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Weak and entropy solutions for a class of nonlinear inhomogeneous Neumann boundary value problem with variable exponent

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ABSTRACT

We study the existence and uniqueness of weak and entropy solutions for the nonlinear inhomogeneous Neumann boundary value problem involving the $p(x)$-Laplace of the form $-$ div $a(x, \nabla u) + |u|^{p(x)-2}\ u = f$ in Ω, $a(x, \nabla u).\eta = \varphi$ on $\partial\Omega$, where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N \geq 3$, $p \in C(\overline{\Omega})$ and $p(x) > 1$ for $x \in \overline{\Omega}$. We prove the existence and uniqueness of a weak solution for data $\varphi \in L^{(p_-)'}(\partial\Omega)$ and $f \in L^{(p_-)'}(\Omega)$, the existence and uniqueness of an entropy solution for L^1-data f and φ independent of u and the existence of weak solutions for f dependent on u and $\varphi \in L^{(p_-)'}(\Omega)$.

RESUMEN

Estudiamos la existencia y unicidad de soluciones y entropía débil para el problema no lineal inhomogéneos de Neumann con valores de frontera que involucra el $p(x)$- Laplace de la forma $-$ div $a(x, \nabla u) + |u|^{p(x)-2}\ u = f$ en Omega, $a(x, \nabla u).\eta = \varphi$ sobre $\partial\Omega$, donde Omega es en un dominio abierto suave y acotado en $\mathbb{R}^N$, $N \geq 3$, $p \in C(\overline{\Omega})$ y $p(x) > 1$ para $x \in \overline{\Omega}$. Probamos la existencia y unicidad de una solución débil para $\varphi \in L^{(p_-)'}(\partial\Omega)$ and $f \in L^{(p_-)'}(\Omega)$, la existencia y unicidad de una solución de entropía para L^1-data f y φ independiente de u y la existencia de soluciones débiles para f dependiente sobre u y $\varphi \in L^{(p_-)'}(\Omega)$.

Keywords and Phrases: Generalized Lebesgue and Sobolev spaces; Weak solution; Entropy solution; $p(x)$-Laplace operator.

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1 Introduction

The purpose of this paper is to study the existence and uniqueness of weak and entropy solutions to the following nonlinear inhomogeneous Neumann problem involving the $p(x)$-Laplace

$$\begin{cases} -\text{div } a(x, \nabla u) + |u|^{p(x)-2}\, u = f \text{ in } \Omega, \\ \\ a(x, \nabla u).\eta = \varphi \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded open domain with smooth boundary and η is the unit outward normal on $\partial\Omega$.
The study of various mathematical problems with variable exponent has recieved considerable attention in recent years (see [4,7,8-15,17,19,24-27,29,30,33,34]). These problems concern applications (see [21,22,31,32,35]) and raise many difficult mathematical problems.
The operator $-\text{div } a(x, \nabla u)$ is called $p(x)$-Laplace, which becomes p-Laplace when $p(x) \equiv p$ (a constant). It possesses more complicated nonlinearities than the p-Laplace. For related results involving the p-Laplace, see [2,3]. In [2], the authors studied the problem

$$\begin{cases} -\text{div } a(x, \nabla u) + \gamma(u) \ni \phi \text{ in } \Omega, \\ \\ a(x, \nabla u).\eta + \beta(u) \ni \psi \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where η is the unit outward normal on $\partial\Omega$, $\psi \in L^1(\partial\Omega)$ and $\phi \in L^1(\Omega)$. The nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$. They proved under a range condition the existence and uniqueness of weak and entropy solutions to the problem (1.2).
Following these ideas, Ouaro and Soma [24] proved the existence and uniqueness of weak and entropy solutions for a class of homogeneous nonlinear Neumann boundary value problem of the form

$$\begin{cases} -\text{div } a(x, \nabla u) + |u|^{p(x)-2}\, u = f \text{ in } \Omega, \\ \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded open domain with smooth boundary and $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$.
In this paper, our aim is to prove the existence and uniqueness of weak and entropy solutions to the nonlinear Neumann boundary value problem (1.1) in order to generalize the results in [24].
The paper is presented as follows. In section 2, we introduce some fundamental preliminary results that we use in this work. The existence and the uniqueness of weak solution for (1.1) is proved in section 3 when the data f and φ belongs to $L^{(p_-)'}$. In section 4, we prove some existence results of weak solution to the problem (1.1) for an f assumed to depend on u and for a boundary datum $\varphi \in L^{(p_-)'}(\partial\Omega)$. Finally, in section 5, we prove the existence and the uniqueness of an entropy solution of (1.1) when the data f and φ belongs to L^1.

2 Assumptions and preliminaries

In this work, we study the problem (1.1) for a variable exponent $p(.)$ which is continuous, more precisely, we assume that

$$\begin{cases} p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that} \\ \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{2.1}$$

where $p_- := \operatorname{ess} \inf_{x \in \Omega} p(x)$.
We denote $p_+ := \operatorname{ess} \sup_{x \in \Omega} p(x)$.
For the vector fields $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory and is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that:

- The following equality holds true

$$A(x,0) = 0, \tag{2.2}$$

for almost every $x \in \Omega$.

- There exists a positive constant C_1 such that

$$|a(x,\xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}) \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where j is a nonnegative function in $L^{p'(.)}(\Omega)$, with $1/p(x) + 1/p'(x) = 1$.

- There exists a positive constant C_2 such that for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$,

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0. \tag{2.4}$$

- The following inequalities hold true

$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi) \tag{2.5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.
Remark 2.1. *Since for almost every $x \in \Omega$, $a(x,.)$ is a gradient and is monotone then the primitive $A(x,.)$ of $a(x,.)$ is necessarily convex.*
As the exponent $p(.)$ appearing in (2.3) and (2.5) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) |u|_{p(.)}\, |v|_{p'(.)}\,, \tag{2.6}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}\,.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(.)}\right)$ is a separable and reflexive Banach space; more details can be found in [17].

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (cf. [15]):

Lemma 2.2. *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*

(i) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_+}$;

(ii) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_-}$;

(iii) $|u|_{p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);

(iv) $|u_n|_{p(.)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho_{p(.)}(u_n) \to 0$ (*respectively* $\to +\infty$);

(v) $\rho_{p(.)}\left(u/\,|u|_{p(.)}\right) = 1.$

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the following notation:

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

We have the following lemma (cf. [33]):

Lemma 2.3. *If $u \in W^{1,p(\cdot)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{1,p(.)} < 1$(*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{1,p(.)}(u) < 1$(*respectively* $= 1; > 1$);

(ii) $\|u\|_{1,p(.)} < 1 \Leftrightarrow \|u\|_{1,p(.)}^{p_+} \leq \rho_{1,p(.)}(u) \leq \|u\|_{1,p(.)}^{p_-}$;

(iii) $\|u\|_{1,p(.)} > 1 \Leftrightarrow \|u\|_{1,p(.)}^{p_-} \leq \rho_{1,p(.)}(u) \leq \|u\|_{1,p(.)}^{p_+}$.

Put

$$p^{\partial}(x) := (p(x))^{\partial} := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x) < N \\ \\ \infty, \text{ if } p(x) \geq N. \end{cases}$$

We have the following useful result (cf. [13,34]).
Proposition 2.4. *Let* $p \in C(\overline{\Omega})$ *and* $p_- > 1$. *If* $q \in C(\partial\Omega)$ *satisfies the condition*

$$1 \leq q(x) < p^{\partial}(x), \ \forall x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. *In particular, there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.
Let us introduce the following notation: given two bounded measurable functions $p(.), q(.) : \Omega \to \mathbb{R}$, we write

$$q(.) \ll p(.) \text{ if } ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

3 Weak solution

In this section, we study the existence and uniqueness of a weak solution of (1.1) where the data $\varphi \in L^{(p_-)'}(\partial\Omega)$ and $f \in L^{(p_-)'}(\Omega)$. The definition of weak solution is the following:
Definition 3.1. *A weak solution of (1.1) is a measurable function* $u : \Omega \longrightarrow \mathbb{R}$ *such that*

$$u \in W^{1,p(.)}(\Omega),$$

and

$$\int_{\Omega} a(x, \nabla u).\nabla v dx + \int_{\Omega} |u|^{p(x)-2}\, uv dx - \int_{\partial\Omega} \varphi v d\sigma = \int_{\Omega} f v dx, \quad \forall\, v \in W^{1,p(.)}(\Omega), \tag{3.1}$$

where $d\sigma$ *is the surface measure on* $\partial\Omega$.
Let E denote the generalized Sobolev space $W^{1,p(.)}(\Omega)$.
If we denote the functional $J : E \to \mathbb{R}$ by

$$J(u) = \int_{\Omega} A(x, \nabla u)dx + \int_{\Omega} \frac{1}{p(x)}|u|^{p(x)}dx - \int_{\partial\Omega} \varphi u d\sigma - \int_{\Omega} f u dx,$$

then

$$\langle J'(u), v\rangle = \int_{\Omega} a(x, \nabla u).\nabla v dx + \int_{\Omega} |u|^{p(x)-2}\, uv dx - \int_{\partial\Omega} \varphi v d\sigma - \int_{\Omega} f v dx, \ \text{ for all } u, v \in E.$$

Therefore, the weak solution of (1.1) corresponds to the critical point of the functional J.
The main result of this section is the following:
Theorem 3.2. *Assume that (2.1)-(2.5) hold. Then there exists a unique weak solution of (1.1).*
Proof. * Existence. With the techniques that became standard by now, it is not difficult to

verify that J is well-defined on E, is of class $C^1(E,\mathbb{R})$ and is weakly lower semi-continuous (see for example [6,19,24,25,26,28]). To end the proof of the existence part, we just have to prove that J is bounded from below and coercive.
Using (2.5) and since E is continuously embedded in $L^{p_-}(\Omega)$, we have

$$\begin{aligned} J(u) &= \int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_{\partial\Omega} \varphi u d\sigma - \int_\Omega fu dx \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \|\varphi\|_{(p_-)',\partial\Omega}\|u\|_{p_-,\partial\Omega} - \|f\|_{(p_-)',\Omega}\|u\|_{p_-,\Omega} \\ &\geq \frac{1}{p^+}\rho_{1,p(.)}(u) - c\,\|\varphi\|_{(p_-)',\partial\Omega}\|u\|_{1,p(.)} - C\|u\|_{1,p(.)}, \end{aligned}$$

where $\|u\|_{p_-,\Omega} = \left(\int_\Omega |u|^{p_-}dx\right)^{\frac{1}{p_-}}$ and $\|u\|_{p_-,\partial\Omega} = \left(\int_{\partial\Omega} |u|^{p_-}d\sigma\right)^{\frac{1}{p_-}}$.
As $\varphi \in L^{(p_-)'}(\partial\Omega)$, then $\|\varphi\|_{(p_-)',\partial\Omega} < +\infty$. Also, for the coercivity of J, we will work with u such that $\|u\|_{1,p(.)} > 1$. Then, by Lemma 2.3 we obtain that

$$J(u) \geq \frac{1}{p^+}\|u\|_{1,p(.)}^{p_-} - C_3\|u\|_{1,p(.)}.$$

As $p_- > 1$, then J is coercive.
If $\|u\|_{1,p(.)} < 1$, we have that

$$\begin{aligned} J(u) &\geq \frac{1}{p^+}\|u\|_{1,p(.)}^{p_+} - C_3\|u\|_{1,p(.)} \\ &\geq -C_3 > -\infty. \end{aligned}$$

Therefore, J is bounded from below.
Since the functional J is proper, lower semi-continuous and coercive, then it has a minimizer which is a weak solution of (1.1).

$*$ **Uniqueness**. Let u_1 and u_2 be two weak solutions of (1.1).
With u_1 as weak solution, we take $v = u_1 - u_2$ in (3.1) to get

$$\int_\Omega a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_\Omega |u_1|^{p(x)-2}u_1(u_1-u_2)dx - \int_{\partial\Omega}\varphi(u_1-u_2)d\sigma = \int_\Omega f(x)(u_1-u_2)dx. \tag{3.2}$$

Similarly, with u_2 as weak solution, we take $\varphi = u_2 - u_1$ to obtain

$$\int_\Omega a(x,\nabla u_2).\nabla(u_2-u_1)dx + \int_\Omega |u_2|^{p(x)-2}u_2(u_2-u_1)dx - \int_{\partial\Omega}\varphi(u_2-u_1)d\sigma = \int_\Omega f(x)(u_2-u_1)dx. \tag{3.3}$$

After adding (3.2) and (3.3), we obtain

$$\int_\Omega (a(x,\nabla u_1) - a(x,\nabla u_2)).(\nabla u_1 - \nabla u_2) + \int_\Omega \left(|u_1|^{p(x)}u_1 - |u_2|^{p(x)}u_2\right)(u_1-u_2)dx = 0. \tag{3.4}$$

Using (2.4), we deduce from (3.4) that

$$\int_\Omega \left(|u_1(x)|^{p(x)}\ u_1(x) - |u_2(x)|^{p(x)}\ u_2(x)\right)(u_1(x) - u_2(x))dx = 0. \tag{3.5}$$

Since $p_- > 1$, the following relation is true for any $\xi, \eta \in \mathbb{R}$, $\xi \neq \eta$ (cf. [14])

$$\left(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta\right)(\xi - \eta) > 0. \tag{3.6}$$

Therefore, from (3.5), we get

$$\left(|u_1(x)|^{p(x)}\ u_1(x) - |u_2(x)|^{p(x)}\ u_2(x)\right)(u_1(x) - u_2(x)) = 0,\ \text{a.e. } x \in \Omega. \tag{3.7}$$

Now, we use (3.6) to get

$$u_1(x) = u_2(x)\ \text{a.e. } x \in \Omega. \tag{3.8}$$

and uniqueness is true □

4 Weak solutions for a right-hand side dependent on u

In this section, we show the existence result of weak solution to some general problem. More precisely, we prove that there exists at least one weak solution to the problem

$$\begin{cases} -\operatorname{div}\ a(x, \nabla u) + |u|^{p(x)-2}\ u = f(x,u) & \text{in } \Omega, \\ \\ a(x,\nabla u).\eta = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\varphi \in L^{(p_-)'}(\partial\Omega)$.

We study (4.1) under the assumptions (2.1)-(2.5) and the following additional assumptions on f.
$f(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory and there exists two positive constants C_4, C_5 such that

$$|f(x,t)| \leq C_4 + C_5|t|^{\beta(x)-1}, \tag{4.2}$$

for every $t \in \mathbb{R}$ and for almost every $x \in \Omega$ with $0 \leq \beta(.) \ll p(.)$.
Let

$$F(x,t) = \int_0^t f(x,s)ds.$$

As mentioned before, we look for distributional solution of (4.1) in the following sense:
Definition 4.1. *A weak solution of (4.1) is a measurable function* $u : \Omega \longrightarrow \mathbb{R}$ *such that* $u \in W^{1,p(.)}(\Omega)$ *and for all* $v \in W^{1,p(.)}(\Omega)$

$$\int_\Omega a(x,\nabla u).\nabla v dx + \int_\Omega |u|^{p(x)-2}\ uv dx - \int_{\partial\Omega} \varphi v d\sigma = \int_\Omega f(x,u)v dx. \tag{4.3}$$

We have the following existence result:

Theorem 4.2. *Assume that (2.1)-(2.5) and (4.2) hold. Then, the problem (4.1) admits at least one weak solution.*

Proof. Let $g(u) = \int_\Omega F(x,u)dx$, for all $u \in E$. The functional g is of class $C^1(E,\mathbb{R})$ with the derivative given by $\left\langle g'(u), v\right\rangle = \int_\Omega f(x,u)v dx, \quad \forall u, v \in E$.

Consequently,

$$J(u) = \int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_{\partial\Omega}\varphi u d\sigma - \int_\Omega F(x,u)dx, \ u \in E$$

is such that J is of class $C^1(E,\mathbb{R})$ and is lower semi-continuous.

We then have to prove that J is bounded from below and coercive in order to complete the proof.

From (4.2), we have $|F(x,t)| \le C\left(1 + |t|^{\beta(x)}\right)$ and then

$$J(u) \ge \frac{1}{p_+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p_+}\int_\Omega |u|^{p(x)}dx - \int_{\partial\Omega}\varphi u d\sigma - C\int_\Omega |u|^{\beta(x)}dx - C meas(\Omega).$$

Let $M > 1$ be a fixed real number (to be chosen later) and $\epsilon := ess \inf_{x\in\Omega}(p(x) - \beta(x))$. We have

$$\begin{aligned}
J(u) \quad & \ge \frac{1}{2p_+}\rho_{1,p(.)}(u) + \int_{\{|u|\le M\}}\left(\frac{1}{2p_+}|u|^{p(x)} - C|u|^{\beta(x)}\right)dx + \\
& \int_{\{|u|>M\}}\left(\frac{1}{2p_+}|u|^{p(x)} - C|u|^{\beta(x)}\right)dx - C meas(\Omega) - C''\|u\|_{1,p(.)} \\
& \ge \frac{1}{2p_+}\rho_{1,p(.)}(u) + \int_{\{|u|>M\}}\left(\frac{1}{2p_+}|u|^{p(x)} - C|u|^{\beta(x)}\right)dx - C''\|u\|_{1,p(.)} - (M^{\beta_+} + 1)C meas(\Omega) \\
& \ge \frac{1}{2p_+}\rho_{1,p(.)}(u) + \int_{\{|u|>M\}}|u|^{\beta(x)}\left(\frac{1}{2p_+}|u|^{p(x)-\beta(x)} - C)\right)dx - C''\|u\|_{1,p(.)} - (M^{\beta_+} + 1)C meas(\Omega) \\
& \ge \frac{1}{2p_+}\rho_{1,p(.)}(u) + \left(\frac{1}{2p_+}M^\epsilon - C\right)\int_{\{|u|>M\}}|u|^{\beta(x)}dx - C''\|u\|_{1,p(.)} - (M^{\beta_+} + 1)C meas(\Omega) \\
& \ge \frac{1}{p_+}\|u\|_{1,p(.)}^{p_-} - C''\|u\|_{1,p(.)} - (M^{\beta_+} + 1)C meas(\Omega),
\end{aligned}$$

For all $M > \max((2p_+C)^{\frac{1}{\epsilon}}, 1)$ and all $u \in E$ with $\|u\|_{1,p(.)} > 1$.

Since $1 < p_-$ it follows that $J(u) \longrightarrow +\infty$ as $\|u\|_E \longrightarrow +\infty$.

Consequently, J is bounded from below and coercive. The proof is then complete.

Assume now that $F^+(x,t) = \int_0^t f^+(x,s)ds$ is such that there exists $C_6 > 0, \ C_7 > 0$ such that

$$|f^+(x,t)| \le C_6 + C_7|t|^{\beta(x)-1}, \tag{4.4}$$

where $0 \le \beta(.) \ll p(.)$.

Then we have the following result:

Theorem 4.3 *Under assumptions (2.1)-(2.5) and (4.4), the problem (4.1) admits at least one weak solution.*

Proof. As $f = f^+ - f^-$, let $F^-(x,t) = \int_0^t f^-(x,s)ds$.

Then

$$\begin{aligned} J(u) &= \int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega F^-(x,u)dx - \int_\Omega F^+(x,u)dx - \int_{\partial\Omega} \varphi u d\sigma \\ &\geq \int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega F^+(x,u)dx - \int_{\partial\Omega} \varphi v d\sigma. \end{aligned}$$

Therefore, similarly as in the proof of Theorem 4.2, the result of Theorem 4.3 follows immediately.

5 Entropy solutions

In this section, we study the existence of entropy solution for the problem (1.1) when the data $f \in L^1(\Omega)$ and $\varphi \in L^1(\partial\Omega)$.

We first recall some notations.

For any $k > 0$, we define the truncation function T_k by $T_k(s) := \max\{-k, \min\{k,s\}\}$.

Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1 and $1 \leq p(.) < +\infty$. It is well known(see [20] or [23]) that if $u \in W^{1,p(\cdot)}(\Omega)$, it is possible to define the trace of u on $\partial\Omega$. More precisely, there is a bounded operator τ from $W^{1,p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\partial\Omega)$ such that $\tau(u) = u|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega})$.

Set

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable such that } T_k(u) \in W^{1,p(\cdot)}(\Omega), \text{ for any } k > 0 \Big\}.$$

In [1], the authors have proved the following

Proposition 5.1 *Let* $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. *Then there exists a unique measurable function* $v : \Omega \longrightarrow \mathbb{R}^N$ *such that* $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, *for all* $k > 0$. *The function* v *is denoted by* ∇u. *Moreover if* $u \in W^{1,p(\cdot)}(\Omega)$ *then* $v \in \left(L^{p(\cdot)}(\Omega)\right)^N$ *and* $v = \nabla u$ *in the usual sense.*

It is easy to see that, in general, it is not possible to define the trace of an element of $\mathcal{T}^{1,p(\cdot)}(\Omega)$. In demension one it is enough to consider the function $u(x) = \frac{1}{x}$ for $x \in]0,1[$. Therefore, we are going to define following [2,3], the trace for the elements of a subset $\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ of $\mathcal{T}^{1,p(\cdot)}(\Omega)$.

$\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ will be the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:

(C_1) $u_n \to u$ a.e in Ω.

(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$.

(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [2,3]. In the sequel the trace of $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(\cdot)}(\Omega), tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u-\varphi) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $tr(u-\varphi) = tr(u) - tr(\varphi)$.

We can now introduce the notion of entropy solution of (1.1).

Definition 5.2. *A measurable function* u *is an entropy solution to problem (1.1) if* $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $|u|^{p(x)-2}\, u \in L^1(\Omega)$ *and for every* $k > 0$,

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-v)dx + \int_\Omega |u|^{p(x)-2}\, uT_k(u-v)dx \leq \int_{\partial\Omega} \varphi T_k(u-v)d\sigma + \int_\Omega f(x)T_k(u-v)dx \tag{5.1}$$

for all $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Our main result in this section is the following:

Theorem 5.3. *Assume (2.1)-(2.5),* $f \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Then, there exists a unique entropy solution* u *to problem (1.1).*

The following propositions are useful for the proof of Theorem 5.3.

Proposition 5.4. *Assume (2.1)-(2.5),* $f \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Let* u *be an entropy solution of (1.1). If there exists a positive constant* M *such that*

$$\int_{\{|u|>k\}} k^{q(x)}dx \leq M \tag{5.2}$$

then

$$\int_{\{|\nabla u|^{\alpha(.)}>k\}} k^{q(x)}dx \leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)} + M, \quad \textit{for all } k > 0,$$

where $\alpha(.) = p(.)/(q(.)+1)$.

Proof. Taking $v = 0$ in the entropy inequality (5.1) and using (2.5), we get

$$\int_\Omega |\nabla T_k(u)|^{p(x)}dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) \quad \text{for all } k > 0.$$

Therefore, defining $\psi := \dfrac{1}{k}T_k(u)$, we have for all $k > 0$,

$$\int_\Omega k^{p(x)-1}|\nabla\psi|^{p(x)}dx = \frac{1}{k}\int_\Omega |\nabla T_k(u)|^{p(x)}dx \leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}.$$

From the above inequality, from the definition of $\alpha(.)$ and (5.2), we get

$$\begin{aligned}
\int_{\{|\nabla u|^{\alpha(.)}>k\}} k^{q(x)}dx &\leq \int_{\{|\nabla u|^{\alpha(.)}>k\}\cap\{|u|\leq k\}} k^{q(x)}dx + \int_{\{|u|>k\}} k^{q(x)}dx \\
&\leq \int_{\{|u|\leq k\}} k^{q(x)}\left(\frac{|\nabla u|^{\alpha(x)}}{k}\right)^{\frac{p(x)}{\alpha(x)}} dx + M \\
&\leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)} + M, \quad \text{for all } k > 0.
\end{aligned}$$

Proposition 5.5. *Assume (2.1)-(2.5),* $f \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Let* u *be an entropy solution of (1.1), then*

$$\int_\Omega |\nabla T_k(u)|^{p(x)}dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) \textit{ for all } k > 0 \tag{5.3}$$

and

$$\left\| |u|^{p(x)-2}\ u \right\|_1 = \left\| |u|^{p(x)-1} \right\|_1 \leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}. \tag{5.4}$$

Proof. The inequality (5.3) is already obtained in the proof of Proposition 5.2. Let's prove (5.4). Taking $\varphi = 0$ in (5.1), we get for all $k > 0$,

$$\int_\Omega |u|^{p(x)-2}\ uT_k(u)dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right),$$

then

$$\int_{\{|u|>k\}} |u|^{p(x)-2}\ uT_k(u)dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right).$$

From the inequality above, we obtain

$$k\int_{\{u>k\}} |u|^{p(x)-2}\ udx - k\int_{\{u<-k\}} |u|^{p(x)-2}\ udx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right),$$

which imply

$$\int_{\{u>k\}} |u|^{p(x)-2}\ udx - \int_{\{u<-k\}} |u|^{p(x)-2}\ udx \leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}.$$

The last inequality means

$$\int_{\{|u|>k\}} |u|^{p(x)-1}dx \leq \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)} \text{ for all } k > 0. \tag{5.5}$$

We use Fatou's Lemma in (5.5) by letting k goes to 0 to obtain (5.4).

Proposition 5.6. *Assume that (2.1)-(2.5) hold,* $f \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Let* u *be an entropy solution of (1.1), then*

$$\int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p^-}dx \leq C(k+1) \textit{ for all } k > 0. \tag{5.6}$$

Proof. Note that

$$\begin{aligned}
\int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p^-}dx &= \int_{\{|u|\leq k, |\nabla u|>1\}} |\nabla T_k(u)|^{p^-}dx + \int_{\{|u|\leq k, |\nabla u|\leq 1\}} |\nabla T_k(u)|^{p^-}dx \\
&\leq \int_{\{|u|\leq k, |\nabla u|>1\}} |\nabla T_k(u)|^{p^-}dx + \text{meas}(\Omega) \\
&\leq \int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p(x)}dx + \text{meas}(\Omega).
\end{aligned}$$

Since $\displaystyle\int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p(x)}dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right)$, we obtain

$$\int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p^-}dx \leq k\left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) + \text{meas}(\Omega) \text{ for all } k > 0.$$

Proposition 5.7. *Assume that (2.1)-(2.5) hold,* $f \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Let* u *be an entropy solution of (1.1). Then*

$$\textit{meas}\{|u| > h\} \leq \frac{\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}}{h^{p_- - 1}} \textit{ for all } h \geq 1, \tag{5.7}$$

and

$$\textit{meas}\{|\nabla u| > h\} \leq \frac{\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}}{h^{p_- - 1}} \textit{ for all } h \geq 1. \tag{5.8}$$

Proof.

$$\begin{aligned} \int_\Omega |u|^{p(x)-1} dx &= \int_{\{|u|\leq h\}} |u|^{p(x)-1} dx + \int_{\{|u|>h\}} |u|^{p(x)-1} dx \\ &\geq \int_{\{|u|>h\}} |u|^{p(x)-1} dx \\ &\geq \int_{\{|u|>h\}} h^{p(x)-1} dx \\ &\geq h^{p_- - 1} \text{meas}\{|u| > h\} \text{ since } h \geq 1. \end{aligned}$$

Then, by (5.4) we deduce (5.7).
We next prove (5.8).
For $k, \lambda \geq 0$, set

$$\Phi(k, \lambda) = \text{meas}\{|\nabla u|^{p_-} > \lambda, |u| > k\}.$$

We have

$$\Phi(k, 0) \leq \text{meas}\{|u| > k\}.$$

For $k \geq 1$, we obtain by (5.7)

$$\Phi(k, 0) \leq \left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) k^{1-p_-}.$$

Using the fact that the function $\lambda \longmapsto \Phi(k, \lambda)$ is nonincreasing, we get for $k > 0$ and $\lambda > 0$, that

$$\begin{aligned} \Phi(0, \lambda) &= \frac{1}{\lambda} \int_0^\lambda \Phi(0, \lambda) ds \leq \frac{1}{\lambda} \int_0^\lambda \Phi(0, s) ds \\ &\leq \frac{1}{\lambda} \int_0^\lambda \Big[\Phi(0, s) + (\Phi(k, 0) - \Phi(k, s))\Big] ds \\ &\leq \Phi(k, 0) + \frac{1}{\lambda} \int_0^\lambda (\Phi(0, s) - \Phi(k, s)) ds. \end{aligned}$$

Now, let us observe that

$$\Phi(0, s) - \Phi(k, s) = \text{meas}\{|u| \leq k, |\nabla u|^{p_-} > s\}.$$

Then, thanks to (5.6), we get

$$\int_0^{+\infty} (\Phi(0, s) - \Phi(k, s)) ds = \int_{\{|u|\leq k\}} |\nabla u|^{p_-} dx \leq C(k+1),$$

where $C = \max\left(\text{meas}(\Omega), \|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right)$.
It follows that

$$\Phi(0,\lambda) \leq \frac{C(k+1)}{\lambda} + \left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) k^{1-p_-}, \text{ for all } k \geq 1, \lambda > 0.$$

In particular, we have

$$\Phi(0,\lambda) \leq \frac{C(k+1)}{\lambda} + \left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) k^{1-p_-}, \text{ for all } k \geq 1, \lambda \geq 1.$$

We now set

$$f_\lambda(k) = \frac{C(k+1)}{\lambda} + \left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) k^{1-p_-}, \text{ for all } k \geq 1,$$

where $\lambda \geq 1$ is a fixed real number.
The minimization of f_λ in k gives

$$\Phi(0,\lambda) \leq \left(\|f\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right) \lambda^{-(1/(p_-)')}, \tag{5.9}$$

for all $\lambda \geq 1$.
Setting $\lambda = h^{p_-}$ in (5.9) gives (5.8).
Proof of Theorem 5.3. $*$ **Uniqueness of entropy solution.** Let $h > 0$ and u_1, u_2 be two entropy solutions of (1.1). We write the entropy inequality (5.1) corresponding to the solution u_1, with $T_h(u_2)$ as a test function, and to the solution u_2, with $T_h(u_1)$ as a test function. Upon addition, we get

$$\begin{cases} \displaystyle\int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx + \int_{\{|u_2-T_h(u_1)|\leq k\}} a(x,\nabla u_2).\nabla(u_2 - T_h(u_1))dx \\ \displaystyle + \int_\Omega |u_1|^{p(x)-2}\, u_1 T_k(u_1 - T_h(u_2))dx + \int_\Omega |u_2|^{p(x)-2}\, u_2 T_k(u_2 - T_h(u_1))dx \leq \\ \displaystyle\int_{\partial\Omega} \varphi\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)d\sigma + \int_\Omega f\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)dx. \end{cases} \tag{5.10}$$

Define now

$$E_1 := \{|u_1 - u_2| \leq k, |u_2| \leq h\}, \quad E_2 := E_1 \cap \{|u_1| \leq h\}, \quad \text{and } E_3 := E_1 \cap \{|u_1| > h\}.$$

We start with the first integral in (5.10). By (2.5), we have

$$\left\{\begin{array}{l}
\displaystyle\int_{\{|u_1-T_h(u_2)|\le k\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\
\displaystyle= \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\
\qquad\qquad\displaystyle+\int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|>h\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\
\displaystyle= \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{\{|u_1-h\mathrm{sign}(u_2)|\le k\}\cap\{|u_2|>h\}} a(x,\nabla u_1).\nabla u_1\,dx \\
\displaystyle\ge \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1-u_2)dx = \int_{E_1} a(x,\nabla u_1).\nabla(u_1-u_2)dx \\
\displaystyle= \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla(u_1-u_2)dx \\
\displaystyle= \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla u_1\,dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2\,dx \\
\displaystyle\ge \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2\,dx.
\end{array}\right. \tag{5.11}$$

Using (2.3) and (2.6), we estimate the last integral in (5.11) as follows:

$$\left\{\begin{array}{l}
\displaystyle\left|\int_{E_3} a(x,\nabla u_1).\nabla u_2 dx\right| \le C_1\int_{E_3}\left(j(x)+|\nabla u_1|^{p(x)-1}\right)|\nabla u_2|dx \\
\displaystyle\le C_1\left(|j|_{p'(.)}+\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right)|\nabla u_2|_{p(.),\{h-k<|u_1|\le h\}},
\end{array}\right. \tag{5.12}$$

where $\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}} = \left\||\nabla u_1|^{p(x)-1}\right\|_{L^{p'(.)}(\{h<|u_1|\le h+k\})}$.
The quantity $C_1\left(|j|_{p'(.)}+\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right)$ can be written as follows

$$C_1\left(|j|_{p'(.)}+\left||\nabla T_{h+k}(u_1)|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right) < +\infty,$$

since $T_{h+k}(u_1)\in W^{1,p(.)}(\Omega)$ and $j\in L^{p'(.)}(\Omega)$.
We deduce by Proposition 5.7 that

$C_1\left(|j|_{p'(.)}+\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right)|\nabla u_2|_{p(.),\{h-k<|u_1|\le h\}}$ converges to 0 as $h\to+\infty$.

Therefore, from (5.11) and (5.12), we obtain

$$\int_{\{|u_1-T_h(u_2)|\le k\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \ge I_h + \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx, \tag{5.13}$$

where I_h converges to zero as $h\to+\infty$.
We may adopt the same procedure to treat the second term in (5.10) to obtain

$$\int_{\{|u_2-T_h(u_1)|\le k\}} a(x,\nabla u_2).\nabla(u_2-T_h(u_1))dx \ge J_h - \int_{E_2} a(x,\nabla u_2).\nabla(u_1-u_2)dx, \tag{5.14}$$

where J_h converges to zero as $h \to +\infty$.
Now set for all $h, k > 0$

$$K_h = \int_\Omega |u_1|^{p(x)-2}\ u_1 T_k(u_1 - T_h(u_2))dx + \int_\Omega |u_2|^{p(x)-2}\ u_2 T_k(u_2 - T_h(u_1))dx.$$

We have

$$|u_1|^{p(x)-2}\ u_1 T_k(u_1 - T_h(u_2)) \longrightarrow |u_1|^{p(x)-2}\ u_1 T_k(u_1 - u_2) \text{ a.e in } \Omega \text{ as } h \to +\infty,$$

and

$$\Big||u_1|^{p(x)-2}\ u_1 T_k(u_1 - T_h(u_2))\Big| \leq k|u_1|^{p(x)-1} \in L^1(\Omega).$$

Then by Lebesgue Theorem, we deduce that

$$\lim_{h\to+\infty} \int_\Omega |u_1|^{p(x)-2}\ u_1 T_k(u_1 - T_h(u_2))dx = \int_\Omega |u_1|^{p(x)-2}\ u_1 T_k(u_1 - u_2)dx. \tag{5.15}$$

Similarly, we have

$$\lim_{h\to+\infty} \int_\Omega |u_2|^{p(x)-2}\ u_2 T_k(u_2 - T_h(u_1))dx = \int_\Omega |u_2|^{p(x)-2}\ u_2 T_k(u_2 - u_1)dx. \tag{5.16}$$

Using (5.15) and (5.16), we get

$$\lim_{h\to+\infty} K_h = \int_\Omega \Big(|u_1|^{p(x)-2}\ u_1 - |u_2|^{p(x)-2}\ u_2\Big) T_k(u_1 - u_2)dx. \tag{5.17}$$

We next examine the right-hand side of (5.10).
For all $k > 0$,

$$f\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big) \longrightarrow f\Big(T_k(u_1 - u_2) + T_k(u_2 - u_1)\Big) = 0 \text{ a.e in } \Omega \text{ as } h \to +\infty,$$

$$\varphi\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big) \longrightarrow \varphi\Big(T_k(u_1 - u_2) + T_k(u_2 - u_1)\Big) = 0 \text{ a.e in } \partial\Omega \text{ as } h \to +\infty,$$

and

$$\Big|f(x)\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)\Big| \leq 2k|f| \in L^1(\Omega),$$

$$\Big|\varphi\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)\Big| \leq 2k|\varphi| \in L^1(\partial\Omega).$$

Lebesgue Theorem allows us to write

$$\lim_{h\to+\infty} \left[\int_{\partial\Omega} \varphi\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)d\sigma + \int_\Omega f\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)dx\right] = 0. \tag{5.18}$$

Using (5.13), (5.14), (5.17) and (5.18), we get

$$\begin{cases} \displaystyle\int_{\{|u_1 - u_2| \leq k\}} \Big(a(x, \nabla u_1) - a(x, \nabla u_2)\Big).\Big(\nabla u_1 - \nabla u_2\Big)dx \\ \displaystyle + \int_\Omega \Big(|u_1|^{p(x)-2}\ u_1 - |u_2|^{p(x)-2}\ u_2\Big) T_k(u_1 - u_2)dx \leq 0. \end{cases} \tag{5.19}$$

Therefore
$$\int_\Omega \left(|u_1|^{p(x)-2}\,u_1-|u_2|^{p(x)-2}\,u_2\right)T_k(u_1-u_2)dx=0. \tag{5.20}$$
For x fixed in Ω, $s\longmapsto |s|^{p(x)-2}\,s$ is nondecreasing and vanishes at 0. Then,
$$\left(|u_1|^{p(x)-2}\,u_1-|u_2|^{p(x)-2}\,u_2\right)T_k(u_1-u_2)\geq 0,\ \forall x\in\Omega \text{ and } \forall k>0.$$
Now, using inequality above and (5.20), for all $k\in\mathbb{R}^+$ there exist $\Omega_k\subset\Omega$ with $meas(\Omega_k)=0$ such that for all $x\in\Omega\backslash\Omega_k$,
$$\left(|u_1(x)|^{p(x)-2}\,u_1(x)-|u_2(x)|^{p(x)-2}\,u_2(x)\right)T_k(u_1(x)-u_2(x))=0.$$
Therefore,
$$\left(|u_1(x)|^{p(x)-2}\,u_1(x)-|u_2(x)|^{p(x)-2}\,u_2(x)\right)(u_1(x)-u_2(x))=0, \text{ for all } x\in\Omega\backslash\bigcup_{k\in\mathbb{N}^*}\Omega_k. \tag{5.21}$$
Now, using (5.21) and (3.6), we get
$$u_1=u_2 \text{ a.e. in } \Omega.$$

$*$ **Existence of entropy solution.** Let $f_n=T_n(f)$ and $\varphi_n=T_n(\varphi)$; then $(f_n)_n$ and $(\varphi_n)_n$ are in $L^{(p_-)'}(\Omega)$ and $L^{(p_-)'}(\partial\Omega)$ respectively and are strongly converging to f in $L^1(\Omega)$ and to φ in $L^1(\partial\Omega)$ respectively. Moreover $\|f_n\|_{L^1(\Omega)}\leq\|f\|_{L^1(\Omega)}$ and $\|\varphi_n\|_{L^1(\partial\Omega)}\leq\|\varphi\|_{L^1(\partial\Omega)}$, for all $n\in\mathbb{N}$.
Next, we consider the problem
$$\begin{cases} -\text{div } a(x,\nabla u_n)+|u_n|^{p(x)-2}\,u_n=f_n \text{ in } \Omega,\\[2ex] a(x,\nabla u_n).\eta=\varphi_n \quad \text{on } \partial\Omega. \end{cases} \tag{5.22}$$
It follows from Theorem 3.2 that there exists a unique $u_n\in W^{1,p(.)}(\Omega)$ such that
$$\int_\Omega a(x,\nabla u_n).\nabla v dx+\int_\Omega |u_n|^{p(x)-2}\,u_n v dx=\int_{\partial\Omega}\varphi_n v d\sigma+\int_\Omega f_n v dx \tag{5.23}$$
for all $v\in W^{1,p(.)}(\Omega)$.
Our aim is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution to the limit problem (1.1). To start with, we prove the following lemma:
Lemma 5.8. *For any* $k>0$, $\|T_k(u_n)\|_{1,p(.)}\leq 1+C$ *where* $C=C(k,\varphi,f,p_-,p_+,meas(\Omega))$ *is a positive constant.*
Proof. By taking $v=T_k(u_n)$ in (5.23), we get
$$\int_\Omega a(x,\nabla u_n).\nabla T_k(u_n)+\int_\Omega |u_n|^{p(x)-2}\,u_n T_k(u_n)dx=\int_{\partial\Omega}\varphi_n T_k(u_n)d\sigma+\int_\Omega f_n T_k(u_n)dx.$$

Since all the terms in the left-hand side of equality above are nonnegative and

$$\int_{\partial\Omega} \varphi_n T_k(u_n) d\sigma + \int_\Omega f_n T_k(u_n) dx \leq k \left(\|\varphi_n\|_{L^1(\partial\Omega)} + \|f_n\|_{L^1(\Omega)} \right) \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right);$$

by using (2.5) we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right) \tag{5.24}$$

and

$$\int_\Omega |u_n|^{p(x)-2}\, u_n T_k(u_n) dx \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right). \tag{5.25}$$

The inequality (5.25) is equivalent to

$$\int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx + \int_{\{|u_n| > k\}} |u_n|^{p(x)-2}\, u_n T_k(u_n) dx \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right).$$

Therefore,

$$\int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right). \tag{5.26}$$

Furthermore

$$\begin{aligned} \int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx &= \int_{\{|u_n| > k\}} k^{p(x)} dx \\ &\leq \begin{cases} k^{p_+} \text{meas}(\Omega) \text{ if } k \geq 1, \\ \\ \text{meas}(\Omega) \text{ if } k < 1. \end{cases} \end{aligned}$$

This allows us to write

$$\int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx \leq (1 + k^{p_+}) \text{meas}(\Omega). \tag{5.27}$$

Relations (5.26) and (5.27) give

$$\int_\Omega |T_k(u_n)|^{p(x)} dx \leq k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right) + (1 + k^{p_+}) \text{meas}(\Omega). \tag{5.28}$$

Hence, adding (5.24) and (5.28), it yields

$$\rho_{1,p(.)}(T_k(u_n)) \leq 2k \left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} \right) + (1 + k^{p_+}) \text{meas}(\Omega) = C(k, \varphi, f, p_+, \text{meas}(\Omega)). \tag{5.29}$$

If $\|T_k(u_n)\|_{1,p(.)} \geq 1$, we have

$$\|T_k(u_n)\|_{1,p(.)}^{p_-} \leq \rho_{1,p(.)}(T_k(u_n)) \leq C(k, \varphi, f, p_+, \text{meas}(\Omega)),$$

which is equivalent to

$$\|T_k(u_n)\|_{1,p(.)} \leq \Big(C(k, \varphi, f, p_+, \text{meas}(\Omega)) \Big)^{\frac{1}{p_-}} = C(k, \varphi, f, p_-, p_+, \text{meas}(\Omega)).$$

The above inequality gives

$$\|T_k(u_n)\|_{1,p(.)} \leq 1 + C(k, \varphi, f, p_-, p_+, \text{meas}(\Omega)).$$

Then, the proof of Lemma 5.8. is complete.
From Lemma 5.8. we deduce that for any $k > 0$, the sequence $(T_k(u_n))$ is uniformly bounded in $W^{1,p(.)}(\Omega)$ and so in $W^{1,p_-}(\Omega)$. Then, up to a subsequence we can assume that for any $k > 0$, $T_k(u_n)$ converges weakly to σ_k in $W^{1,p_-}(\Omega)$, and so $T_k(u_n)$ converges strongly to σ_k in $L^{p_-}(\Omega)$.
We next prove the following proposition:
Proposition 5.9. *Assume that (2.1)-(2.5) hold and $u_n \in W^{1,p(.)}(\Omega)$ is the weak solution of (5.22). Then the sequence $(u_n)_n$ is Cauchy in measure. In particular, there exists a measurable function u and a subsequence still denoted $(u_n)_n$ such that $u_n \longrightarrow u$ in measure.*
Proof. Let $s > 0$ and define

$$E_1 := \{|u_n| > k\}, \quad E_2 := \{|u_m| > k\} \quad \text{and} \quad E_3 := \{|T_k(u_n) - T_k(u_m)| > s\}$$

where $k > 0$ is to be fixed. We note that

$$\{|u_n - u_m| > s\} \subset E_1 \cup E_2 \cup E_3,$$

and hence

$$\text{meas}\{|u_n - u_m| > s\} \leq \text{meas}(E_1) + \text{meas}(E_2) + \text{meas}(E_3). \tag{5.30}$$

Let $\epsilon > 0$. Using Proposition 5.7, we choose $k = k(\epsilon)$ such that

$$\text{meas}(E_1) \leq \epsilon/3 \quad \text{and meas}(E_2) \leq \epsilon/3. \tag{5.31}$$

Since $T_k(u_n)$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$.
Thus

$$\text{meas}(E_3) \leq \frac{1}{s^{p_-}} \int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-} dx \leq \frac{\epsilon}{3}, \tag{5.32}$$

for all $n, m \geq n_0(s, \epsilon)$.
Finally, from (5.30), (5.31) and (5.32), we obtain

$$\text{meas}\{|u_n - u_m| > s\} \leq \epsilon \text{ for all } n, m \geq n_0(s, \epsilon). \tag{5.33}$$

Relations (5.33) mean that the sequence $(u_n)_n$ is Cauchy sequence in measure and the proof of Proposition 5.9. is complete.
Note that as $u_n \longrightarrow u$ in measure, up to a subsequence, we can assume that $u_n \longrightarrow u$ a.e. in Ω.
In the sequel, we need the following two technical lemmas.

Lemma 5.10. **(cf.[30, Lemma 5.4])** *Let $(v_n)_n$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then $v_n \longrightarrow v$ strongly in $L^1(\Omega)$.*

The second technical lemma is a well known result in measure theory (cf. [16]).

Lemma 5.11. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \longrightarrow [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$, such that

$$\mu(A) < \epsilon, \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

We now set to prove that the function u in the Proposition 5.9 is an entropy solution of (1.1). Let $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. For any $k > 0$, choose $T_k(u_n - v)$ as a test function in (5.23). We get

$$\begin{aligned}\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - v)dx \qquad &+ \int_\Omega |u_n|^{p(x)-2}\, u_n T_k(u_n - v)dx = \\ &\int_{\partial\Omega} \varphi_n(x) T_k(u_n - v)d\sigma + \int_\Omega f_n(x) T_k(u_n - v)dx. \quad (5.34)\end{aligned}$$

We have the following proposition:

Proposition 5.12. *Assume that (2.1)-(2.5) hold and $u_n \in W^{1,p(.)}(\Omega)$ be the weak solution of (5.22). Then*

(i) ∇u_n converges in measure to the weak gradient of u;

(ii) For all $k > 0$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in $(L^1(\Omega))^N$.

(iii) For all $t > 0$, $a(x, \nabla T_t(u_n))$ converges to $a(x, \nabla T_t(u))$ in $\left(L^1(\Omega)\right)^N$ strongly and in $\left(L^{p'(.)}(\Omega)\right)^N$ weakly.

(iv) u_n converges to some function v a.e. on $\partial\Omega$.

Proof.

(i) We claim that the sequence $(\nabla u_n)_n$ is Cauchy in measure.

Indeed, let $s > 0$, and consider

$$E_1 := \{|\nabla u_n| > h\} \cup \{|\nabla u_m| > h\}, \quad E_2 := \{|u_n - u_m| > k\}$$

and

$$E_3 := \{|\nabla u_n| \leq h, |\nabla u_m| \leq h, \ |u_n - u_m| \leq k, \ |\nabla u_n - \nabla u_m| > s\},$$

where h and k will be chosen later.

Note that

$$\{|\nabla u_n - \nabla u_m| > s\} \subset E_1 \cup E_2 \cup E_3. \tag{5.35}$$

Let $\epsilon > 0$. By Proposition 5.7 (relation (5.8)), we may choose $h = h(\epsilon)$ large enough such that

$$\text{meas}(E_1) \leq \epsilon/3, \tag{5.36}$$

for all $n, m \geq 0$.
On the other hand, by Proposition 5.9

$$\text{meas}(E_2) \leq \epsilon/3, \tag{5.37}$$

for all $n, m \geq n_0(k, \epsilon)$.
Moreover, since $a(x,\xi)$ is continuous with respect to ξ for a.e every $x \in \Omega$, by assumption (2.5) there exists a real valued function $\gamma : \Omega \longrightarrow [0, +\infty]$ such that $\text{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0$ and

$$(a(x,\xi) - a(x,\xi')).(\xi - \xi') \geq \gamma(x), \tag{5.38}$$

for all $\xi, \xi' \in \mathbb{R}^N$ such that $|\xi| \leq h, \ |\xi'| \leq h, \ |\xi - \xi'| \geq s$, for a.e $x \in \Omega$.
Let $\delta = \delta(\epsilon)$ be given by Lemma 5.11., replacing ϵ and A by $\epsilon/3$ and E_3 respectively.
As u_n is a weak solution of (5.22), using $T_k(u_n - u_m)$ as a test function, we get

$$\begin{aligned}\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - u_m)dx \quad & + \int_\Omega |u_n|^{p(x)-2}\, u_n T_k(u_n - u_m)dx = \\ & \int_{\partial\Omega} \varphi_n T_k(u_n - u_m)d\sigma + \int_\Omega f_n T_k(u_n - u_m)dx \\ & \leq k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right).\end{aligned}$$

Similarly for u_m, we have

$$\begin{aligned}\int_\Omega a(x, \nabla u_m).\nabla T_k(u_m - u_n)dx \quad & + \int_\Omega |u_m|^{p(x)-2}\, u_m T_k(u_m - u_n)dx = \\ & \int_{\partial\Omega} \varphi_m T_k(u_m - u_n)d\sigma + \int_\Omega f_m T_k(u_m - u_n)dx \\ & \leq k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right).\end{aligned}$$

After adding the last two inequalities, it yields

$$\begin{cases} \displaystyle\int_{\{|u_n - u_m| \leq k\}} (a(x, \nabla u_n) - a(x, \nabla u_m)).(\nabla u_n - \nabla u_m)dx \\ \\ \displaystyle + \int_\Omega \Big(|u_n|^{p(x)-2}\, u_n - |u_m|^{p(x)-2}\, u_m\Big) T_k(u_n - u_m)dx \leq 2k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right). \end{cases}$$

Since the second term of the above inequality is nonnegative, we obtain by using (5.38)

$$\int_{E_3} \gamma(x)dx \leq \int_{E_3} (a(x, \nabla u_n) - a(x, \nabla u_m)).(\nabla u_n - \nabla u_m)dx \leq 2k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) < \delta,$$

where $k = \delta/4\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right)$.

From Lemma 5.11, it follows that

$$\text{meas}(E_3) \leq \epsilon/3. \tag{5.39}$$

Thus using (5.35), (5.36), (5.37) and (5.39), we get

$$\operatorname{meas}(\{|\nabla u_n - \nabla u_m| > s\}) \leq \epsilon, \text{ for all } n, m \geq n_0(s, \epsilon) \tag{5.40}$$

and then the claim is proved.
Consequently, $(\nabla u_n)_n$ converges in measure to some measurable function v. In order to end the proof of (i), we need the following lemma:

Lemma 5.13.
(a) For a.e. $t \in \mathbb{R}$, $\nabla T_t(u_n)$ converges in measure to $v\chi_{\{|u|<t\}}$;
(b) for a.e. $t \in \mathbb{R}$, $\nabla T_t(u) = v\chi_{\{|u|<t\}}$;
(c) $\nabla T_t(u) = v\chi_{\{|u|<t\}}$ holds for all $t \in \mathbb{R}$.

Proof of Lemma 5.13.

- Proof of (a).

We know that $\nabla u_n \to v$ in measure. Thus, $\chi_{\{|u|<t\}}\nabla u_n \to \chi_{\{|u|<t\}}v$ in measure.
Now, let us show that $\left(\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right)\nabla u_n \to 0$ in measure. For that, it is sufficient to show that $\left(\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right) \to 0$ in measure. Now, for all $\delta > 0$,

$$\left\{\left|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right| |\nabla u_n| > \delta\right\} \subset \left\{\left|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right| \neq 0\right\}$$
$$\subset \{|u| = t\} \cup \{u_n < t < u\} \cup \{u < t < u_n\} \cup \{u_n < -t < u\} \cup \{u < -t < u_n\}.$$

Thus,

$$\begin{cases} meas\left\{\left|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right| |\nabla u_n| > \delta\right\} \leq meas\{|u| = t\} + meas\{u_n < t < u\} + \\ meas\{u < t < u_n\} + meas\{u_n < -t < u\} + meas\{u < -t < u_n\}. \end{cases} \tag{5.41}$$

Note that
$meas\{|u| = t\} \leq meas\{t - h < u < t + h\} + meas\{-t - h < u < -t + h\} \to 0$ as $h \to 0$
for a.e. t, since u is a fixed function. Next,

$$meas\{u_n < t < u\} \leq meas\{t < u < t + h\} + meas\{|u - u_n| > h\}, \text{ for all } h > 0.$$

Due to Proposition 5.9, we have for all fixed $h > 0$, $meas\{|u - u_n| > h\} \to 0$ as $n \to +\infty$. Since $meas\{t < u < t + h\} \to 0$ as $h \to 0$, for all $\epsilon > 0$, one can find N such that for all $n > N$, $meas\{u_n < t < u\} < \epsilon/2 + \epsilon/2 = \epsilon$ by choosing h and then N. Each of the other terms in the right-hand side of (5.41) can be treated in the same way as for $meas\{u_n < t < u\}$. Thus, $meas\left\{\left|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\right| |\nabla u_n| > \delta\right\} \to 0$ as $n \to +\infty$.
Finally, since $\nabla T_t(u_n) = \nabla u_n \chi_{\{|u_n|<t\}}$, the claim (a) follows.

- Proof of (b).

Let ψ_t be the weak $W^{1,p(\cdot)}$-limit of $T_t(u_n)$, then it is also the strong L^1-limit of $T_t(u_n)$. But, as T_t is a Lipschitz function, the convergence in measure of u_n to u implies the convergence in measure of $T_t(u_n)$ to $T_t(u)$. Thus, by the uniqueness of the limit in measure, ψ_t is identified with $T_t(u)$, we conclude that $\nabla T_t(u_n) \to \nabla T_t(u)$ weakly in $L^{p(\cdot)}(\Omega)$.
The previous convergence also ensures that $\nabla T_t(u_n)$ converges to $\nabla T_t(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\nabla T_t(u_n)$ converges to $v\chi_{\{|u|<t\}}$ in measure. By lemma 5.10, since $\nabla T_t(u_n)$ is uniformly bounded in $L^{p-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of a weak L^1-limit, $v\chi_{\{|u|<t\}}$ coincides with $\nabla T_t(u)$.

- Proof of (c)

Let $0 < t < s$, and s be such that $v\chi_{\{|u|<s\}}$ coincides with $\nabla T_s(u)$. Then

$$\nabla T_t(u) = \nabla T_t(T_s(u)) = \nabla T_s(u)\chi_{\{|T_s(u)|<t\}} = v\chi_{\{|u|<s\}}\chi_{\{|u|<t\}} = v\chi_{\{|u|<t\}}.$$

Now, we can end the proof of (i). Indeed, combining Lemma 5.13-(c) and Proposition 5.1, (i) follows.
(ii) Let $s > 0$, $k > 0$ and consider

$$E_4 = \{|\nabla u_n - \nabla u_m| > s, |u_n| \leq k, |u_m| \leq k\},\ E_5 = \{|\nabla u_m| > s, |u_n| > k, |u_m| \leq k\},$$

$$E_6 = \{|\nabla u_n| > s, |u_m| > k, |u_n| \leq k\} \text{ and } E_7 = \{0 > s, |u_m| > k, |u_n| > k\}.$$

Note that

$$\{|\nabla T_k(u_n) - \nabla T_k(u_m)| > s\} \subset E_4 \cup E_5 \cup E_6 \cup E_7. \tag{5.42}$$

Let $\epsilon > 0$. By Proposition 5.7, we may choose $k(\epsilon)$ such that

$$\text{meas}(E_5) \leq \frac{\epsilon}{4}, \text{meas}(E_6) \leq \frac{\epsilon}{4} \text{ and } \text{meas}(E_7) \leq \frac{\epsilon}{4}. \tag{5.43}$$

Therefore, using (5.40), (5.42) and (5.43), we get

$$\text{meas}(\{|\nabla T_k(u_n) - \nabla T_k(u_m)| > s\}) \leq \epsilon, \text{ for all } n, m \geq n_1(s, \epsilon). \tag{5.44}$$

Consequently, $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$.
Then, using lemmas 5.8 and 5.10, (ii) follows.

(iii) By lemmas 5.10 and 5.13, we have that for all $t > 0$, $a(x, \nabla T_t(u_n))$ converges to $a(x, \nabla T_t(u))$ in $\left(L^1(\Omega)\right)^N$ strongly and $a(x, \nabla T_t(u_n))$ converges to $\chi_t \in (L^{p'(\cdot)}(\Omega))^N$ in $(L^{p'(\cdot)}(\Omega))^N$ weakly. Since each of the convergences implies the weak L^1-convergence, χ_t can be identified with $a(x, \nabla T_t(u))$; thus, $a(x, \nabla T_t(u)) \in (L^{p'(\cdot)}(\Omega))^N$. The proof of (iii) is then complete.
(iv) As u_n is a weak solution of (5.22), using $T_k(u_n)$ as a test function, we get

$$\int_\Omega |T_k(u_n)|^{p(x)}\, dx \leq \int_\Omega |u_n|^{p(x)-2} u_n T_k(u_n) dx \leq k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right). \tag{5.45}$$

We deduce from (5.24) and (5.45) that

$$\int_\Omega |T_k(u_n)|^{p_-}\,dx \le k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega), \tag{5.46}$$

and

$$\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx \le k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega). \tag{5.47}$$

Furthermore, $T_k(u_n)$ converges weakly to $T_k(u)$ in $W^{1,p_-}(\Omega)$ and since for every $1 \le p \le +\infty$,

$$\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega), u \mapsto \tau(u) = u|_{\partial\Omega}$$

is compact, we deduce that $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^{p_-}(\partial\Omega)$ and so, up to a subsequence, we can assume that $T_k(u_n)$ converges to $T_k(u)$, a.e. on $\partial\Omega$. In other words, there exists $C \subset \partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega\backslash C$ with $\mu(C) = 0$ where μ is the area measure on $\partial\Omega$.

Now, we use Hölder Inequality, (5.46) and (5.47) to get

$$\int_\Omega |T_k(u_n)|\,dx \le (\mathrm{meas}(\Omega))^{\frac{1}{(p_-)'}} \left(k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega)\right)^{\frac{1}{p_-}}, \tag{5.48}$$

and

$$\int_\Omega |\nabla T_k(u_n)|\,dx \le (\mathrm{meas}(\Omega))^{\frac{1}{(p_-)'}} \left(k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega)\right)^{\frac{1}{p_-}}. \tag{5.49}$$

By using Fatou's Lemma in (5.48) and (5.49) we get as n goes to $+\infty$,

$$\int_\Omega |T_k(u)|\,dx \le (\mathrm{meas}(\Omega))^{\frac{1}{(p_-)'}} \left(k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega)\right)^{\frac{1}{p_-}}, \tag{5.50}$$

and

$$\int_\Omega |\nabla T_k(u)|\,dx \le (\mathrm{meas}(\Omega))^{\frac{1}{(p_-)'}} \left(k\left(\|\varphi\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right) + \mathrm{meas}(\Omega)\right)^{\frac{1}{p_-}}. \tag{5.51}$$

For every $k > 0$, let $A_k := \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $C' = \partial\Omega\backslash \bigcup_{k>0} A_k$.

We have

$$\begin{aligned}
\mu(C') = \frac{1}{k}\int_{C'} |T_k(u)|\,dx \quad &\le \frac{1}{k}\int_{\partial\Omega} |T_k(u)|\,dx \\
&\le \frac{C_1}{k}\,\|T_k(u)\|_{W^{1,1}(\Omega)} \\
&\le \frac{C_1}{k}\,\|T_k(u)\|_{L^1(\Omega)} + \frac{C_1}{k}\,\|\nabla T_k(u)\|_{L^1(\Omega)}.
\end{aligned}$$

According to (5.50) and (5.51), we deduce by letting $k \to +\infty$ that $\mu(C') = 0$.

Let us define in $\partial\Omega$ the function v by

$$v(x) := T_k(u(x)) \text{ if } x \in A_k.$$

We take $x \in \partial\Omega \setminus (C \cup C')$; then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x)))\,.$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$.
Therefore

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to +\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$.
The proof of the Proposition 5.12 is then complete.

We are now able to pass to the limit in the identity (5.34).

For the right-hand side, the convergence is obvious since f_n converges strongly to f in $L^1(\Omega)$, φ_n converges strongly to φ in $L^1(\partial\Omega)$ and $T_k(u_n - v)$ converges weakly-$*$ to $T_k(u - v)$ in $L^\infty(\Omega)$ and a.e in Ω and to $T_k(u - v)$ in $L^\infty(\partial\Omega)$ and a.e in $\partial\Omega$.
For the second term of (5.34), we have

$$\begin{aligned}\int_\Omega |u_n|^{p(x)-2}\, u_n T_k(u_n - v)dx &= \int_\Omega \left(|u_n|^{p(x)-2}\, u_n - |v|^{p(x)-2}\, v\right) T_k(u_n - v)dx \\ &\quad + \int_\Omega |v|^{p(x)-2}\, v T_k(u_n - v)dx.\end{aligned}$$

The quantity $\left(|u_n|^{p(x)-2}\, u_n - |v|^{p(x)-2}\, v\right) T_k(u_n - v)$ is nonnegative and since for all $x \in \Omega$, $\;s \longmapsto |s|^{p(x)-2}\, s\;$ is continuous, we get

$$\left(|u_n|^{p(x)-2}\, u_n - |v|^{p(x)-2}\, v\right) T_k(u_n - v) \longrightarrow \left(|u|^{p(x)-2}\, u - |v|^{p(x)-2}\, v\right) T_k(u - v)dx \text{ a.e in } \Omega.$$

Then, it follows by Fatou's Lemma that

$$\liminf_{n\to+\infty} \int_\Omega \left(|u_n|^{p(x)-2}\, u_n - |v|^{p(x)-2}\, v\right) T_k(u_n - v)dx \geq \int_\Omega \left(|u|^{p(x)-2}\, u - |v|^{p(x)-2}\, v\right) T_k(u - v)dx.$$

Let us show that $|v|^{p(x)-2}\, v \in L^1(\Omega)$.
We have

$$\int_\Omega \left||v|^{p(x)-2}\, v\right| dx = \int_\Omega |v|^{p(x)-1} dx \leq \int_\Omega \left(\|v\|_\infty\right)^{p(x)-1} dx.$$

If $\|v\|_\infty \leq 1$, then $\displaystyle\int_\Omega \left||v|^{p(x)-2}\, v\right| dx \leq \text{meas}(\Omega) < +\infty$.

If $\|v\|_\infty > 1$, then $\displaystyle\int_\Omega \left||v|^{p(x)-2}\, v\right| dx \leq \int_\Omega \left(\|v\|_\infty\right)^{p_+-1} dx = \left(\|v\|_\infty\right)^{p_+-1} \text{meas}(\Omega) < +\infty$.

Hence $|v|^{p(x)-2}\ v \in L^1(\Omega)$.

Since $T_k(u_n - v)$ converges weakly-$*$ to $T_k(u-v)$ in $L^\infty(\Omega)$ and $|v|^{p(x)-2}\ v \in L^1(\Omega)$, it follows that

$$\lim_{n\to+\infty}\int_\Omega |v|^{p(x)-2}\ vT_k(u_n - v)dx = \int_\Omega |v|^{p(x)-2}\ vT_k(u-v)dx.$$

Next, we write the first term in (5.34) in the following form

$$\int_{\{|u_n - v|\le k\}} a(x,\nabla u_n).\nabla u_n\, dx - \int_{\{|u_n - v|\le k\}} a(x,\nabla u_n).\nabla v dx. \tag{5.52}$$

Set $l = k + \|v\|_\infty$, the second integral in (5.52) equals to

$$\int_{\{|u_n - v|\le k\}} a(x,\nabla T_l(u_n)).\nabla v dx.$$

Since $a(x,\nabla T_l(u_n))$ is uniformly bounded in $\left(L^{p'(\cdot)}(\Omega)\right)^N$ (by (2.3) and (5.24)),
by Proposition 5.12$-$(iii), it converges weakly to $a(x,\nabla T_l(u))$ in $\left(L^{p'(\cdot)}(\Omega)\right)^N$.
Therefore

$$\lim_{n\to+\infty}\int_{\{|u_n - v|\le k\}} a(x,\nabla T_l(u_n)).\nabla v dx = \int_{\{|u-v|\le k\}} a(x,\nabla T_l(u)).\nabla v dx.$$

Moreover $a(x,\nabla u_n).\nabla u_n$ is nonnegative and converges a.e in Ω to $a(x,\nabla u).\nabla u$.
Thanks to Fatou's Lemma, we obtain

$$\liminf_{n\to+\infty}\int_{\{|u_n - v|\le k\}} a(x,\nabla u_n).\nabla u_n\, dx \ge \int_{\{|u-v|\le k\}} a(x,\nabla u).\nabla u dx.$$

Gathering results, we obtain

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-v)dx + \int_\Omega |u|^{p(x)-2}\ uT_k(u-v)dx \le \int_{\partial\Omega}\varphi T_k(u-v)d\sigma + \int_\Omega fT_k(u-v)dx.$$

We conclude that u is an entropy solution of (1.1).

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