equivalent:

(i) S^* is a generalized $\epsilon(S^*)-$efficient solution to (Pλ).

(ii) There exists $S \in Q$ such that

$$\Sigma_{i=1}^{p}[F_i(S) - (\frac{F_i(S^*)}{G_i(S)} - \epsilon_i(S^*))G_i(S)] \geq \Sigma_{i=1}^{p}[F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S^*)] - \Sigma_{i=1}^{p}\epsilon_i(S^*)G_i(S^*).$$

3 Parametric Sufficient $\epsilon-$ Optimality Conditions

This section deals with some parametric sufficient $\epsilon-$ optimality conditions for problem (P) under the generalized frameworks for generalized $(\rho, \eta)-$invexity of higher order $m \geq 1$. We begin with real-valued functions $A_i(.; \lambda, u)$ and $B_j(., v)$ defined by

$$A_i(.; \lambda, u) = u_i[F_i(S) - \lambda_i G_i(S)] \text{ for } i = 1, ..., p, \text{ and for fixed } \lambda, u \text{ and } v$$

and

$$B_j(., v) = v_j H_j(S),\ j = 1, ..., m.$$

Theorem 3.1. Let $S^* \in Q = \{S \in X : H_j(S) \leq 0 \text{ for } j \in \{1, ..., m\}$, the feasible set of (P). Let $F_i, G_i,\ i \in \{1, ..., p\}$, and $H_j,\ j \in \{1, ..., m\}$, be differentiable at $S^* \in Q$, and let there exist $u^* \in U = \{u \in \mathfrak{R}^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $v^* \in R_+^m$ such that

$$\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)] + \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(S^*), \eta(S, S^*)\rangle \geq 0 \forall S \in Q, \tag{3.1}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \text{ for } i \in \{1, ..., p\}, \tag{3.2}$$

$$\nu_j^* H_j(S^*) = 0 \text{ for } j \in \{1, ..., m\}, \tag{3.3}$$

where $\lambda_i^* = (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))$.

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(.;\lambda^*, u^*)$ $(\forall i = 1, ..., p)$ are $(\rho, \eta)-$pseudo-invex at S^* of higher order and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$quasi-invex at S^* of higher order.

(ii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-pseudo-invex at S^* of higher order and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are strictly-quasi-invex at S^* of higher order.

(iii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-quasi-invex at S^* of higher order and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-pseudo-invex at S^* of higher order.

Then S^* is an $\epsilon-$efficient solution to (P).

Proof. If (i) holds, and if $S^* \in Q$, then it follows from (3.1) that

$$\begin{aligned} & \langle \Sigma_{i=1}^p u_i^* [\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)], \eta(S, S^*) \rangle \\ + \quad & \langle \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(S^*), \eta(S, S^*) \rangle \geq 0 \, \forall S \in Q. \end{aligned} \tag{3.4}$$

Since $\nu^* \geq 0$, $S \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^m \nu_j^* H_j(S) \leq 0 = \Sigma_{j=1}^m \nu_j^* H_j(S^*),$$

and in light of the $(\rho, \eta)-$quasi-invexity of $B_j(., \nu^*)$ at S^*, we arrive at

$$\langle \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(S^*), \eta(S, S^*) \rangle \leq -\rho \|S - S^*\|^m. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\langle \Sigma_{i=1}^p u_i^* [\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)], \eta(S, S^*) \rangle \geq \rho \|S - S^*\|^m. \tag{3.6}$$

This further implies

$$\langle \Sigma_{i=1}^p u_i^* [\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)], \eta(S, S^*) \rangle \geq -\rho \|S - S^*\|^m. \tag{3.7}$$

Next, applying the $(\rho, \eta)-$pseudo-invexity at S^* to (3.6), we have

$$\Sigma_{i=1}^p u_i^* [F_i(S) - \lambda_i^* G_i(S)] \geq \Sigma_{i=1}^p u_i^* [F_i(S^*) - \lambda_i^* G_i(S^*)],$$

that is equivalent to

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq \Sigma_{i=1}^{p} u_i^*[F_i(S^*) - \lambda_i^* G_i(S^*)] \\ - \; & \Sigma_{i=1}^{p} u_i^* \epsilon_i(S^*) G_i(S^*) \\ = \; & 0, \end{aligned}$$

where $\lambda_i^* = (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))$. Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geq 0. \tag{3.8}$$

Since $u_i^* > 0$ for each $i \in \{1, ..., p\}$, we conclude that there does not exist an $S \in Q$ such that

$$\frac{F_i(S)}{G_i(S)} - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) \leq 0 \, \forall \, i = 1, ..., p,$$

$$\frac{F_j(S)}{G_j(S)} - (\frac{F_j(S^*)}{G_j(S^*)} - \epsilon_j(S^*)) < 0 \, \forall \, j \in \{1, ..., p\}.$$

Hence, S^* is an $\epsilon-$efficient solution to (P). Similar proofs hold for (ii) and (iii).

When m=2, we have

Theorem 3.2. Let $S^* \in Q = \{S \in X : H_j(S) \leq 0 \, \text{for} \, j \in \{1, ..., m\}$, the feasible set of (P). Let F_i, G_i, $i \in \{1, ..., p\}$, and H_j, $j \in \{1, ..., m\}$, be differentiable at $S^* \in Q$, and let there exist $u^* \in U = \{u \in \mathfrak{R}^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $\nu^* \in R_+^m$ such that

$$\langle \Sigma_{i=1}^{p} u_i^*[\triangledown F_i(S^*) - \lambda_i^* \triangledown G_i(S^*)] + \Sigma_{j=1}^{m} \nu_j^* \triangledown H_j(S^*), \eta(S, S^*) \rangle \geq 0 \forall S \in Q, \tag{3.9}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \, \text{for} \, i \in \{1, ..., p\}, \tag{3.10}$$

$$\nu_j^* H_j(S^*) = 0 \, \text{for} \, j \in \{1, ..., m\}, \tag{3.11}$$

where $\lambda_i^* = (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))$.

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(.; \lambda^*, u^*)$ $(\forall i = 1, ..., p)$ are $(\rho, \eta)-$pseudo-invex at S^* and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$quasi-invex at S^*.

(ii) $A_i(.; \lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-pseudo-invex at S^* and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are strictly-quasi-invex at S^*.

(iii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-quasi-invex at S^* and $B_j(., \nu^*)$ $\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-pseudo-invex at S^*.

Then S^* is an $\epsilon-$efficient solution to (P).

For $\epsilon = 0$, we have

Theorem 3.3. Let $S^* \in Q$, let F_i, G_i, $i \in \{1, ..., p\}$, and H_j, $j \in \{1, ..., m\}$, be differentiable at $S^* \in \Lambda$, and let there exist $u^* \in U = \{u \in \mathfrak{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $\nu^* \in R_+^m$ such that

$$\begin{aligned} &\langle \Sigma_{i=1}^p u_i^* [\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)] + \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(S^*), \eta(S, S^*)\rangle \\ + \quad & \rho \|S - S^*\|^2 \geq 0 \, \forall S \in \Lambda^n, \end{aligned} \tag{3.12}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \text{ for } i \in \{1, ..., p\}, \tag{3.13}$$

$$\nu_j^* H_j(S^*) = 0 \text{ for } j \in \{1, ..., m\}. \tag{3.14}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(.;\lambda^*, u^*)$ $(\forall i = 1, ..., p)$ are $(\rho, \eta)-$pseudo-invex at S^* of higher order and $B_j(., \nu^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$quasi-invex at S^* of order.

(ii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-pseudo-invex at S^* of higher order and $B_j(., \nu^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-quasi-invex at S^* of higher order.

(iii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-quasi-invex at S^* of higher order and $B_j(., \nu^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-pseudo-invex at S^* of higher order.

Then S^* is an efficient solution to (P).

Theorem 3.4. ([11], Theorem 3.1) Let $S^* \in Q$, let F_i, G_i, $i \in \{1, ..., p\}$, and H_j, $j \in \{1, ..., m\}$, be differentiable at $S^* \in Q$, and let there exist $u^* \in U = \{u \in \mathfrak{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $\nu^* \in R_+^m$ such that

$$\begin{aligned} &\langle \Sigma_{i=1}^p u_i^* [\bigtriangledown F_i(S^*) - \lambda_i^* \bigtriangledown G_i(S^*)] + \Sigma_{j=1}^m \nu_j^* \bigtriangledown H_j(S^*), \eta(S, S^*)\rangle \\ + \quad & \rho \|S - S^*\|^2 \geq 0 \, \forall S \in \Lambda^n, \end{aligned} \tag{3.15}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0 \text{ for } i \in \{1, ..., p\}, \tag{3.16}$$

$$\nu_j^* H_j(S^*) = 0 \text{ for } j \in \{1, ..., m\}. \tag{3.17}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $A_i(.;\lambda^*, u^*)$ $(\forall i = 1, ..., p)$ are pseudo-invex at S^* and $B_j(., \nu*)$ $(\forall j \in \{1, ..., m\}$ are quasi-invex at S^*.

(ii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are prestrictly-pseudo-invex at S^* and $B_j(., \nu^*)$ $(\forall j \in \{1, ..., m\}$ are strictly-quasi-invex at S^*.

(iii) $A_i(.;\lambda^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are prestrictly-quasi-invex at S^* and $B_j(., \nu^*)$ $(\forall j \in \{1, ..., m\}$ are strictly-pseudo-invex at S^*.

Then S^* is an efficient solution to (P).

4 Semi-Parametric Sufficient $\epsilon-$ Optimality Conditions

This section deals with some semi-parametric sufficient $\epsilon-$ optimality conditions for problem (P) under the generalized frameworks for generalized invexity. We start with real-valued functions $E_i(., S^*, u^*)$, $B_j(., \nu)$, and $H_i(., S^*, u^*, \nu^*)$ defined by

$$E_i(S, S^*, u^*) = u_i[F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S)] \text{ for } i \in \{1, ..., p\},$$

$$L_i(S, S^*, u^*, \nu^*) = u_i^*[F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S)] + \Sigma_{j \in J_0} \nu_j^* H_j(S) \text{ for } i \in \{1, ..., p\},$$

and

$$B_j(., \nu) = \nu_j H_j(S),\ j = 1, ..., m.$$

Theorem 4.1. Let $S^* \in Q = \{S \in X : H_j(S) \leq 0 \text{ for } j \in \{1, ..., m\}$, the feasible set of (P). Let F_i, G_i, $i \in \{1, ..., p\}$, and H_j, $j \in \{1, ..., m\}$, be differentiable at $S^* \in Q$, and let there exist

$u^* \in U = \{u \in \mathfrak{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in R_+^m$ such that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) \nabla G_i(S^*)] + \Sigma_{j\in J_0} v_j^* \nabla H_j(S^*), \eta(S, S^*)\rangle \geq 0\,, \tag{4.1}$$

and

$$v_j^* H_j(S^*) = 0 \text{ for } j \in \{1, ..., m\}. \tag{4.2}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $E_i(.; S^*, u^*)$ $(\forall i = 1, ..., p)$ are $(\rho, \eta)-$pseudo-invex at S^* of higher order and $B_j(., v^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$quasi-invex at S^* of higher order.

(ii) $E_i(.; S^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-pseudo-invex at S^* of higher order and $B_j(., v^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-quasi-invex at S^* of higher order.

(iii) $E_i(.; S^*, u^*)$ $(\forall i \in \{1, ..., p\}$ are $(\rho, \eta)-$prestrictly-quasi-invex at S^* of higher order and $B_j(., v^*)$ $(\forall j \in \{1, ..., m\}$ are $(\rho, \eta)-$strictly-pseudo-invex at S^* of higher order.

Then S^* is an $\epsilon-$efficient solution to (P).

Proof. If (i) holds, and if $S \in Q$, then it follows from (4.1) that

$$\begin{aligned} &\langle \Sigma_{i=1}^p u_i^* [\nabla F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) \nabla G_i(S^*)], \eta(S, S^*)\rangle \\ + \quad &\langle \Sigma_{j=1}^m v_j^* \nabla H_j(S^*), \eta(S, S^*)\rangle \geq 0\, \forall S \in \Lambda^n. \end{aligned} \tag{4.3}$$

Since $v^* \geq 0$, $S \in Q$ and (4.2) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(S) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(S^*),$$

and in light of the $(\rho, \eta)-$quasi-invexity of $B_j(., v^*)$ at S^*, we arrive at

$$\langle \Sigma_{j=1}^m v_j^* \nabla H_j(S^*), \eta(S, S^*)\rangle \leq -\rho \|S - S^*\|^m. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) \nabla G_i(S^*)], \eta(S, S^*)\rangle \geq \rho \|S - S^*\|^m. \tag{4.5}$$

Next, applying the $(\rho, \eta)-$pseudo-invexity at S^* to (4.5), we have

$$\Sigma_{i=1}^p u_i^* [F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) G_i(S)] \geq \Sigma_{i=1}^p u_i^* [F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) G_i(S^*)],$$

that is equivalent to

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^* [F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S)] \geq \Sigma_{i=1}^{p} u_i^* [F_i(S^*) - (\frac{F_i(S^*)}{G_i(S^*)} \\ - & \epsilon_i(S^*))G_i(S^*)] \\ - & \Sigma_{i=1}^{p} u_i^* \epsilon_i G_i(S^*) \\ = & 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^* [F_i(S) - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*))G_i(S)] \geq 0. \tag{4.6}$$

Since $u_i^* > 0$ for each $i \in \{1, ..., p\}$, we conclude that there does not exist an $S \in Q$ such that

$$\frac{F_1(S)}{G_1(S)} - (\frac{F_i(S^*)}{G_i(S^*)} - \epsilon_i(S^*)) \leq 0 \, \forall \, i = 1, ..., p,$$

$$\frac{F_j(S)}{G_j(S)} - (\frac{F_j(S^*)}{G_j(S^*)} - \epsilon_j(S^*)) < 0 \, \forall \, j \in \{1, ..., p\}.$$

Hence, S^* is an $\epsilon-$efficient solution to (P). Similar proofs hold for (ii) and (iii).

Received: August 2011. Revised: August 2011.

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