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Partial Fractions and q-Binomial Determinant Identities

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ABSTRACT

Partial fraction decomposition method is applied to evaluate a general determinant of shifted factorial fractions, which contains several Gaussian binomial determinant identities.

RESUMEN

El método de descomposición en fracción parciales aplicado para evaluar un determinante general de fracciones factoriales trasladadas, la cual contiene varias identidades determinante binomial Gaussiano.

Key words and phrases: *The Cauchy double alternant, Partial fractions, q-Binomial coefficients.*

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Binomial determinant evaluation plays an important role in combinatorial enumeration, particularly in plane partitions. This paper will establish a very general determinant identity through partial fraction decomposition method. It will be shown to be useful in q-binomial determinant evaluations with several interesting known and new formulae being exemplified.

1 Partial Fraction Decomposition

For two sequences $\{\alpha_k, \gamma_k\}_{k\geq 0}$, define the generalized shifted factorials by

$$(x|\alpha)_0 = 1 \quad \text{and} \quad (x|\alpha)_n = \prod_{k=0}^{n-1}(1-x\alpha_k) \quad \text{with} \quad n \in \mathbb{N}, \tag{1a}$$

$$(y|\gamma)_0 = 1 \quad \text{and} \quad (y|\gamma)_n = \prod_{k=0}^{n-1}(1-y\gamma_k) \quad \text{with} \quad n \in \mathbb{N}. \tag{1b}$$

When $\alpha_k = \gamma_k = q^k$ for $k \in \mathbb{N}_0$, they will reduce to the usual shifted factorials

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{with} \quad n \in \mathbb{N}. \tag{2}$$

For the triangular matrix given by $\alpha = [\alpha_{ij}]_{0\leq i\leq j<\infty}$, denote its j-th column by $\alpha_j = (\alpha_{0j}, \alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{jj})$. Then the main result may be stated as follows.

Theorem 1 (Generalized Cauchy determinant)**.** *Let* $\{x_k\}_{k=0}^n$ *be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\leq i,j\leq n}\left[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}}\right] = \frac{\prod_{0\leq i<j\leq n}(x_i - x_j)(\alpha_{ij} - \gamma_j)}{\prod_{k=0}^{n}(x_k|\gamma)_{n+1}}.$$

The very special case of this theorem with $\alpha_{ij} = \gamma_i$ for $i,j \in \mathbb{N}_0$ results in the celebrated Cauchy's double alternant (cf. [6, 7]):

$$\det_{0\leq i,j\leq n}\left[\frac{1}{1-x_i\gamma_j}\right] = \frac{\prod_{0\leq i<j\leq n}(x_i - x_j)(\gamma_i - \gamma_j)}{\prod_{0\leq i,j\leq n}(1-x_i\gamma_j)}. \tag{3}$$

Proof. Expanding the rational function in partial fractions, we have

$$\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(1-x_i\alpha_{\iota j})}{\prod_{k=0}^{j}(1-x_i\gamma_k)} = \sum_{k=0}^{j}\frac{w_{kj}}{1-x_i\gamma_k}$$

where the connected coefficients are determined by the following limit relation

$$w_{kj} = \lim_{x_i\to\frac{1}{\gamma_k}}(1-x_i\gamma_k)\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(\alpha_{\iota j}-\gamma_k)}{\prod_{\iota=0:\,\iota\neq k}^{j}(\gamma_\iota-\gamma_k)}.$$

This leads us to the following determinant factorization

$$\det_{0\le i,j\le n}\left[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}}\right] = \det_{0\le i,k\le n}\left[\frac{1}{1-x_i\gamma_k}\right]\times\det_{0\le k,j\le n}\left[w_{kj}\right].$$

For the matrix $\left[w_{kj}\right]_{0\le k,j\le n}$ is upper triangular, its determinant is equal to the product of its diagonal entries:

$$\det_{0\le k,j\le n}\left[w_{kj}\right] = \prod_{j=0}^{n}w_{jj} = \prod_{0\le i<j\le n}\frac{\alpha_{ij}-\gamma_j}{\gamma_i-\gamma_j}.$$

While the first determinant can be evaluated by Cauchy's double alternant (2). Their combination yields the determinant identity stated in Theorem 1. ☐

Shifting the γ-parameters by $\gamma_k\to\gamma_{k-1}$, we may state the determinant identity in Theorem 1 in the following more convenient form.

Proposition 2 (Determinant identity)**.** *Let $\{x_k\}_{k=0}^{n}$ be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\le i,j\le n}\left[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)(\alpha_{ij}-\gamma_{j-1})}{\prod_{k=0}^{n}(x_k|\gamma)_n}.$$

Letting $\alpha_{ij}=p^iy_j$ and $\gamma_k=q^k$ further in Proposition 2, we have the identity.

Corollary 3 (Bibasic determinant evaluation formula)**.**

$$\det_{0\le i,j\le n}\left[\frac{(x_iy_j;p)_j}{(x_i;q)_j}\right] = q^{2\binom{n+1}{3}}\prod_{0\le i<j\le n}(x_j-x_i)\prod_{k=0}^{n}\frac{(q^{1-k}y_k;p)_k}{(x_k;q)_n}.$$

From this corollary, we can derive numerous q-binomial determinant identities.

2 q-Binomial Determinant Identities

Define the Gaussian binomial coefficients by

$$\begin{bmatrix}x\\n\end{bmatrix} = \frac{(q^{1+x-n};q)_n}{(q;q)_n}\quad\text{where}\quad n\in\mathbb{N}_0\quad\text{and}\quad x\in\mathbb{C}.$$

Applying Corollary 3, we show now ten classes of q-binomial determinant identities.

2.1 Expressing the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} X_i - j \\ A \end{bmatrix} = q^{-Aj} \begin{bmatrix} X_i \\ A \end{bmatrix} \frac{(q^{A-X_i};q)_j}{(q^{-X_i};q)_j}$$

we derive the corresponding determinant formula

$$\det_{0\le i,j\le n} \left[\begin{bmatrix} X_i - j \\ A \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_j} - q^{-X_i})(1-q^{1+A+i-j}) \tag{4a}$$

$$\times \frac{q^{2\binom{n+1}{3}-A\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} X_k \\ A \end{bmatrix} \begin{bmatrix} n-1-X_k \\ n \end{bmatrix}^{-1}. \tag{4b}$$

2.2 Rewriting the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} A \\ X_i - j \end{bmatrix} = (-1)^j q^{-\binom{j}{2}+jX_i} \begin{bmatrix} A \\ X_i \end{bmatrix} \frac{(q^{-X_i};q)_j}{(q^{1+A-X_i};q)_j}$$

we get the corresponding determinant identity

$$\det_{0\le i,j\le n} \left[q^{-jX_i} \begin{bmatrix} A \\ X_i - j \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_i} - q^{-X_j})(1-q^{-A+i-j}) \tag{5a}$$

$$\times \frac{q^{\binom{n+1}{3}+(1+A)\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} A \\ X_k \end{bmatrix} \begin{bmatrix} A+n-X_k \\ n \end{bmatrix}^{-1}. \tag{5b}$$

2.3 Reformulating the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} A + X_i - j \\ X_i - j \end{bmatrix} = q^{-Aj} \begin{bmatrix} A + X_i \\ A \end{bmatrix} \frac{(q^{-X_i};q)_j}{(q^{-A-X_i};q)_j}$$

we obtain the following determinant evaluation formula

$$\det_{0\le i,j\le n} \left[\begin{bmatrix} A + X_i - j \\ X_i - j \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_j} - q^{-X_i})(1-q^{1+A+i-j}) \tag{6a}$$

$$\times \frac{q^{2\binom{n+1}{3}-2A\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} A + X_k \\ A \end{bmatrix} \begin{bmatrix} -1-A+n-X_k \\ n \end{bmatrix}^{-1}. \tag{6b}$$

2.4 Applying the q-binomial relation

$$\begin{bmatrix} X_i + j \\ A \end{bmatrix} = \begin{bmatrix} X_i \\ A \end{bmatrix} \frac{(q^{1+X_i};q)_j}{(q^{1-A+X_i};q)_j}$$

we find the corresponding determinant formula

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+j\\A\end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+i-j}) \tag{7a}$$

$$\times\frac{q^{2\binom{n+1}{3}+(1-A)\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}X_k\\A\end{bmatrix}\begin{bmatrix}X_k-A+n\\n\end{bmatrix}^{-1}. \tag{7b}$$

2.5 Observing the q-binomial relation

$$\begin{bmatrix}A\\X_i+j\end{bmatrix}=(-1)^j q^{(A-X_i)j-\binom{j}{2}}\begin{bmatrix}A\\X_i\end{bmatrix}\frac{(q^{-A+X_i};q)_j}{(q^{1+X_i};q)_j}$$

we recover the determinant identity due to Carlitz [4] (cf. Chu [5] also)

$$\det_{0\le i,j\le n}\left[q^{jX_i}\begin{bmatrix}A\\X_i+j\end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{-A+i-j}) \tag{8a}$$

$$\times\frac{q^{\binom{n+1}{3}+(1+A)\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}A\\X_k\end{bmatrix}\begin{bmatrix}X_k+n\\n\end{bmatrix}^{-1}. \tag{8b}$$

2.6 By invoking the q-binomial relation

$$\begin{bmatrix}A+X_i+j\\X_i+j\end{bmatrix}=\begin{bmatrix}A+X_i\\A\end{bmatrix}\frac{(q^{1+A+X_i};q)_j}{(q^{1+X_i};q)_j}$$

we recover another determinant identity due to Carlitz [4] (see Menon [9] also)

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}A+X_i+j\\X_i+j\end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+i-j}) \tag{9a}$$

$$\times\frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}A+X_k\\A\end{bmatrix}\begin{bmatrix}X_k+n\\n\end{bmatrix}^{-1} \tag{9b}$$

which reduces, for $q\to 1$, to the binomial determinant of Ostrowski [10].

Furthermore for $\delta=0,1$, we can show the following determinant identity

$$\det_{0\le i,j\le n}\left[C^{(\delta)}_{X_i+j}(q)\right] = (2q)^{(1+n)(1+n+\delta)+2\sum_{\iota=0}^{n}X_\iota}q^{n(n+1)(1+2n+6\delta)/6} \tag{10a}$$

$$\times\prod_{k=0}^{n}\frac{(q;q^2)_{1+k}(q;q^2)_{\delta+X_k}}{(q^2;q^2)_{1+\delta+X_k+n}}\prod_{0\le i<j\le n}(q^{2X_i}-q^{2X_j}) \tag{10b}$$

where the q-Catalan numbers due to Andrews [2] has been slightly extended by

$$C_n^{(\delta)}(q):=\frac{(2q)^{1+\delta+2n}}{1-q^{2+2\delta+2n}}\begin{bmatrix}\delta+2n\\n\end{bmatrix}\frac{1-q}{(-q;q)_n(-q;q)_{\delta+n}}. \tag{11}$$

When $x_k = k+\ell$, we get the following Hankel determinant identity

$$\det_{0\le i,j\le n}\Big[C^{(\delta)}_{i+j+\ell}(q)\Big] = (2q)^{(1+n)(1+\delta+2n+2\ell)}q^{n(n+1)(4n+6\ell+6\delta-1)/6} \tag{12a}$$

$$\times \prod_{k=0}^{n}\frac{(q;q)_{1+2k}(q;q^2)_{\delta+k+\ell}}{(q^2;q^2)_{1+\delta+k+n+\ell}}. \tag{12b}$$

Letting $\delta=0$ and $q\to 1$, we recover further the related results [1, 8, 11] on the classical Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$:

$$\det_{0\le i,j\le n}\Big[C_{i+j}\Big]=1,\quad \det_{0\le i,j\le n}\Big[C_{i+j+1}\Big]=1,\quad \det_{0\le i,j\le n}\Big[C_{i+j+2}\Big]=n+2. \tag{13}$$

2.7 By means of the q-binomial relation

$$\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ j\end{bmatrix}^{-1} = q^{(Y_j-A)j}\frac{(q^{-X_i-Y_j};q)_j}{(q^{-A-X_i};q)_j}$$

we get the following determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ j\end{bmatrix}^{-1}\right] = \frac{q^{2\binom{n+1}{3}-\sum_{k=0}^{n}(2kA+nX_k-kY_k)}}{(q;q)_n^{n+1}\prod_{k=0}^{n}\begin{bmatrix}n-1-A-X_k\\ n\end{bmatrix}} \tag{14a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{1+A-Y_j+i-j}). \tag{14b}$$

2.8 In view of the q-binomial relation

$$\begin{bmatrix}X_i+Y_j+j\\ Y_j\end{bmatrix}\begin{bmatrix}X_i+Y_j\\ Y_j\end{bmatrix}^{-1} = \frac{(q^{1+X_i+Y_j};q)_j}{(q^{1+X_i};q)_j}$$

we obtain the corresponding determinant formula

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j+j\\ Y_j\end{bmatrix}\begin{bmatrix}X_i+Y_j\\ Y_j\end{bmatrix}^{-1}\right] = \frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}X_k+n\\ n\end{bmatrix}^{-1} \tag{15a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+Y_j+i-j}). \tag{15b}$$

2.9 According to the q-binomial relation

$$\begin{bmatrix}A+X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}X_i+j\\ j\end{bmatrix}^{-1} = \frac{(q^{1+A+X_i+Y_j-j};q)_j}{(q^{1+X_i};q)_j}$$

we derive the corresponding determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}A+X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}X_i+j\\ j\end{bmatrix}^{-1}\right]=\frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}X_k+n\\ n\end{bmatrix}^{-1} \tag{16a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+Y_j+i-2j}). \tag{16b}$$

2.10 Similarly, the q-binomial relation

$$\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i-j\\ n-j\end{bmatrix}=q^{(Y_j-A)j}\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ n\end{bmatrix}\frac{(q^{-X_i-Y_j};q)_j}{(q^{-A-X_i};q)_j}$$

leads us to the following binomial determinant evaluation formulae

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i-j\\ n-j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_j}-q^{-X_i})(1-q^{1+A+i-j-Y_j}) \tag{17a}$$

$$\times\frac{q^{\sum_{k=0}^{n}(k-1-2A+Y_k)k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}A+X_k\\ n\end{bmatrix}}{\begin{bmatrix}-1-A+n-X_k\\ n\end{bmatrix}}, \tag{17b}$$

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+j\\ j\end{bmatrix}\begin{bmatrix}A+X_i+Y_j\\ n-j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_i}-q^{-X_j})(1-q^{1+n-A-Y_{n-j}+i-j}) \tag{18a}$$

$$\times\frac{q^{\sum_{k=0}^{n}(k-1+A-2n+Y_{n-k})k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}n+X_k\\ n\end{bmatrix}}{\begin{bmatrix}-1-X_k\\ n\end{bmatrix}}; \tag{18b}$$

where the last identity is derived from the first one under substitution $j\to n-j$ on the column index.

3 Duplicate Determinant Identities

Performing the parameter replacements in Proposition 2

$$\begin{aligned}x_k &\to ax_k+c/x_k,\\ \gamma_k &\to d\gamma_k/(1+acd^2\gamma_k^2),\\ \alpha_{ij} &\to b\alpha_{ij}/(1+ab^2c\alpha_{ij}^2);\end{aligned}$$

and then applying factorizations

$$\begin{aligned} x_i - x_j &\to (x_i - x_j)(a - c/x_i x_j), \\ \alpha_{ij} - \gamma_k &\to \frac{(b\alpha_{ij} - d\gamma_k)(1 - abcd\alpha_{ij}\gamma_k)}{(1 + ab^2c\alpha_{ij}^2)(1 + acd^2\gamma_k^2)}, \\ 1 - x_i\gamma_k &\to \frac{(1 - ad\gamma_k x_i)(1 - cd\gamma_k/x_i)}{1 + acd^2\gamma_k^2}, \\ 1 - x_k\alpha_{ij} &\to \frac{(1 - abx_k\alpha_{ij})(1 - bc\alpha_{ij}/x_k)}{1 + ab^2c\alpha_{ij}^2}; \end{aligned}$$

we find the following duplicate determinant identity.

Proposition 4. *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\le i,j\le n}\Big[\frac{(abx_i|\alpha_j)_j(bc/x_i|\alpha_j)_j}{(adx_i|\gamma)_j(cd/x_i|\gamma)_j}\Big] = \prod_{0\le i<j\le n}(b\alpha_{ij} - d\gamma_{j-1})(1 - abcd\alpha_{ij}\gamma_{j-1}) \times \frac{\prod_{0\le i<j\le n}(x_i - x_j)(a - c/x_i x_j)}{\prod_{k=0}^n (adx_k|\gamma)_n(cd/x_k|\gamma)_n}.$$

This identity contains the following three determinant evaluations.

Corollary 5 ($a = b = 1$ and $\gamma_k \to 0$ in Proposition 4)**.**

$$\det_{0\le i,j\le n}\Big[(x_i|\alpha_j)_j(c/x_i|\alpha_j)_j\Big] = \prod_{0\le i<j\le n}\Big\{\alpha_{ij}(x_i - x_j)(1 - c/x_i x_j)\Big\}.$$

Corollary 6 ($d = 1$ and $\alpha_{ij} \to 0$ in Proposition 4)**.**

$$\det_{0\le i,j\le n}\Big[\frac{1}{(ax_i|\gamma)_j(c/x_i|\gamma)_j}\Big] = \frac{\prod_{0\le i<j\le n}(x_j - x_i)(a - c/x_i x_j)}{\prod_{k=0}^n (ax_k|\gamma)_n(c/x_k|\gamma)_n}\prod_{\ell=1}^n \gamma_{\ell-1}^\ell.$$

Putting $\alpha_{ij} = p^i y_j$ and $\gamma_k = q^k$ in Proposition 4, we find the following determinant evaluation formula of factorial fractions with two different bases.

Corollary 7 (Bibasic determinant identity)**.**

$$\det_{0\le i,j\le n}\Big[\frac{(abx_i y_j;p)_j(bcy_j/x_i;p)_j}{(adx_i;q)_j(cd/x_i;q)_j}\Big] = d^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(x_j - x_i)(a - c/x_i x_j) \times q^{2\binom{n+1}{3}}\prod_{k=0}^n \frac{(q^{1-k}by_k/d;p)_k(q^{k-1}abcdy_k;p)_k}{(adx_k;q)_n(cd/x_k;q)_n}.$$

When $p = q$ and $y_k = 1$, it reduces to the following determinant identity

$$\det_{0\le i,j\le n}\left[\frac{(abx_i;q)_j(bc/x_i;q)_j}{(adx_i;q)_j(cd/x_i;q)_j}\right] = b^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(x_i-x_j)(a-c/x_ix_j) \tag{19a}$$

$$\times\ q^{\binom{n+1}{3}}\prod_{k=0}^{n}\frac{(d/b;q)_k(q^{k-1}abcd;q)_k}{(adx_k;q)_n(cd/x_k;q)_n}. \tag{19b}$$

The determinant evaluation formulae established in this section contain numerous q-binomial determinant identities as special cases, which will be illustrated by the following five examples.

3.1 Expressing the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+A\\ j\end{bmatrix}\begin{bmatrix}X_i-B-C\\ n-j\end{bmatrix}}{\begin{bmatrix}X_i+B\\ j\end{bmatrix}\begin{bmatrix}X_i-A-C\\ n-j\end{bmatrix}} = q^{(A-B)j}\frac{\begin{bmatrix}X_i-B-C\\ n\end{bmatrix}}{\begin{bmatrix}X_i-A-C\\ n\end{bmatrix}}\times\frac{(q^{1+X_i-A-C-n};q)_j(q^{-X_i-A};q)_j}{(q^{1+X_i-B-C-n};q)_j(q^{-X_i-B};q)_j}$$

we establish from Corollary 7 the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+A\\ j\end{bmatrix}\begin{bmatrix}X_i-B-C\\ n-j\end{bmatrix}}{\begin{bmatrix}X_i+B\\ j\end{bmatrix}\begin{bmatrix}X_i-A-C\\ n-j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{n-1+C-X_i-X_j}) \tag{20a}$$

$$\times\frac{q^{B\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n+A+B+C-k\\ k\end{bmatrix}\begin{bmatrix}B-A\\ k\end{bmatrix}\begin{bmatrix}X_k-B-C\\ n\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}X_k+B\\ n\end{bmatrix}\begin{bmatrix}n-1+B+C-X_k\\ n\end{bmatrix}\begin{bmatrix}X_k-A-C\\ n\end{bmatrix}} \tag{20b}$$

which contains, as special case, the following q-binomial determinant identity

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}\lambda i+A\\ j\end{bmatrix}\begin{bmatrix}\lambda i-B\\ n-j\end{bmatrix}}{\begin{bmatrix}\lambda i+B\\ j\end{bmatrix}\begin{bmatrix}\lambda i-A\\ n-j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{\lambda i}-q^{\lambda j})(1-q^{n-1-\lambda i-\lambda j}) \tag{21a}$$

$$\times\frac{q^{B\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n+A+B-k\\ k\end{bmatrix}\begin{bmatrix}B-A\\ k\end{bmatrix}\begin{bmatrix}\lambda k-B\\ n\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}\lambda k+B\\ n\end{bmatrix}\begin{bmatrix}n-1+B-\lambda k\\ n\end{bmatrix}\begin{bmatrix}\lambda k-A\\ n\end{bmatrix}}. \tag{21b}$$

3.2 Rewriting the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A-X_i+Y_j\\ j\end{bmatrix}}{\begin{bmatrix}B+X_i\\ j\end{bmatrix}\begin{bmatrix}A+B-X_i\\ j\end{bmatrix}} = q^{2j(Y_j-B)}\frac{(q^{-X_i-Y_j};q)_j(q^{X_i-A-Y_j};q)_j}{(q^{-X_i-B};q)_j(q^{X_i-A-B};q)_j}$$

we recover from Corollary 7 the determinant identity due to Joris Van Jeugt

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A-X_i+Y_j\\ j\end{bmatrix}}{\begin{bmatrix}B+X_i\\ j\end{bmatrix}\begin{bmatrix}A+B-X_i\\ j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{A-X_i-X_j}) \tag{22a}$$

$$\times\frac{q^{\sum_{k=0}^{n}kY_k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}1+A+B+Y_k-k\\ k\end{bmatrix}\begin{bmatrix}B-Y_k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}B+X_k\\ n\end{bmatrix}\begin{bmatrix}A+B-X_k\\ n\end{bmatrix}}. \tag{22b}$$

This identity can further be specialized to the q-binomial determinant evaluation

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}A+\lambda i+j\\ j\end{bmatrix}\begin{bmatrix}A-\lambda i+j\\ j\end{bmatrix}}{\begin{bmatrix}B+\lambda i\\ j\end{bmatrix}\begin{bmatrix}B-\lambda i\\ j\end{bmatrix}}q^{\binom{j}{2}}\right]=\prod_{0\le i<j\le n}(q^{-\lambda i}-q^{-\lambda j})(1-q^{\lambda i+\lambda j}) \tag{23a}$$

$$\times\frac{q^{\sum_{k=0}^{n}k(A+k)}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}1+A+B\\ k\end{bmatrix}\begin{bmatrix}B-A-k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}B+\lambda k\\ n\end{bmatrix}\begin{bmatrix}B-\lambda k\\ n\end{bmatrix}}. \tag{23b}$$

3.3 Reformulating the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j-A-j\end{bmatrix}}{\begin{bmatrix}X_i+Y_j\\ X_i-Y_j-A\end{bmatrix}\begin{bmatrix}X_i+C+j\\ X_i-A-C-j\end{bmatrix}}=q^{(C-Y_j)j}\frac{\begin{bmatrix}A+2C+2j\\ 2C-2Y_j\end{bmatrix}}{\begin{bmatrix}A+2C\\ A+2Y_j\end{bmatrix}\begin{bmatrix}C+X_i\\ A+2C\end{bmatrix}}\times\frac{(q^{1+X_i+Y_j};q)_j(q^{A-X_i+Y_j};q)_j}{(q^{1+X_i+C};q)_j(q^{A-X_i+C};q)_j}$$

we derive from Corollary 7 the following determinant formula

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j-A-j\end{bmatrix}}{\begin{bmatrix}X_i+Y_j\\ X_i-Y_j-A\end{bmatrix}\begin{bmatrix}X_i+C+j\\ X_i-A-C-j\end{bmatrix}}\right]=\prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{A-1-X_i-X_j}) \tag{24a}$$

$$\times\frac{q^{2\binom{n+2}{3}+\binom{n+1}{2}(C-A)+n\sum_{k=0}^{n}X_k}}{(q;q)_n^{n+1}\begin{bmatrix}A+2C+2n\\ 2n\end{bmatrix}^{n+1}\begin{bmatrix}2n\\ n\end{bmatrix}^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}A+2C+2k\\ 2C-2Y_k\end{bmatrix}\begin{bmatrix}Y_k-C\\ k\end{bmatrix}\begin{bmatrix}-k-A-C-Y_k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}A+2C\\ A+2Y_k\end{bmatrix}\begin{bmatrix}X_k+C+n\\ A+2C+2n\end{bmatrix}} \tag{24b}$$

which reduces, for $X_i=bi$ and $Y_j=0$, to the q-binomial determinant identity:

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}bi+j\\ 2j\end{bmatrix}\begin{bmatrix}bi+c+j\\ 2c+2j\end{bmatrix}^{-1}\right]=\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{1+bi+bj}) \tag{25a}$$

$$\times\frac{q^{\sum_{k=0}^{n}k(nb+c+k)}}{(q;q)_n^{n+1}\begin{bmatrix}2c+2n\\ 2n\end{bmatrix}^{n+1}\begin{bmatrix}2n\\ n\end{bmatrix}^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}2c+2k\\ 2c\end{bmatrix}\begin{bmatrix}-c\\ k\end{bmatrix}\begin{bmatrix}-k-c\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}bk+c+n\\ 2c+2n\end{bmatrix}}. \tag{25b}$$

3.4 According to Corollary 6, the q-binomial relation

$$q^{jX_i}\begin{bmatrix}X_i+A-j\\ X_i+j\end{bmatrix}=\frac{(-1)^jq^{\binom{j}{2}-Aj}(q;q)_{X_i+A}}{(q;q)_{X_i}(q;q)_{A-2j}}\times\frac{1}{(q^{1+X_i};q)_j(q^{-A-X_i};q)_j}$$

yields the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[q^{jX_i}\begin{bmatrix}X_i+A-j\\ X_i+j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_i}-q^{-X_j})(1-q^{1+A+X_i+X_j}) \tag{26a}$$

$$\times\prod_{k=0}^{n}\begin{bmatrix}X_k+A-n\\ X_k+n\end{bmatrix}\frac{q^{nX_k}}{(q^{1+A-2n};q)_{2k}}. \tag{26b}$$

In particular for $X_i = c + bi$, it becomes the q-binomial determinant identity

$$\det_{0\le i,j\le n}\left[q^{bij}\begin{bmatrix}a+bi-j\\ c+bi+j\end{bmatrix}\right]\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{1+a+c+bi+bj}) \tag{27a}$$

$$\times\, q^{nb\binom{n+1}{2}}\prod_{k=0}^{n}\frac{(q;q)_{a+bk-n}}{(q;q)_{a-c-2k}(q;q)_{c+bk+n}} \tag{27b}$$

which is the q-analogue of the determinant evaluated by Amdeberhan and Zeilberger [3, Eq 2].

3.5 In view of Corollary 5, the q-binomial relation

$$q^{-jX_i}\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j+A-j\end{bmatrix}\begin{bmatrix}X_i-Y_j+A\\ X_i+Y_j\end{bmatrix}=\frac{(-1)^j q^{(A-Y_j)j-\binom{j}{2}}}{(q;q)_{2Y_j-A+2j}(q;q)_{A-2Y_j}}\times(q^{1+X_i+Y_j};q)_j(q^{Y_j-X_i-A};q)_j$$

leads to the following determinant evaluation

$$\det_{0\le i,j\le n}\left[q^{-jX_i}\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j+A-j\end{bmatrix}\begin{bmatrix}X_i-Y_j+A\\ X_i+Y_j\end{bmatrix}\right] \tag{28a}$$

$$=\frac{\prod_{0\le i<j\le n}(q^{-X_j}-q^{-X_i})(1-q^{1+A+X_i+X_j})}{\prod_{k=0}^{n}(q;q)_{A-2Y_k}(q;q)_{2Y_k-A+2k}}. \tag{28b}$$

In particular for $X_i = a + bi$ and $Y_j = 0$, the last identity gives

$$\det_{0\le i,j\le n}\left[q^{-bij}\begin{bmatrix}a+bi+j\\ c+bi-j\end{bmatrix}\right]=\frac{\prod_{0\le i<j\le n}(q^{-bj}-q^{-bi})(1-q^{1+a+c+bi+bj})}{\prod_{k=0}^{n}(q^{1+a+bk};q)_{c-a}(q;q)_{a-c+2k}} \tag{29}$$

which results in the q-analogue of the binomial determinant identity due to Amdeberhan and Zeilberger [3, Eq 1].

Similarly, letting $x_i = q^{a+bi}$ and $\alpha_{ij} = q^{dj-i}$, we find from Corollary 5 another determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}a+bi+dj\\ j\end{bmatrix}\begin{bmatrix}c-bi+dj\\ j\end{bmatrix}q^{\binom{j}{2}}\right] \tag{30a}$$

$$=\prod_{k=0}^{n}\frac{q^{k(c+dk)}}{(q;q)_k^2}\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{a-c+bi+bj}) \tag{30b}$$

which is the q-analogue of the result in Amdeberhan and Zeilberger [3, Eq 14]. The list of examples can be endless. However, we are not going further to prolong it due to the space limitation.

References

[1] AIGNER, M., *Catalan-like numbers and determinants,* J. Combin. Theory (Ser. A) **87** (1999), 33–51.

[2] ANDREWS, G.E., *Catalan numbers, q-Catalan numbers and hypergeometric series,* J. Combin. Theory (Ser. A) **44** (1987), 267–273.

[3] AMDEBERHAN, T. AND ZEILBERGER, D., *Determinants through the looking glass,* Adv. in Appl. Math. **27** (2001), 225–230.

[4] CARLITZ, L., *Some determinants of q-binomial coefficients,* J. Reine Angew. Math. **226** (1967), 216–220.

[5] CHU, W., *On the evaluation of some determinants with q-binomial coefficients,* J. Systems Science & Math. Science 8:4 (1988), 361–366.

[6] CHU, W., *Generalizations of the Cauchy determinant,* Publicationes Mathematicae Debrecen 58:3 (2001), 353–365.

[7] CHU, W., *The Cauchy double alternant and divided differences,* Electronic Journal of Linear Algebra **15** (2006), 14–21.

[8] CIGLER, J., *Operatormethoden für q-Identitäten. VII: q-Catalan-Determinanten,* Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **208** (1999), 123–142.

[9] MENON, K.V., *Note on some determinants of q-binomial numbers,* Discrete Math. 61:2-3 (1986), 337–341.

[10] OSTROWSKI, A.M., *On some determinants with combinatorial numbers,* J. Reine Angew. Math. **216** (1964), 25–30.

[11] RADOUX, C., *Nombres de Catalan généralisés,* Bull. Belg. Math. Soc. Simon Stevin 4:2 (1997), 289–292.