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# Units in Abelian Group Algebras Over Direct Products of Indecomposable Rings

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**ABSTRACT**

Let $R$ be a commutative unitary ring of prime characteristic $p$ which is a direct product of indecomposable subrings and let $G$ be a multiplicative Abelian group such that $G_0/G_p$ is finite. We characterize the isomorphism class of the unit group $U(RG)$ of the group algebra $RG$. This strengthens recent results due to Mollov-Nachev (Commun. Algebra, 2006) and Danchev (Studia Babes Bolyai - Mat., 2011).

**RESUMEN**

Sea $R$ un anillo conmutativo y unitario de característica prima $p$, que es producto directo de subanillos indescomponibles y sea $G$ un grupo multiplicativo y abeliano tal que $G_0/G_p$ es finito. Caracterizamos las clases de isomorfismo del grupo unitario $U(RG)$ del álgebra del grupo $RG$. Estos fuertes y recientes resultados se deben a Mollov-Nachev (Commun. Algebra, 2006) and Danchev (Studia Babes Bolyai - Mat., 2011).

**Keywords and Phrases:** groups, rings, group rings, indecomposable rings, units, direct decompositions, isomorphisms.

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# 1 Introduction

Throughout the current paper, suppose $R$ is a commutative unitary (i.e., with identity) ring of prime characteristic $p$ and suppose $G$ is a multiplicative Abelian group as is the custom when discussing group rings. For such $R$ and $G$, we denote by $RG$ the group ring of $G$ over $R$ with unit group $U(RG)$, normalized subgroup $V(RG)$ of units (with augmentation $1$) and its idempotent subgroup $Id(RG)$. Note that the decomposition $U(RG) = V(RG) \times U(R)$ is valid, where $U(R)$ is the unit group of $R$. As usual, $G_0$ is the maximal torsion subgroup of $G$ with $p$-torsion component $G_p$, and $S(RG) = V_p(RG)$ is the $p$-torsion component of $V(RG)$. Besides, for any natural number $n$, $\zeta_n$ denotes the primitive $n$th root of unity and $R[\zeta_n]$ is the free $R$-module, generated algebraically as a ring by $\zeta_n$, with dimension $[R[\zeta_n] : R]$. As it is well-known, a ring is said to be *indecomposable* if it cannot be decomposed into a direct sum of two or more non-trivial subrings (ideals), that is, this ring possesses only the trivial idempotents $0, 1$.

The algebraic structures of $V(RG)$ and $U(RG)$ have been very intensively explored in the past twenty years (see, e.g., [K]). In this aspect, some isomorphism description results were obtained in [Da] and [MN], respectively. The purpose of this work is to improve considerably one of the central achievements in the second citation by giving a more direct and conceptual proof (some of parts of the proof of the corresponding result in [MN] are unnecessary intricated). Likewise, we generalize the main result in [Dg] to a ring which is an arbitrary direct product of indecomposable rings.

Notice that our method suggested below gives a new perspective for establishing some other results of this form, because it leads the general case to the $p$-mixed one.

## II. Main Results

As noted above, Mollov and Nachev obtained in ([MN], Theorem 5.8) the following statement.

**Theorem** (2006). *Let $R$ be a commutative indecomposable ring with identity of prime characteristic $p$ and let $G$ be a splitting Abelian group. Suppose that $G_0/G_p$ is a finite group of exponent $n$ and $n \in U(R)$. Then*

$$U(RG) \cong \coprod_{d/n} \coprod_{\lambda(d)} U(R[\zeta_d]) \times \coprod_{b} G/G_0 \times \coprod_{d/n} \coprod_{\lambda(d)} S(R[\zeta_d](G_p \times G/G_0))$$

*where $\lambda(d) = \frac{(G_0/G_p)(d)}{[R[\zeta_d]:R]}$, with $(G_0/G_p)(d)$ the number of elements of $G_0/G_p$ of order $d$, and $b = \sum_{d/n} \lambda(d)$.*

Note that since $char(R) = p$ is a prime integer, it is self-evident that $exp(G_0/G_p)$ inverts in $R$, so that the condition $n \in U(R)$ is always fulfilled and hence it is a superfluously stated in the theorem.

In [Dg] we dropped the limitation that $G$ is a splitting group. Specifically, we list the following:

**Theorem** (2011). *Suppose* $R$ *is an indecomposable ring of* $\mathrm{char}(R) = p$ *and* $G$ *is a group for which* $G_0/G_p$ *is finite. Then the following isomorphism is true:*

(*)

$$U(RG) \cong \coprod_{d/\exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times [(G/\coprod_{q\neq p} G_q)V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))]]$$

*where* $a(d) = \frac{|\{g\in G_0/G_p : order(g)=d\}|}{[R[\zeta_d]:R]}$.

*In particular:*

*(1) if* $G$ *is* $p$*-splitting, then*

$$U(RG) \cong \coprod_{d/\exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))] \times \coprod_{\sum_{d/\exp(G_0/G_p)} a(d)} G/G_0.$$

*(2) if* $G_p$ *is a direct sum of cyclic groups, then*

$$U(RG) \cong \coprod_{d/\exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times (V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p)] \times$$

$$\times \coprod_{\sum_{d/\exp(G_0/G_p)} a(d)} G/\coprod_{q\neq p} G_q.$$

*Moreover, the quotient* $V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p)$ *is a direct sum of cyclic groups by [D] and can be characterized via the Ulm-Kaplansky invariants calculated in [Df].*

Before stating and proving our chief attainment, we need two more preliminaries.

**Proposition 1**. *Let* $R = \prod_{i\in I} R_i$ *be a direct product of subrings* $R_i$ *where* $I$ *is an index set, and* $F$ *is a finite abelian group. Then the following isomorphism holds:*

$$RF \cong \prod_{i\in I} R_i F.$$

*Proof.* It is straightforward and we leave it to the reader. △

**Lemma 2**. *Suppose* $G_0/G_p$ *is bounded. Then the following decomposition is true:*

$$G = M \times B$$

*where* $M \cong G/\coprod_{q\neq p} G_q$ *is* $p$*-mixed and* $B \cong \coprod_{q\neq p} G_q \cong G_0/G_p$ *is bounded.*

*Proof.* Since $\coprod_{q\neq p} G_q$ is bounded and is pure in $G_0$ as its direct factor, whence pure in $G$, it follows that $\coprod_{q\neq p} G_q$ is a direct factor of $G$ as well. Denoting $B = \coprod_{q\neq p} G_q$, one may write $G = B \times M$ where $M \cong G/B$. It is obvious that $M$ is $p$-mixed, i.e., $M_0 = M_p$. △

So, we come to our main achievement.

**Theorem 3**. *Let* $R$ *be a ring of prime characteristic* $p$ *which is a direct product of indecomposable rings* $R_i$ *for some index set* $I$*, and let* $G$ *be an abelian group such that* $G_0/G_p$ *is finite. Then the following isomorphism formula is fulfilled*:

(*)

$$U(RG) \cong [\coprod_{i\in I} U(R_i(G_0/G_p))] \times [Id(LM)V_p(LM)]$$

*for some commutative unitary ring* $L$ *of prime characteristic* $p$ *which is a direct product of indecomposable rings, and where, for all indices* $i \in I$,

$$U(R_i(G_0/G_p)) \cong \coprod_{d/\exp(G_0/G_p)} \coprod_{a_i(d)} U(R_i[\zeta_d])$$

*with* $a_i(d) = \frac{|\{g\in G_0/G_p : order(g)=d\}|}{[R_i[\zeta_d]:R_i]}$.

*In particular, the maximal divisible subgroup* $dU(RG)$ *of* $U(RG)$ *is completely described up to isomorphism.*

*Proof.* According to Lemma 2 one may write $G = F \times M$ where $F \cong \coprod_{q\neq p} G_q$ is finite and $M$ is $p$-mixed. Thus $RG = (RF)M = LM$ where we put $RF = L$. Therefore, $U(RG) = U(LM) = U(RF) \times V(LM)$. Concerning $V(LM)$ we may write $V(LM) = Id(LM)V_p(LM)$ (see, e.g., [Dd] or [De]).

On the other hand, owing to Proposition 1, $L = RF = (\prod_{i\in I} R_i)F \cong \prod_{i\in I} R_iF$ where each $R_i$ is an indecomposable ring of characteristic $p$. Furthermore, since $F$ is finite of exponent that inverts in $R$, and hence it inverts in each $R_i$, appealing to Theorem 4.4 and Remark 4.5 of [MN], every $R_iF$ is a finite direct sum of indecomposable subrings. Consequently, $L$ is a commutative unitary ring of prime characteristic $p$ which can be interpreted as a ring that is a direct product of indecomposable subrings. Moreover, $U(RF) \cong \coprod_{i\in I} U(R_iF)$, where $U(R_iF)$ has an explicit description for any index $i$. Thus formula (*) is deduced.

Finally, observe that $dU(RG) = dU(RF) \times dV(LM) \cong \coprod_{i\in I} dU(R_iF) \times dV(LM)$. Since $U(R_iF)$, and hence $dU(R_iF)$, is already characterized above, and $dV(LM)$ is classified in [Dd] and [De], we infer that the same can be said of $dU(RG)$. $\triangle$

**Remark**. The proof of Theorem 2.7 from [MMN] contains a gap and so it is uncomplete. In fact, the authors claimed that they will assume that the splitting group is $p$-mixed. The reason is that the $K$-algebras isomorphism $KG \cong KH$ yields that $K(G/\coprod_{q\neq p} G_q) \cong K(H/\coprod_{q\neq p} H_q)$ whenever $K$ is a field of $char(K) = p$. But they need to show that $G$ being splitting ensures that so is $G/\coprod_{q\neq p} G_q$. However, this was already done in [Db].

We close the work with the following problem.

**Conjecture**. Suppose $R$ is an indecomposable ring and $G$ is a finite group of exponent which inverts in $R$. Then $RG \cong RH$ for some group $H$ if, and only if, $H$ is finite with the same exponent as that of $G$ and $RG_p \cong RH_p$ for each prime number $p$.

Notice that the sufficiency is trivial, because $G$ and $H$ being both bounded implies that $G = \coprod_p G_p$ and $H = \coprod_p H_p$, whence $RG \cong \otimes_R RG_p$ and $RH \cong \otimes_R RH_p$. Thus $RG_p \cong RH_p$ forces that $RG \cong RH$, as desired.

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