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K-theory for the C*-algebras of continuous functions on certain homogeneous spaces in semi-simple Lie groups

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ABSTRACT

We study K-theory for the C^*-algebras of all continuous functions on certain homogeneous spaces in the semi-simple connected Lie groups $SL_n(\mathbb{R})$ by the discrete subgroups $SL_n(\mathbb{Z})$, mainly. As a byproduct, we also consider a certain nilpotent case similarly.

RESUMEN

Estudiamos la K-teoría para las C^*-álgebras de todas las funciones continuas sobre ciertos espacios homogéneos, principalmente en los grupos de Lie conexos semi- simples $SL_n(\mathbb{R})$ y subgrupos discretos $SL_n(\mathbb{Z})$. Como subproducto consideramos un caso nilpotente en forma análoga.

Keywords and Phrases: C*-algebra, K-theory, homogeneous space, semi-simple Lie group, discrete subgroup.

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1 Introduction

This work is started with an attempt to find a candidate for the K-theory groups for the full or reduced group C^*-algebras of the discrete groups $SL_n(\mathbb{Z})$. Our idea comes from the fact that K-theory for the group C^*-algebra of the discrete groups $\mathbb{Z}^n$ of integers is the same as that for the C^*-algebra of all continuous functions on the tori $\mathbb{T}^n$ viewed as the quotient $\mathbb{R}^n/\mathbb{Z}^n$, via the Fourier transform, and that this picture should have some similar meanings in more general or noncommutative setting, at least in K-theory level.

Refer to [5] for some basics of K-theory and C^*-algebras.

After a quick review in Section 2 about the abelian case of commutative connected Lie groups, we consider in Section 3 homogeneous spaces in $SL_2(\mathbb{R})$ a semi-simple connected Lie group and compute the K-theory groups of the C^*-algebras of all continuous functions on those spaces. Moreover, we consider the case of $SL_n(\mathbb{R})$ $(n \geq 3)$ in Section 4. The results obtained would be useful for further research in this direction. Furthermore, as a byproduct, we consider a certain nilpotent case of discrete Heisenberg groups.

2 Abelian case

For convenience, recall that we have the following short exact sequence of abelian (or commutative Lie) groups:

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{T}^n \to 0.$$

Consider their group C^*-algebras $C^*(\mathbb{Z}^n)$, $C^*(\mathbb{R}^n)$, and $C^*(\mathbb{T}^n)$. By Fourier transform, they are isomorphic respectively to $C(\mathbb{T}^n)$, $C_0(\mathbb{R}^n)$, and $C_0(\mathbb{Z}^n)$ the C^*-algebras of all continuous functions on $\mathbb{T}^n$, on $\mathbb{R}^n$ and $\mathbb{Z}^n$ vanishing at infinity. Their K-theory groups are well known as follows ([5]):

$$\begin{aligned}
&K_j(C^*(\mathbb{Z}^n)) \cong K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}, \quad (j = 0, 1);\\
&K_0(C^*(\mathbb{R}^{2n})) \cong K_0(C_0(\mathbb{R}^{2n})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}, \quad K_1(C^*(\mathbb{R}^{2n})) \cong K_1(\mathbb{C}) \cong 0,\\
&K_0(C^*(\mathbb{R}^{2n-1})) \cong K_0(C_0(\mathbb{R}^{2n-1})) \cong K_1(\mathbb{C}) \cong 0, \quad K_1(C^*(\mathbb{R}^{2n-1})) \cong K_0(\mathbb{C}) \cong \mathbb{Z},\\
&K_0(C^*(\mathbb{T}^n)) \cong K_0(C_0(\mathbb{Z}^n)) \cong \oplus^{\mathbb{Z}^n}\mathbb{Z}, \quad K_1(C^*(\mathbb{T}^n)) \cong K_1(C_0(\mathbb{Z}^n)) \cong 0,
\end{aligned}$$

where $\oplus^k$ means the k-times direct sum. Observe that K-theory of the group C^*-algebra of the discrete group $\mathbb{Z}^n$ is the same as that of the C^*-algebra of all continuous functions on the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

3 Homogeneous spaces in $SL_2(\mathbb{R})$

Consider the following inclusion and its homogeneous space denoted as:

$$0 \to SL_2(\mathbb{Z}) \to SL_2(\mathbb{R}), \quad SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \equiv H_2.$$

Let $SL_2(\mathbb{R}) = KAN$ be the Iwasawa decomposition. More precisely, we have the following homeomorphism:

$$\begin{aligned} SL_2(\mathbb{R}) &\approx KAN = SO(2)A_2N_2, \\ \text{where} \quad SO(2) &= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,|\, \theta \in \mathbb{R} \right\} \cong S^1 = \{e^{i\theta} \,|\, \theta \in \mathbb{R}\}, \\ A_2 &= \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \,|\, a_1 a_2 = 1, a_1 > 0, a_2 > 0 \right\}, \\ N_2 &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,|\, b \in \mathbb{R} \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} SL_2(\mathbb{Z}) &\approx K_{\mathbb{Z}} A_{\mathbb{Z}} N_{\mathbb{Z}} = SO(2)_{\mathbb{Z}} A_{2,\mathbb{Z}} N_{2,\mathbb{Z}}, \\ \text{where} \quad SO(2)_{\mathbb{Z}} &\cong S^1_{\mathbb{Z}} = \{e^{i\theta} \in \mathbb{Z}^2 \,|\, \theta \in \mathbb{R}\} = \{(\pm 1, 0), (0, \pm 1)\}, \\ A_{2,\mathbb{Z}} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\ N_{2,\mathbb{Z}} &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,|\, b \in \mathbb{Z} \right\}. \end{aligned}$$

It follows from considering quotient spaces that the homogeneous space H_2 is homeomorphic to the following product space:

$$H_2 \approx (\sqcup^4 \mathbb{R})^+ \times \mathbb{R} \times \mathbb{T},$$

where $\sqcup^k \mathbb{R}$ means the disjoint union of k copies of $\mathbb{R}$, and X^+ means the one-point compactification of X, and $SO(2)/SO(2)_{\mathbb{Z}} \approx (\sqcup^4 \mathbb{R})^+$, and $A_2 \approx \mathbb{R}$, and $N_2/N_{2,\mathbb{Z}} \approx \mathbb{T}$.

Let $C_0(H_2)$ be the C*-algebra of all continuous functions on H_2 vanishing at infinity. We compute its K-theory groups as follows. First of all, we have

$$\begin{aligned} K_j(C_0(H_2)) &\cong K_j(C_0((\sqcup^4 \mathbb{R})^+ \times \mathbb{R} \times \mathbb{T})) \\ &\cong K_{j+1}(C((\sqcup^4 \mathbb{R})^+ \times \mathbb{T})), \end{aligned}$$

by the Bott periodicity, where $j + 1$ (mod 2). Consider the following short exact sequence of C*-algebras:

$$0 \longrightarrow C_0((\sqcup^4 \mathbb{R}) \times \mathbb{T}) \xrightarrow{i} C((\sqcup^4 \mathbb{R})^+ \times \mathbb{T}) \xrightarrow{q} C(\mathbb{T}) \longrightarrow 0.$$

Note that this extension of C*-algebras splits, clearly. We then have the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(C_0((\sqcup^4 \mathbb{R}) \times \mathbb{T})) & \xrightarrow{i_*} & K_0(C((\sqcup^4 \mathbb{R})^+ \times \mathbb{T})) & \xrightarrow{q_*} & K_0(C(\mathbb{T})) \\ \uparrow & & & & \downarrow \\ K_1(C(\mathbb{T})) & \xleftarrow{q_*} & K_1(C((\sqcup^4 \mathbb{R})^+ \times \mathbb{T})) & \xleftarrow{i_*} & K_1(C_0((\sqcup^4 \mathbb{R}) \times \mathbb{T})), \end{array}$$

with

$$\begin{aligned} K_j(C_0((\sqcup^4\mathbb{R})\times\mathbb{T})) &\cong \oplus^4 K_j(C_0(\mathbb{R}\times\mathbb{T})) \\ &\cong \oplus^4 K_{j+1}(C(\mathbb{T})) \cong \mathbb{Z}^4 \end{aligned}$$

for $j = 0, 1$, where $\oplus^k$ means the direct sum of k copies. The commutative diagram also splits into two short exact sequences of K_0 and K_1-groups, by the splitting short exact sequence of C^*-algebras. Therefore, we obtain

$$0 \to \mathbb{Z}^4 \to K_j(C((\sqcup^4\mathbb{R})^+ \times \mathbb{T})) \to \mathbb{Z} \to 0$$

for $j = 0, 1$. Since extensions of groups by $\mathbb{Z}$ also split, certainly known, we obtain that $K_j(C((\sqcup^4\mathbb{R})^+ \times \mathbb{T})) \cong \mathbb{Z}^5$ for $j = 0, 1$. Hence we get

Theorem 3.1. *Let* $H_2 = SL_2(\mathbb{R})/SL_2(\mathbb{Z}) = KAN/K_{\mathbb{Z}}A_{\mathbb{Z}}N_{\mathbb{Z}}$ *be the homogeneous space via the Iwasawa decomposition. Then* H_2 *is homeomorphic to the product space* $(\sqcup^4\mathbb{R})^+ \times \mathbb{R} \times \mathbb{T}$*, and*

$$K_j(C_0(H_2)) \cong \mathbb{Z}^5, \quad (j = 0, 1).$$

Moreover, we obtain

Proposition 3.2. Let $K/K_{\mathbb{Z}} = SO(2)/SO(2)_{\mathbb{Z}} = KAN/K_{\mathbb{Z}}AN$ be the homogeneous space of the compact group $SO(2)$. Then $K/K_{\mathbb{Z}}$ is the compact space $(\sqcup^4\mathbb{R})^+$, and

$$K_0(C(K/K_{\mathbb{Z}})) \cong \mathbb{Z} \quad \text{and} \quad K_1(C(K/K_{\mathbb{Z}})) \cong \mathbb{Z}^4.$$

Proof. Consider the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\sqcup^4\mathbb{R}) \xrightarrow{i} C((\sqcup^4\mathbb{R})^+) \xrightarrow{q} \mathbb{C} \longrightarrow 0.$$

Note that this extension of C^*-algebras splits. We then have the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(C_0(\sqcup^4\mathbb{R})) & \xrightarrow{i_*} & K_0(C((\sqcup^4\mathbb{R})^+)) & \xrightarrow{q_*} & K_0(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}) & \xleftarrow{q_*} & K_1(C((\sqcup^4\mathbb{R})^+)) & \xleftarrow{i_*} & K_1(C_0(\sqcup^4\mathbb{R})), \end{array}$$

with

$$K_j(C_0((\sqcup^4\mathbb{R}))) \cong \oplus^4 K_j(C_0(\mathbb{R})) \cong \oplus^4 K_{j+1}(\mathbb{C})$$

for $j = 0, 1$. The commutative diagram also splits into two short exact sequences of K_0 and K_1-groups. Therefore, we obtain that $K_0(C((\sqcup^4\mathbb{R})^+)) \cong \mathbb{Z}$ and $K_1(C((\sqcup^4\mathbb{R})^+)) \cong \mathbb{Z}^4$. □

Remark. Note that the quotient space $N/N_{\mathbb{Z}}$ is isomorphic to $\mathbb{T}$ as a group. Thus, $K_j(C(N/N_{\mathbb{Z}})) \cong \mathbb{Z}$ for $j = 0, 1$.

Furthermore, we have

Proposition 3.3. The homogeneous space $\mathsf{SL}_2(\mathbb{R})/\mathsf{K} = \mathsf{AN}$ is homeomorphic to the product space $\mathbb{R} \times \mathbb{T}$, and $\mathsf{K}_j(\mathsf{C}_0(\mathsf{AN})) \cong \mathbb{Z}$ for $j = 0, 1$.

Proof. We have

$$\mathsf{K}_j(\mathsf{C}_0(\mathbb{R} \times \mathbb{T})) \cong \mathsf{K}_{j+1}(\mathsf{C}(\mathbb{T})) \cong \mathbb{Z}$$

for $j = 0, 1$. □

Notes. It is shown by Natsume [2] that for $\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))$ the full group C^*-algebra of $\mathsf{SL}_2(\mathbb{Z})$,

$$\mathsf{K}_0(\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))) \cong \mathbb{Z}^8, \quad \mathsf{K}_1(\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))) \cong 0,$$

and the same holds by replacing $\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))$ with its reduced group C^*-algebra of the regular representation of $\mathsf{SL}_2(\mathbb{Z})$.

More precisely, since $\mathsf{SL}_2(\mathbb{Z})$ is isomorphic to the amalgam $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ of cyclic groups with orders $2, 4, 6$, we have $\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))$ isomorphic to the amalgam $\mathsf{C}^*(\mathbb{Z}_4) *_{\mathsf{C}^*(\mathbb{Z}_2)} \mathsf{C}^*(\mathbb{Z}_6)$ of their group C^*-algebras, so that

$$\mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_4) *_{\mathsf{C}^*(\mathbb{Z}_2)} \mathsf{C}^*(\mathbb{Z}_6)) \cong (\mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_4)) \oplus \mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_6)))/\mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_2))$$

for $j = 0, 1$. In particular, $\mathsf{K}_0(\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))) \cong \mathbb{Z}^8 \cong \mathbb{Z}^{10}/\mathbb{Z}^2$. Also,

$$\mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_4) * \mathsf{C}^*(\mathbb{Z}_6)) \cong \mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_4)) \oplus \mathsf{K}_j(\mathsf{C}^*(\mathbb{Z}_6))$$

for $j = 0, 1$, where $\mathsf{C}^*(\mathbb{Z}_4) * \mathsf{C}^*(\mathbb{Z}_6)$ is the full free product of C^*-algebras. More generally, for $\mathfrak{A} * \mathfrak{B}$ the full free product of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we have ([1])

$$\mathsf{K}_j(\mathfrak{A} * \mathfrak{B}) \cong \mathsf{K}_j(\mathfrak{A}) \oplus \mathsf{K}_j(\mathfrak{B}), \quad (j = 0, 1).$$

Corollary 1. *We have*

$$\mathsf{K}_0(\mathsf{C}_0(\mathsf{H}_2)) \oplus \mathsf{K}_1(\mathsf{C}_0(\mathsf{H}_2)) \cong \mathsf{K}_0(\mathsf{C}^*(\mathbb{Z}_4) * \mathsf{C}^*(\mathbb{Z}_6)) \oplus \mathsf{K}_1(\mathsf{C}^*(\mathbb{Z}_4) * \mathsf{C}^*(\mathbb{Z}_6)),$$

as a group, but

$$\mathsf{K}_0(\mathsf{C}_0(\mathsf{H}_2)) \oplus \mathsf{K}_1(\mathsf{C}_0(\mathsf{H}_2)) \not\cong \mathsf{K}_0(\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))) \oplus \mathsf{K}_1(\mathsf{C}^*(\mathsf{SL}_2(\mathbb{Z}))).$$

Remark. Since $10 > 8$, it may say to be possible that K-theory data of the homogeneous space C^*-algebra contains that of the group C^*-algebra of $\mathsf{SL}_2(\mathbb{Z})$. In fact, in the group non-isomorphic equation above, the right hand side can be a quotient of the left hand side. This picture might be extended to the more general setting.

4 Homogeneous spaces in $SL_n(\mathbb{R})$

Consider the following inclusion and its homogeneous space denoted as:

$$0 \to SL_n(\mathbb{Z}) \to SL_n(\mathbb{R}), \quad SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \equiv H_n.$$

Let $SL_n(\mathbb{R}) = KAN$ be the Iwasawa decomposition. More precisely, we have the following homeomorphism:

$$SL_n(\mathbb{R}) \approx KAN = SO(n)A_nN_n,$$

$$\text{where} \quad A_n = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \mid \Pi_{j=1}^n a_j = 1, a_j > 0 \right\},$$

$$N_n = \left\{ \begin{pmatrix} 1 & b_{12} & \cdots & b_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & b_{n-1,n} \\ 0 & & & 1 \end{pmatrix} \mid b_{i,j} \in \mathbb{R}, (i < j) \right\}.$$

It follows that

$$SL_n(\mathbb{Z}) \approx K_{\mathbb{Z}}A_{\mathbb{Z}}N_{\mathbb{Z}} = SO(n)_{\mathbb{Z}}A_{n,\mathbb{Z}}N_{n,\mathbb{Z}},$$

where $SO(n)_{\mathbb{Z}}$ consists of all matrices of $SO(n)$ with components of integers, $A_{n,\mathbb{Z}}$ of only the n-th identity matrix, and $N_{n,\mathbb{Z}}$ of all matrices of N_n with components of integers. It follows from considering quotient spaces that the homogeneous space H_n is homeomorphic to the following product space:

$$H_n \approx (SO(n)/SO(n)_{\mathbb{Z}}) \times \mathbb{R}^{n-1} \times \mathbb{T}^{\frac{(n-1)n}{2}},$$

where $A_n \approx \mathbb{R}^{n-1}$ and $N_n/N_{n,\mathbb{Z}} \approx \mathbb{T}^{\frac{(n-1)n}{2}}$.

Recall that as a topological space,

$$SO(n)/SO(n-1) \approx S^{n-1},$$

where S^{n-1} is the $n-1$ dimensional sphere. Indeed, $SO(n)$ acts transitively on S^{n-1} by matrix multiplication, and the isotropy group for the n-th standard basis vector in S^{n-1} is $SO(n-1)$, from which the homeomorphism is obtained. However, these quotient spaces do not split in general into the product spaces:

$$SO(n) \approx SO(n-1) \times S^{n-1},$$

but this is certainly true if and only if there is a continuous section from S^{n-1} to $SO(n)$. This is just the cases where $n = 4$ or $n = 8$, a well-known, non-tirvial, important result in algebraic topology. Note that what is necessary in what follows may be the isomorphisms in topological K-theory level:

$$K^j(SO(n)) \cong K^j(SO(n-1) \times S^{n-1})$$

(or mere replacements).

We have shown that $SO(2)/SO(2)_{\mathbb{Z}} \approx S^1/S^1_{\mathbb{Z}}$. If we assume the homeomorphisms for $SO(n)$, inductively we have

$$SO(n)/SO(n)_{\mathbb{Z}} \approx (SO(n-1)/SO(n-1)_{\mathbb{Z}}) \times (S^{n-1}/S^{n-1}_{\mathbb{Z}}),$$

where $S^{n-1}_{\mathbb{Z}}$ means the set of all integral points in S^{n-1}, and the equivalence relation on S^{n-1} by $S^{n-1}_{\mathbb{Z}}$ is defined as: for $\xi, \eta \in S^{n-1}$, we have $\xi \sim \eta$ if and only if $\xi = \eta$, or $\xi, \eta \in S^{n-1}_{\mathbb{Z}}$. Therefore, we obtain

$$SO(n)/SO(n)_{\mathbb{Z}} \approx (S_1/S_{\mathbb{Z}}) \times \cdots \times (S^{n-1}/S^{n-1}_{\mathbb{Z}}).$$

However, this may not be true in general, but even in such a case, we may replace $SO(n)/SO(n)_{\mathbb{Z}}$ by the product space in the right hand side, as a reasonable candidate, and we continue. But what is necessary in what follows may be the isomorphisms in topological K-theory level:

$$K^j(SO(n)/SO(n)_{\mathbb{Z}}) \cong K^j((SO(n-1)/SO(n-1)_{\mathbb{Z}}) \times (S^{n-1}/S^{n-1}_{\mathbb{Z}}))$$

(or mere replacements).

We also have

$$S^{n-1}_{\mathbb{Z}} = \{(\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1) \in \mathbb{R}^n\}.$$

Hence we identify $S^{n-1}_{\mathbb{Z}}$ with $\sqcup^n \mathbb{Z}_2$ the n-fold disjoint union of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Therefore, we get

$$S^{n-1}/S^{n-1}_{\mathbb{Z}} \approx S^{n-1}/\sqcup^n \mathbb{Z}_2.$$

Let $C_0(H_n)$ be the C*-algebra of all continuous functions on H_n vanishing at infinity. We compute its K-theory groups as follows. First of all, we have

$$\begin{aligned} K_j(C_0(H_n)) &\cong K_j(C_0((SO(n)/SO(n)_{\mathbb{Z}}) \times \mathbb{R}^{n-1} \times \mathbb{T}^{\frac{(n-1)n}{2}})) \\ &\cong K_{j+n-1}(C(SO(n)/SO(n)_{\mathbb{Z}}) \times \mathbb{T}^{\frac{(n-1)n}{2}})), \end{aligned}$$

by the Bott periodicity, where $j + n - 1$ (mod 2).

Now let $S_n = SO(n)/SO(n)_{\mathbb{Z}}$ and $T_n = \mathbb{T}^{\frac{(n-1)n}{2}}$. Since $C(S_n \times T_n) \cong C(S_n) \otimes C(T_n)$ a C*-tensor product, the Künneth formula implies

$$\begin{aligned} K_0(C(S_n \times T_n)) &\cong (K_0(C(S_n)) \otimes K_0(C(T_n))) \oplus (K_1(C(S_n)) \otimes K_1(C(T_n))), \\ K_1(C(S_n \times T_n)) &\cong (K_0(C(S_n)) \otimes K_1(C(T_n))) \oplus (K_1(C(S_n)) \otimes K_0(C(T_n))). \end{aligned}$$

For $j = 0, 1$, we have

$$K_j(C(T_n)) = K_j(C(\mathbb{T}^{\frac{(n-1)n}{2}})) \cong \mathbb{Z}^{2^{2^{-1}(n-1)n-1}} = \mathbb{Z}^{2^{2^{-1}(n-2)(n+1)}}.$$

Let $S^k/S^k_{\mathbb{Z}} = V_k$ for $1 \le k \le n-1$ and $(S^1/S^1_{\mathbb{Z}}) \times \cdots \times (S^k/S^k_{\mathbb{Z}}) = U_k$. Since we have

$$C((S_1/S_{\mathbb{Z}}) \times \cdots \times (S^{n-1}/S^{n-1}_{\mathbb{Z}})) \cong C(S_1/S_{\mathbb{Z}}) \otimes \cdots \otimes C(S^{n-1}/S^{n-1}_{\mathbb{Z}}),$$

the Künneth formula implies that, for instance,

$$K_0(C(U_3)) \cong \oplus_{(i_1,i_2,i_3)\in I_3} K_{i_1}(C(V_1)) \otimes K_{i_2}(C(V_2)) \otimes K_{i_3}(C(V_3)),$$
$$K_1(C(U_3)) \cong \oplus_{(j_1,j_2,j_3)\in J_3} K_{j_1}(C(V_1)) \otimes K_{j_2}(C(V_2)) \otimes K_{j_3}(C(V_3)),$$

where

$$I_3 = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\},$$
$$J_3 = \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\},$$

where note that for each tuple in I_3, the number of 0 is 3 or 1 odd, while for each tuple in J_3, the number of 0 is 2 or 0 even, and the cardinal numbers of I_3 and J_3 are computed as:

$$|I_3| = {}_3C_3 + {}_3C_1 = 1 + 3 = 2^2, \quad |J_3| = {}_3C_2 + {}_3C_0 = 3 + 1 = 2^2,$$

where ${}_nC_k$ means the combination of k elements in n elements. As one more example, similarly,

$$|I_4| = {}_4C_4 + {}_4C_2 + {}_4C_0 = 1 + 6 + 1 = 2^3,$$
$$|J_4| = {}_4C_3 + {}_4C_1 = 4 + 4 = 2^3.$$

Therefore, more generally, we have

$$K_0(C(U_k)) \cong \oplus_{(i_1,\cdots,i_k)\in I_k} K_{i_1}(C(V_1)) \otimes \cdots \otimes K_{i_k}(C(V_k)),$$
$$K_1(C(U_k)) \cong \oplus_{(j_1,\cdots,j_k)\in J_k} K_{j_1}(C(V_1)) \otimes \cdots \otimes K_{j_k}(C(V_k)),$$

where if k is even, then

$$|I_k| = {}_kC_k + {}_kC_{k-2} + \cdots + {}_kC_0 = 2^k,$$
$$|J_k| = {}_kC_{k-1} + {}_kC_{k-3} + \cdots + {}_kC_1 = 2^k.$$

and if k is odd, then

$$|I_k| = {}_kC_k + {}_kC_{k-2} + \cdots + {}_kC_1 = 2^k,$$
$$|J_k| = {}_kC_{k-1} + {}_kC_{k-3} + \cdots + {}_kC_0 = 2^k,$$

and in both cases, I_k and J_k consist of tuples with elements 0 or 1 chosen accordingly to the above combinatorial sums.

Note that the quotient space V_{k-1} is just

$$V_{k-1} = S^{k-1}/\sqcup^k \mathbb{Z}_2 = (S^{k-1} \setminus (\sqcup^k \mathbb{Z}_2))^+ \equiv \mathcal{V}_k^+$$

the one-point compactification $\mathcal{V}_k^+$ of the open subspace $\mathcal{V}_k$ of S^{k-1}obtained by removing points of $\sqcup^n \mathbb{Z}_2$ from S^{k-1}.

Consider the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\mathcal{V}_k) \xrightarrow{i} C(\mathcal{V}_k^+) \xrightarrow{q} \mathbb{C} \longrightarrow 0.$$

Note that this extension of C^*-algebras splits, clearly. We then have the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(C_0(\mathcal{V}_k)) & \xrightarrow{i_*} & K_0(C(\mathcal{V}_k^+)) & \xrightarrow{q_*} & K_0(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}) & \xleftarrow{q_*} & K_1(C(\mathcal{V}_k^+)) & \xleftarrow{i_*} & K_1(C_0(\mathcal{V}_k)), \end{array}$$

and the commutative diagram also splits into two short exact sequences of K_0 and K_1-groups. It follows that

$$K_0(C(\mathcal{V}_k^+)) \cong K_0(C_0(\mathcal{V}_k)) \oplus \mathbb{Z}, \quad K_1(C(\mathcal{V}_k^+)) \cong K_1(C_0(\mathcal{V}_k)).$$

Moreover consider the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\mathcal{V}_k) \xrightarrow{i} C(S^{k-1}) \xrightarrow{q} \oplus^{2k}\mathbb{C} \longrightarrow 0$$

corresponding to attaching $2k$ points to $2k$ holes in $\mathcal{V}_k$ to make S^{k-1}. We then have the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(C_0(\mathcal{V}_k)) & \xrightarrow{i_*} & K_0(C(S^{k-1})) & \xrightarrow{q_*} & \oplus^{2k}K_0(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ \oplus^{2k}K_1(\mathbb{C}) & \xleftarrow{q_*} & K_1(C(S^{k-1})) & \xleftarrow{i_*} & K_1(C_0(\mathcal{V}_k)). \end{array}$$

Furthermore consider the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}^{k-1}) \xrightarrow{i} C(S^{k-1}) \xrightarrow{q} \mathbb{C} \longrightarrow 0,$$

where note that $S^{k-1} \approx (\mathbb{R}^{n-1})^+$. Note that this extension of C^*-algebras splits, clearly. We then have the following six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{R}^{k-1})) & \xrightarrow{i_*} & K_0(C(S^{k-1})) & \xrightarrow{q_*} & K_0(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}) & \xleftarrow{q_*} & K_1(C(S^{k-1})) & \xleftarrow{i_*} & K_1(C_0(\mathbb{R}^{k-1})) \end{array}$$

and the commutative diagram also splits into two short exact sequences of K_0 and K_1-groups. It follows that for $k \geq 2$,

$$K_0(C(S^{k-1})) \cong K_0(C_0(\mathbb{R}^{k-1})) \oplus \mathbb{Z} \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even}, \\ \mathbb{Z}^2 & \text{if } k \text{ odd}; \end{cases}$$

$$K_1(C(S^{k-1})) \cong K_1(C_0(\mathbb{R}^{k-1})) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even}, \\ 0 & \text{if } k \text{ odd}. \end{cases}$$

Therefore, we obtain that if k is even, then

$$\begin{array}{ccccc} K_0(C_0(\mathcal{V}_k)) & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{q_*} & \oplus^{2k}\mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{q_*} & \mathbb{Z} & \xleftarrow{i_*} & K_1(C_0(\mathcal{V}_k)) \end{array}$$

and if k is odd, then

$$\begin{array}{ccccc} K_0(C_0(\mathcal{V}_k)) & \xrightarrow{i_*} & \mathbb{Z}^2 & \xrightarrow{q_*} & \oplus^{2k}\mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{q_*} & 0 & \xleftarrow{i_*} & K_1(C_0(\mathcal{V}_k)). \end{array}$$

In both cases, the K_0-class corresponding to the unit of $C(S^{k-1})$ is mapped injectively under the map q_*, while the K_0-class corresonding to the Bott projection in a matrix algebra over $C(S^{k-1})$ for k odd is mapped to zero under q_*. It follows that if k is even, then $K_0(C(\mathcal{V}_k)) \cong 0$, while if k is odd, then $K_0(C(\mathcal{V}_k)) \cong \mathbb{Z}$. Therefore, we obtain that if k is even, then $K_1(C_0(\mathcal{V}_k)) \cong \mathbb{Z}^{2k}$, and if k is odd, then $K_1(C_0(\mathcal{V}_k)) \cong \mathbb{Z}^{2k-1}$. Hence we get

$$K_0(C(V_{k-1})) \cong K_0(C(\mathcal{V}_k^+)) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even,} \\ \mathbb{Z}^2 & \text{if } k \text{ odd;} \end{cases}$$

$$K_1(C(V_{k-1})) \cong K_1(C(\mathcal{V}_k^+)) \cong \begin{cases} \mathbb{Z}^{2k} & \text{if } k \text{ even,} \\ \mathbb{Z}^{2k-1} & \text{if } k \text{ odd.} \end{cases}$$

Note that the case where $k = 2$ is considered in the previous section.

Summing up the argument above, we obtain

Theorem 4.1. *Let* $H_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z}) = KAN/K_{\mathbb{Z}}A_{\mathbb{Z}}N_{\mathbb{Z}}$ *be the homogeneous space via the Iwasawa decomposition. Then* H_n *is homeomorphic to the product space* $(SO(n)/SO(n)_{\mathbb{Z}}) \times \mathbb{R}^{n-1} \times \mathbb{T}^{\frac{(n-1)n}{2}}$*, and*

$$\begin{aligned} K_0(C_0(H_n)) &\cong K_1(C_0(H_n)) \\ &\cong \oplus_{j=0,1}(K_j(C(SO(n)/SO(n)_{\mathbb{Z}})) \otimes \mathbb{Z}^{2^{(n-2)(n+1)2^{-1}}}). \end{aligned}$$

Proof. If n is even, then

$$\begin{aligned} K_0(C_0(H_n)) &\cong K_1(C(SO(n)/SO(n)_{\mathbb{Z}}) \otimes C(\mathbb{T}^{\frac{(n-1)n}{2}})) \\ &\cong (K_0(C(T_n)) \otimes \mathbb{Z}^{2^{\frac{(n-2)(n+1)}{2}}}) \oplus K_1(C(T_n)) \otimes \mathbb{Z}^{2^{\frac{(n-2)(n+1)}{2}}}), \\ K_1(C_0(H_n)) &\cong K_0(C(SO(n)/SO(n)_{\mathbb{Z}}) \otimes C(\mathbb{T}^{\frac{(n-1)n}{2}})) \\ &\cong (K_0(C(T_n)) \otimes \mathbb{Z}^{2^{\frac{(n-2)(n+1)}{2}}}) \oplus K_1(C(T_n)) \otimes \mathbb{Z}^{2^{\frac{(n-2)(n+1)}{2}}}), \end{aligned}$$

where $T_n = SO(n)/SO(n)_{\mathbb{Z}}$ for short, and in particular, we get $K_0(C_0(H_n)) \cong K_1(C_0(H_n))$.

If n is odd, then we can deduce the same conclusions by the same calculation as above. □

Remark. The results obtained above and below in K-theory might contain (some of) K-theory data for the (full or reduced) group C*-algebra of $SL_n(\mathbb{Z})$ or the (full or reduced) free product C*-algebra corresponding to the generators of $SL_n(\mathbb{Z})$. It is known that if $n \geq 3$, then $SL_n(\mathbb{Z})$ is not an amalgam, but a certain multi-amalgam of subgroups, by Soulé [4].

Moreover, we obtain

Proposition 4.2. Let $K/K_{\mathbb{Z}} = SO(n)/SO(n)_{\mathbb{Z}} = KAN/K_{\mathbb{Z}}AN$ be the homogeneous space of the compact group $SO(n)$. For convenience, as a candidate, we replace $K/K_{\mathbb{Z}}$ with the compact product space:

$$(S^1/S^1_{\mathbb{Z}}) \times (S^2/S^2_{\mathbb{Z}}) \cdots \times (S^{n-1}/S^{n-1}_{\mathbb{Z}}),$$

which is identified with

$$\begin{aligned}&(S^1/\sqcup^2 \mathbb{Z}_2) \times (S^2/\sqcup^3 \mathbb{Z}_2) \times \cdots \times (S^{n-1}/\sqcup^n \mathbb{Z}_2)\\ &\approx (S^1 \setminus \sqcup^2\mathbb{Z}_2)^+ \times (S^2 \setminus \sqcup^3\mathbb{Z}_2)^+ \times \cdots \times (S^{n-1} \setminus \sqcup^n\mathbb{Z}_2)^+,\end{aligned}$$

or we may assume that we replace the topological K-theory of $K/K_{\mathbb{Z}}$ with that of the product space. Then

$$\begin{aligned}K_0(C(K/K_{\mathbb{Z}})) &\cong \oplus_{(i_1,i_2,\cdots,i_{n-1})\in I_{n-1}}(K_{i_1}(C(V_1)) \otimes \cdots \otimes K_{i_{n-1}}(C(V_{n-1}))),\\ K_1(C(K/K_{\mathbb{Z}})) &\cong \oplus_{(j_1,j_2,\cdots,j_{n-1})\in J_{n-1}}(K_{j_1}(C(V_1)) \otimes \cdots \otimes K_{j_{n-1}}(C(V_{n-1}))),\end{aligned}$$

with $V_k = S^k/S^k_{\mathbb{Z}}$, where if n is odd , then

$$\begin{aligned}|I_{n-1}| &= {}_{n-1}C_{n-1} + {}_{n-1}C_{n-3} + \cdots + {}_{n-1}C_0 = 2^{n-1},\\ |J_{n-1}| &= {}_{n-1}C_{n-2} + {}_{n-1}C_{n-4} + \cdots + {}_{n-1}C_1 = 2^{n-1}.\end{aligned}$$

and if n is even, then

$$\begin{aligned}|I_{n-1}| &= {}_{n-1}C_{n-1} + {}_{n-1}C_{n-3} + \cdots + {}_{n-1}C_1 = 2^{n-1},\\ |J_{n-1}| &= {}_{n-1}C_{n-2} + {}_{n-1}C_{n-4} + \cdots + {}_{n-1}C_0 = 2^{n-1},\end{aligned}$$

and in both cases, I_{n-1} and J_{n-1} consist of the tuples with elements 0 or 1 chosen accordingly to the above combinatorial sums.

Moreover, we obtain

$$K_0(C(V_{k-1})) \cong \begin{cases}\mathbb{Z} & \text{if } k \text{ even},\\ \mathbb{Z}^2 & \text{if } k \text{ odd};\end{cases}$$

$$K_1(C(V_{k-1})) \cong \begin{cases}\mathbb{Z}^{2k} & \text{if } k \text{ even},\\ \mathbb{Z}^{2k-1} & \text{if } k \text{ odd}.\end{cases}$$

Remark. For example, as $n = 5$ we compute

$$\begin{aligned} &K_0(C(V_1)) \otimes K_1(C(V_2)) \otimes K_1(C(V_3)) \otimes K_0(C(V_4)) \\ &\cong \mathbb{Z} \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^6 \otimes \mathbb{Z}^2 \cong \mathbb{Z}^{3\cdot 6\cdot 2} = \mathbb{Z}^{36}, \end{aligned}$$

where $(0, 1, 1, 0) \in I_4$.

Note that the quotient space $N/N_{\mathbb{Z}}$ is homeomorphic to $\mathbb{T}^{(n-1)n2^{-1}}$ as a space. Thus, $K_j(C(N/N_{\mathbb{Z}})) \cong \mathbb{Z}^{2^{(n-2)(n+1)2^{-1}}}$ for $j = 0, 1$.

Furthermore, we have

Proposition 4.3. The homogeneous space $SL_n(\mathbb{R})/K = AN$ is homeomorphic to the product space $\mathbb{R}^{n-1} \times \mathbb{T}^{(n-1)n2^{-1}}$, and $K_j(C_0(AN)) \cong \mathbb{Z}^{2^{2^{-1}(n-2)(n+1)}}$ for $j = 0, 1$.

Proof. We have

$$K_j(C_0(\mathbb{R}^{n-1} \times \mathbb{T}^{\frac{(n-1)n}{2}})) \cong K_{j+n-1}(C(\mathbb{T}^{\frac{(n-1)n}{2}})) \cong \mathbb{Z}^{2^{\frac{(n-2)(n+1)}{2}}}$$

for $j = 0, 1$. □

5 Nilpotent case

Recall that the discrete Heisenberg group $H^{\mathbb{Z}}_{2n+1}$ of rank $2n + 1$ is defined by

$$H^{\mathbb{Z}}_{2n+1} = \left\{ \begin{pmatrix} 1 & a^t & c \\ 0_n & 1_n & b \\ 0 & 0_n^t & 0 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}) \mid a, b \in \mathbb{Z}^n, c \in \mathbb{Z} \right\}$$

where 1_n is the $n \times n$ identity matrix, 0_n is the zero in $\mathbb{Z}^n$, $a, b, 0_n$ are column vectors, and x^t means the transpose of x. The Heisenberg Lie group $H^{\mathbb{R}}_{2n+1}$ with dimension $2n + 1$ is defined by replacing $\mathbb{Z}$ with $\mathbb{R}$ in the definition above. Then we have the homogeneous space:

$$H^{\mathbb{R}}_{2n+1}/H^{\mathbb{Z}}_{2n+1} \approx \mathbb{T}^{2n+1}$$

as a space.

Let $C^*(H^{\mathbb{Z}}_{2n+1})$ be the group C^*-algebra of $H^{\mathbb{Z}}_{2n+1}$. It is shown by the author [3] that for $j = 0, 1$,

$$K_j(C^*(H^{\mathbb{Z}}_{2n+1})) \cong \mathbb{Z}^{3^n}.$$

It follows that

Proposition 5.1. We have

$$K_j(C(H^{\mathbb{R}}_{2n+1}/H^{\mathbb{Z}}_{2n+1})) \cong \mathbb{Z}^{2^{2n}}$$

for $j = 0, 1$, but for $n \geq 1$,

$$K_j(C(H^{\mathbb{R}}_{2n+1}/H^{\mathbb{Z}}_{2n+1})) \not\cong K_j(C^*(H^{\mathbb{Z}}_{2n+1})).$$

Proof. Because $2^{2n} \neq 3^n$ for $n \geq 1$. □

Remark. We have $4^n > 3^n$, so that it may say to be possible that K-theory data of the homogeneous space C^*-algebra contains that of the group C^*-algebra. In fact, in the group non-isomorphic equation above, the right hand side can be a quotient of the left hand side. This picture might be extended to the more general setting.

Conjecture. Let Γ be a nilpotent discrete group with rank n. Then we have

$$\mathrm{rank}_{\mathbb{Z}} K_j(C^*(\Gamma)) \leq 2^{n-1}$$

for $j = 0, 1$, where $\mathrm{rank}_{\mathbb{Z}}(X)$ means the $\mathbb{Z}$-rank of X.

Remark. The equality holds if $\Gamma = \mathbb{Z}^n$ and the estimate is ture if $\Gamma = H^{\mathbb{Z}}_{2n+1}$ as checked above.

It is certainly known that a discrete nilpotent group Γ can be viewed as a subgroup of matrices, i.e. to be linear. Also, it can be viewed as a successive semi-direct products by the abelian groups $\mathbb{Z}^{k_j}$ of integers for some $k_j \geq 1$ $(1 \leq j \leq n)$. In this case, Γ is a subgroup of the connected, simply connected nilpotent Lie group G obtained as a a successive semi-direct products by $\mathbb{R}^{k_j}$, so that the homogeneous space G/Γ is homeomorphic to:

$$G/\Gamma \approx \mathbb{T}^{\sum_{j=1}^n k_j}.$$

Our conjecture says that

$$\mathrm{rank}_{\mathbb{Z}} K_j(C^*(\Gamma)) \leq 2^{-1+\sum_{j=1}^n k_j}.$$

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