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On a Condition for the Nonexistence of W-Solutions of Nonlinear High-Order Equations with L^1-Data

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ABSTRACT

In a bounded open set of $\mathbb{R}^n$ we consider the Dirichlet problem for nonlinear $2m$-order equations in divergence form with L^1-right-hand sides. It is supposed that $2 \leqslant m < n$, and the coefficients of the equations admit the growth of rate $p-1>0$ with respect to the derivatives of order m of unknown function. We establish that under the condition $p \leqslant 2 - m/n$ for some L^1-data the corresponding Dirichlet problem does not have W-solutions.

RESUMEN

En un conjunto abierto y acotado de $\mathbb{R}^n$ consideramos el problema de Dirichlet para ecuaciones no lineales de orden $2m$ en la forma divergente con lados L^1-right-hand. Se supone que $2 \leqslant m < n$, y los coeficientes de las ecuaciones admiten el radio de crecimiento $p-1>0$ con respecto a las derivadas de orden m de la función desconocida. Establecemos que bajo la condición $p \leqslant 2 - m/n$ para algn L^1- data el problema de Dirichlet correspondiente no tiene W-soluciones.

Keywords and Phrases: Nonlinear high-order equations in divergence form, L^1-data, Dirichlet problem, W-solution, nonexistence of W-solutions.

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1 Introduction

It is known that nonlinear elliptic second-order equations in divergence form whose principal coefficients grow with respect to the gradient of unknown function u as $|\nabla u|^{p-1}$ for some L^1-right-hand sides may not have weak solutions if the exponent p is sufficiently close to 1. The fact of the nonexistence of weak solutions was observed in [1] by giving the following example: if Ω is an open set of $\mathbb{R}^n$ with $n \geqslant 2$ and $1 < p \leqslant 2 - 1/n$, then there exists a function $f \in L^1(\Omega)$ such that the problem

$$u \in W^{1,1}_{\mathrm{loc}}(\Omega), \qquad -\Delta_p u + u = f \ \text{ in } \ \mathcal{D}'(\Omega)$$

does not have a solution.

The given observation was one of motivations for the development of the theory of entropy solutions for nonlinear elliptic second-order equations with L^1-data [1]. According to the results of [1], if $1 < p < n$, under natural growth, coercivity and strict monotonicity conditions for coefficients of the equations under consideration an entropy solution exists and is unique for every L^1-right-hand side. Moreover, if $p > 2 - 1/n$, the entropy solution is a weak solution.

Analogous results on the existence of entropy and weak solutions for nonlinear elliptic high-order equations with coefficients satisfying a strengthened coercivity condition and L^1-right-hand sides were obtained in [3, 4]. Conditions of the existence of weak solutions for some classes of degenerate nonlinear elliptic high-order equations with strengthened coercivity and L^1-data were given in [5, 6].

As far as nonlinear elliptic high-order equations with L^1-right-hand sides and coefficients satisfying the natural coercivity condition are concerned, the question on their solvability on the whole is still open. It seems, for these equations the approaches which work in the cases of second-order equations with L^1-data and high-order equations with strengthened coercivity and L^1-data are not suitable.

On the other hand, the use of the known principle of uniform boundedness (see [2, Chapter 2]) one can consider as a general functional tool for the study of conditions for the nonexistence of weak solutions of nonlinear arbitrary even order equations in divergence form with L^1-data. Using this principle, in the present article we give such a condition for high-order equations.

2 Main Results

Let $m, n \in \mathbb{N}$ be numbers such that $2 \leqslant m < n$. Let Ω be a bounded open set of $\mathbb{R}^n$.

We shall use the following notation: Λ is the set of all n-dimensional multi-indicies α such that $|\alpha| = m$, $\mathbb{R}^n_m$ is the space of all functions $\xi : \Lambda \to \mathbb{R}$; if $u \in L^1_{\mathrm{loc}}(\Omega)$ and the function u has the weak derivatives $D^\alpha u$, $\alpha \in \Lambda$, then $\nabla_m u : \Omega \to \mathbb{R}^n_m$ is the mapping such that for every $x \in \Omega$

and for every $\alpha \in \Lambda$, $(\nabla_m u(x))_\alpha = D^\alpha u(x)$.

Let $p > 1$, $c > 0$, $g \in L^{1/(p-1)}(\Omega)$, $g \geqslant 0$ in Ω, and let for every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R}^n_m \to \mathbb{R}$ be a Carathéodory function. We shall assume that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n_m$,

$$\sum_{\alpha\in\Lambda} |A_\alpha(x,\xi)| \leqslant c \sum_{\alpha\in\Lambda} |\xi_\alpha|^{p-1} + g(x). \tag{2.1}$$

For every $f \in L^1(\Omega)$ by $(\mathcal{P}_f)$ we denote the following problem:

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_m u) = f \;\text{ in }\; \Omega,$$

$$D^\alpha u = 0, \quad |\alpha| \leqslant m-1, \quad \text{on } \; \partial\Omega.$$

Definition 2.1. Let $f \in L^1(\Omega)$. A W-solution of problem $(\mathcal{P}_f)$ is a function $u \in \overset{\circ}{W}{}^{m,1}(\Omega)$ such that

(i) for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_m u) \in L^1(\Omega)$;

(ii) for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u) D^\alpha \varphi \Big\} dx = \int_\Omega f\varphi \, dx.$$

Theorem 2.1. *Suppose that*

$$p \leqslant 2 - \frac{m}{n}. \tag{2.2}$$

Then there exists $f \in L^1(\Omega)$ *such that problem* $(\mathcal{P}_f)$ *does not have* W*-solutions.*

Proof. Let us assume that for every $f \in L^1(\Omega)$ there exists a W-solution of problem $(\mathcal{P}_f)$. This implies that if $f \in L^1(\Omega)$, then there exists a function $u_f \in \overset{\circ}{W}{}^{m,1}(\Omega)$ such that for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_m u_f) \in L^1(\Omega)$ and

$$\forall\, \varphi \in C_0^\infty(\Omega), \qquad \int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u_f) D^\alpha \varphi \Big\} dx = \int_\Omega f\varphi \, dx. \tag{2.3}$$

Observe that due to the inequality $p > 1$ and (2.2) we have $0 < 2-p < 1$. We set $p_1 = 1/(2-p)$. Clearly, $p_1 > 1$.

Using (2.1) and the inclusion $g \in L^{1/(p-1)}(\Omega)$, we establish that if $f \in L^1(\Omega)$, then for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_m u_f) \in L^{p_1/(p_1-1)}(\Omega)$.

For every $f \in L^1(\Omega)$ we define the functional $H_f : \overset{\circ}{W}{}^{m,p_1}(\Omega) \to \mathbb{R}$ by

$$\langle H_f, \varphi \rangle = \int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u_f) D^\alpha \varphi \Big\} dx, \qquad \varphi \in \overset{\circ}{W}{}^{m,p_1}(\Omega).$$

It is easy to see that

$$\forall\, f \in L^1(\Omega), \qquad H_f \in (\overset{\circ}{W}{}^{m,p_1}(\Omega))^*. \tag{2.4}$$

Moreover, taking into account (2.3), for every $f \in L^1(\Omega)$ and for every $\varphi \in C_0^\infty(\Omega)$ we get

$$\langle H_f, \varphi\rangle = \int_\Omega f\varphi\, dx. \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\text{for every } \ f_1, f_2 \in L^1(\Omega), \qquad H_{f_1+f_2} = H_{f_1} + H_{f_2}, \tag{2.6}$$

$$\text{for every } \ f \in L^1(\Omega) \ \text{ and for every } \ \lambda \in \mathbb{R}, \qquad H_{\lambda f} = \lambda H_f. \tag{2.7}$$

Next, let $\varphi \in \overset{\circ}{W}{}^{m,p_1}(\Omega)$. We fix a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that

$$\|\varphi_k - \varphi\|_{W^{m,p_1}(\Omega)} \to 0. \tag{2.8}$$

For every $k \in \mathbb{N}$ we define the functional $F_k : L^1(\Omega) \to \mathbb{R}$ by

$$\langle F_k, f\rangle = |\langle H_f, \varphi_k\rangle|, \qquad f \in L^1(\Omega).$$

Using (2.5), we establish the following fact: if $k \in \mathbb{N}$ and $f_1, f_2 \in L^1(\Omega)$, then

$$|\langle F_k, f_1\rangle - \langle F_k, f_2\rangle| \leqslant \big(\max_\Omega |\varphi_k|\big)\|f_1 - f_2\|_{L^1(\Omega)}.$$

This implies that for every $k \in \mathbb{N}$ the functional F_k is continuous on $L^1(\Omega)$. Moreover, with the use of (2.6) and (2.7) we obtain the next properties:

(i) for every $k \in \mathbb{N}$ and for every $f_1, f_2 \in L^1(\Omega)$,

$$\langle F_k, f_1 + f_2\rangle \leqslant \langle F_k, f_1\rangle + \langle F_k, f_2\rangle;$$

(ii) for every $k \in \mathbb{N}$, for every $f \in L^1(\Omega)$ and for every $\lambda \in \mathbb{R}$,

$$\langle F_k, \lambda f\rangle = |\lambda|\langle F_k, f\rangle.$$

Finally, taking into account (2.4) and (2.8), we establish that for every $f \in L^1(\Omega)$ the sequence of the numbers $\langle F_k, f\rangle$ is bounded. This along with the nonnegativity and continuity of the functionals F_k, properties (i) and (ii) and the principle of uniform boundedness [2, Chapter 2] allows us to conclude that there exists $M > 0$ such that for every $k \in \mathbb{N}$ and for every $f \in L^1(\Omega)$,

$$\langle F_k, f\rangle \leqslant M\|f\|_{L^1(\Omega)}.$$

From the result obtained, using the definition of the functionals F_k and (2.4) and (2.8), we deduce that

$$\forall\, f \in L^1(\Omega), \qquad |\langle H_f, \varphi\rangle| \leqslant M\|f\|_{L^1(\Omega)}. \tag{2.9}$$

Now let $F : L^1(\Omega) \to \mathbb{R}$ be the functional such that for every $f \in L^1(\Omega)$,

$$\langle F, f\rangle = \langle H_f, \varphi\rangle. \tag{2.10}$$

Owing to (2.6) and (2.7), the functional F is linear, and by virtue of (2.9) and (2.10), for every $f \in L^1(\Omega)$, $|\langle F, f\rangle| \leqslant M\|f\|_{L^1(\Omega)}$. Therefore, $F \in (L^1(\Omega))^*$. Then there exists a function $\psi \in L^\infty(\Omega)$ such that for every $f \in L^1(\Omega)$,

$$\langle F, f\rangle = \int_\Omega \psi f\, dx.$$

This and (2.10) imply that

$$\forall\, f \in L^1(\Omega), \qquad \langle H_f, \varphi\rangle = \int_\Omega \psi f\, dx. \tag{2.11}$$

Let us show that $\varphi = \psi$ a.e. in Ω. In fact, let $f \in L^{1/(p-1)}(\Omega)$. Then, by (2.5) and (2.8),

$$\langle H_f, \varphi_k\rangle \to \int_\Omega f\varphi\, dx. \tag{2.12}$$

On the other hand, from (2.4), (2.8) and (2.11) it follows that

$$\langle H_f, \varphi_k\rangle \to \int_\Omega f\psi\, dx.$$

This and (2.12) imply that

$$\int_\Omega f(\varphi - \psi)dx = 0.$$

Hence, taking into account the arbitrariness of the function f in $L^{1/(p-1)}(\Omega)$, we obtain that $\varphi = \psi$ a.e. in Ω. Therefore, $\varphi \in L^\infty(\Omega)$.

Thus, we conclude that

$$\overset{\circ}{W}{}^{m,p_1}(\Omega) \subset L^\infty(\Omega). \tag{2.13}$$

However, since, by (2.2), we have $mp_1 \leqslant n$, inclusion (2.13) is not true.

For instance, if $p < 2 - m/n$, $y \in \Omega$, $\nu : \Omega \to \mathbb{R}$ is a function such that for every $x \in \Omega \backslash \{y\}$, $\nu(x) = \ln|x - y|$, B is a closed ball in $\mathbb{R}^n$ with the center y such that $B \subset \Omega$, $\psi_1 \in C_0^\infty(\Omega)$ and $\psi_1 = 1$ in B, then $\nu\psi_1 \in \overset{\circ}{W}{}^{m,p_1}(\Omega) \backslash L^\infty(\Omega)$.

The contradiction obtained allows us to conclude that there exists a function $f \in L^1(\Omega)$ such that problem $(\mathcal{P}_f)$ does not have W-solutions. This completes the proof of the theorem.

Now we give an analogous result for equations with lower-order terms.

Let $c_0 > 0$, $0 < \sigma < n/(n - m)$, $\sigma_1 = \frac{n}{\sigma(n-m)}$, $g_0 \in L^{\sigma_1}(\Omega)$, $g_0 \geqslant 0$ in Ω, and let $A : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$,

$$|A(x, s)| \leqslant c_0|s|^\sigma + g_0(x). \tag{2.14}$$

For every $f \in L^1(\Omega)$ by (P_f) we denote the following problem:

$$\sum_{\alpha\in\Lambda}(-1)^{|\alpha|}D^\alpha A_\alpha(x,\nabla_m u)+A(x,u)=f \ \text{ in } \ \Omega,$$

$$D^\alpha u=0, \ \ |\alpha|\leqslant m-1, \ \text{ on } \partial\Omega.$$

Definition 2.2. Let $f\in L^1(\Omega)$. A W-solution of problem (P_f) is a function $u\in \overset{\circ}{W}{}^{m,1}(\Omega)$ such that

(i) for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_m u)\in L^1(\Omega)$;

(ii) $A(x,u)\in L^1(\Omega)$;

(iii) for every $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega\bigg\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_m u)D^\alpha\varphi+A(x,u)\varphi\bigg\}dx=\int_\Omega f\varphi\,dx.$$

Theorem 2.2. *Suppose that condition* (2.2) *is satisfied. Then there exists* $f\in L^1(\Omega)$ *such that problem* (P_f) *does not have* W*-solutions.*

Proof. Let us assume that for every $f\in L^1(\Omega)$ there exists a W-solution of problem (P_f). This implies that if $f\in L^1(\Omega)$, then there exists a function $u_f\in \overset{\circ}{W}{}^{m,1}(\Omega)$ such that for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_m u_f)\in L^1(\Omega)$, $A(x,u_f)\in L^1(\Omega)$ and

$$\forall\,\varphi\in C_0^\infty(\Omega),\qquad \int_\Omega\bigg\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_m u_f)D^\alpha\varphi+A(x,u_f)\varphi\bigg\}dx=\int_\Omega f\varphi\,dx. \tag{2.15}$$

We set $p_1=1/(2-p)$. As in the proof of the previous theorem, we have: if $f\in L^1(\Omega)$, then for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_m u_f)\in L^{p_1/(p_1-1)}(\Omega)$. Moreover, taking into account that, by Sobolev embedding theorem, $\overset{\circ}{W}{}^{m,1}(\Omega)\subset L^{n/(n-m)}(\Omega)$ and using the inclusion $g_0\in L^{\sigma_1}(\Omega)$ and (2.14), we obtain that for every $f\in L^1(\Omega)$, $A(x,u_f)\in L^{\sigma_1}(\Omega)$.

Next, we define

$$V=\overset{\circ}{W}{}^{m,p_1}(\Omega)\cap L^{\sigma_1/(\sigma_1-1)}(\Omega).$$

The set V is a Banach space with the norm

$$\|u\|_V=\|u\|_{W^{m,p_1}(\Omega)}+\|u\|_{L^{\sigma_1/(\sigma_1-1)}(\Omega)}.$$

For every $f\in L^1(\Omega)$ we define the functional $G_f:V\to\mathbb{R}$ by

$$\langle G_f,\varphi\rangle=\int_\Omega\bigg\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_m u_f)D^\alpha\varphi+A(x,u_f)\varphi\bigg\}dx,\qquad \varphi\in V.$$

Clearly,

$$\forall\, f \in L^1(\Omega), \qquad G_f \in V^*. \tag{2.16}$$

Furthermore, taking into account (2.15), for every $f \in L^1(\Omega)$ and for every $\varphi \in C_0^\infty(\Omega)$ we get

$$\langle G_f, \varphi\rangle = \int_\Omega f\varphi\, dx. \tag{2.17}$$

We denote by V_1 the closure of $C_0^\infty(\Omega)$ in V. Using (2.16) and (2.17) and arguing by analogy with the proof of Theorem 2.1, we establish that

$$V_1 \subset L^\infty(\Omega). \tag{2.18}$$

However, since, by condition (2.2), we have $mp_1 \leqslant n$, inclusion (2.18) is not true.

For instance, if $p < 2 - m/n$ and ν and ψ_1 are the functions described at the end of the proof of Theorem 2.1, then $\nu\psi_1 \in V_1 \backslash L^\infty(\Omega)$.

The contradiction obtained allows us to conclude that there exists $f \in L^1(\Omega)$ such that problem (P_f) does not have W-solutions. This completes the proof of the theorem.

3 Remarks

Simple examples of the functions A_α and A satisfying inequalities (2.1) and (2.14) are as follows:

(i) $A_\alpha(x,\xi) = |\xi_\alpha|^{p-1}$ or $A_\alpha(x,\xi) = |\xi_\alpha|^{p-1}\operatorname{sign}\xi_\alpha$ if $\alpha \in \Lambda$;

(ii) $A(x,s) = as$ or $A(x,s) = a|s|^\sigma$ or $A(x,s) = a|s|^\sigma \operatorname{sign} s$, where $a \in \mathbb{R}$ and $0 < \sigma < n/(n-m)$.

Finally, observe that if $1 < p < n/m$, $A_\alpha(x,\xi) = |\xi_\alpha|^{p-1}\operatorname{sign}\xi_\alpha$, $\alpha \in \Lambda$, $A(x,s) = a|s|^\sigma \operatorname{sign} s$, where $a > 0$ and $\sigma \in (0, n/(n-m))$, then for every $f \in L^{np/(n-mp)}(\Omega)$ problem (P_f) has a W-solution $u \in \overset{\circ}{W}{}^{m,p}(\Omega)$. This fact simply follows from the known results of the theory of monotone operators (see for instance [7]). However, if $1 < p \leqslant 2 - \frac{m}{n}$, according to Theorem 2.2, there exists $f \in L^1(\Omega)$ such that problem (P_f) does not have W-solutions.

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