CUBO A Mathematical Journal
Vol.12, N$^{\underline{o}}$02, (29–42). June 2010

The method of Kantorovich majorants to nonlinear singular integral equations with Hilbert kernel

M. H. Saleh, S. M. Amer [1] and M. H. Ahmed
Departement of Mathematics,
Faculty of Science,
Zagazig University, Zagazig, Egypt.
email: amrsammer@hotmail.com

ABSTRACT

This paper concerned with applicability of the method of Kantorovich majorants to nonlinear singular integral equations with Hilbert kernel . The results are illustrated in Hölder space.

RESUMEN

Este artículo es concerniente a la aplicabilidad del método de mayorantes de Kantorovich para ecuaciones integrales singulares no lineales con núcleo de Hilbert. Los resultados son aplicaciones en espacios de Hölder.

Key words and phrases: *Nonlinear singular integral equations, Kantorovich majorants method, Hölder spaces.*

AMS 2000-Subject classification: *45F15, 45G10.*

1. Introduction

There is a large literature on nonlinear singular integral equations with Hilbert and Cauchy kernel and related Riemann boundary value problems for analytic functions,cf.the monograph by Pogorzel-

[1]Corresponding author

ski [16], Guseinov A.I. and Mukhtarov kh.sh. [9],Kantorovich L.V.[11],Muskhelishvill N.I.[14],and Mikhlin S.G.and Prossdorf S.[13].The method of singular integral equations on closed contour has been intensively investigated by many approximation methods, specially method of modified Newton-Kantorovich, reduction, collocation and mechanical quadratures, (see[1-6,10,12,15,17,19]).
In this paper the method of Kantorovich majorants[7,18,20], has been applied to the following class of nonlinear singular integral equations with Hilbert kernel :

$$\varphi(t)=\lambda G(t,\frac{1}{2\pi}\int_0^{2\pi} g(t,\sigma,\varphi(\sigma))\ cot\frac{\sigma-t}{2}\ d\sigma), \tag{1.1}$$

where λ is a numerical parameter,
where

$$v(t)=Sg(t,\sigma,\varphi(\sigma))=\frac{1}{2\pi}\int_0^{2\pi} g(t,\sigma,\varphi(\sigma))\ cot\frac{\sigma-t}{2}\ d\sigma,$$

then equation (1.1) takes the form:

$$\varphi(t)=\lambda G(t,v(t)).$$

Now, we consider the equation:

$$B(\varphi)=0, \tag{1.2}$$

where

$$(B\varphi)(t)=\varphi(t)-\lambda G(t,v(t)). \tag{1.3}$$

2. Formulation of the problem

Let $f:\bar{S}(\varphi_0,R)\subset X\longrightarrow Y$ be a nonlinear operator defined on the closure of a ball

$$S(\varphi_0,R)=\{\varphi:\varphi\in X,\parallel\varphi-\varphi_0\parallel< R\},$$

in a Banach space X into a Banach space Y.
We give new conditions to ensure the convergence on Newton-Kantorovich approximations toward a solution of $f(\varphi)=0$, under the hypothesis that f is Frechet differentiable in $S(\varphi_0,R)$, and that it's derivative $\grave{f}$ satisfies the local Lipschitz condition :

$$\parallel\grave{f}(\varphi_1)-\grave{f}(\varphi_2)\parallel\le k(r)\parallel\varphi_1-\varphi_2\parallel,\varphi_1,\varphi_2\in\bar{S}(\varphi_0,r),o<r<R, \tag{2.1}$$

where $k(r)$is a non decreasing function on the interval $[0,R]$ and

$$k(r)=sup\{\frac{\parallel\grave{f}(\varphi_1)-\grave{f}(\varphi_2)\parallel}{\parallel\varphi_1-\varphi_2\parallel}\varphi_1,\varphi_2\in\bar{S}(\varphi_0,r)\varphi_1\neq\varphi_2\}. \tag{2.2}$$

Define a scalar function $\psi:[0,R]\to(0,\infty)$ by

$$\psi(r)=a+b\int_0^r w(t)dt-r, \tag{2.3}$$

using the function

$$w(r) = \int_0^r k(t)dt, \tag{2.4}$$

and

$$a = \| \grave{f}(\varphi_0)^{-1} f(\varphi_0) \|, \quad b = \| \grave{f}(\varphi_0)^{-1} \| . \tag{2.5}$$

Theorem 2.1 **[4,7]** Suppose that the equation (2.3)has a unique positive root r_* in $[0, R]$ and $\psi(R) \leq 0$. Then the equation $f(\varphi) = 0$ has a unique solution φ_* in $S(\varphi_0, R)$ and the Newton-Kantorovich approximations:

$$\varphi_n = \varphi_{n-1} - \grave{f}(\varphi_{n-1})^{-1} f(\varphi_{n-1}), \quad n \in N, \tag{2.6}$$

are defined for all $n \in N$, belong to $S(\varphi_0, r_*)$ and converges to φ_*.
Moreover , the following estimate holds

$$\| \varphi_{n+1} - \varphi_n \| \leq r_{n+1} - r_n, \quad \| \varphi_* - \varphi_n \| \leq r_* - r_n, \tag{2.7}$$

where the sequence $(r_n)_{n \in N}$ converges to r_*, is defined by the recurrence formula

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{\psi(r_n)}{\grave{\psi}(r_n)}, \quad n \in N. \tag{2.8}$$

In the present paper , we investigate some sufficient conditions , which ensure that the class of nonlinear singular integral equations (1.1) verifies the hypotheses of theorem (2.1).

3. Some auxiliary results

Definition 3.1[9] We denote by $H_\delta, 0 < \delta < 1$,the Hölder space of continuous functions , which satisfy the Hölder condition with exponent δ with norm

$$\| \varphi \|_\delta = \| \varphi \|_c + H^\delta(\varphi), \tag{3.1}$$

where

$$\| \varphi \|_c = \max_{\sigma \in [0, 2\pi]} | \varphi(\sigma) |,$$

and

$$H^\delta(\varphi) = \sup_{\sigma_1 \neq \sigma_2} \frac{| \varphi(\sigma_1) - \varphi(\sigma_2) |}{| \sigma_1 - \sigma_2 |^\delta}.$$

Lemma 3.1 **[9]** Let the functions $G(t, v(t))$, $g(t, \sigma, \varphi(\sigma))$ and it's partial derivatives up to second order, satisfy the following conditions

$$\left| \frac{\partial^m G(t_1, v(t_1))}{\partial v^m} - \frac{\partial^m G(t_2, v(t_2))}{\partial v^m} \right| \leq c_m(r) \{ | t_1 - t_2 |^\delta + | v(t_1) - v(t_2) | \}, \tag{3.2}$$

and

$$|\ \frac{\partial^m g_{_\varphi}(t_1,\sigma_1,\varphi(\sigma_1))}{\partial\varphi^m} - \frac{\partial^m g_{_\varphi}(t_2,\sigma_2,\varphi(\sigma_2))}{\partial\varphi^m}\ |\leq a_m(r)\{|\ t_{_1}-t_{_2}\ |^\delta + |\ \sigma_{_1}-\sigma_{_2}\ |^\delta + |\ \varphi(t_{_1})-\varphi(t_{_2})\ |\} \tag{3.3}$$

where $c_m(r), a_m(r)$ are positive increasing functions m=0,1,2 and $t_{_i},\sigma_{_i}\in[0,2\pi], i=1,2$. If $\varphi(\sigma)\in H_{_\delta}$, then $G(t,v(t)),\ g(t,\sigma,\varphi(\sigma))\in H_{_\delta}$.

Lemma 3.2 If the functions $G(t,v(t))$ *and* $g(t,\sigma,\varphi(\sigma))$ satisfy the conditions of lemma(3.1), then the operator $B(\varphi)$ has a Frechet derivative at every fixed point in the space $H_{_\delta}$ and its derivative is given by

$$\grave{B}(\varphi)h = h(t) - \lambda G_v(t,v(t))Sg_{_\varphi}(t,\sigma,\varphi(\sigma))h(\sigma). \tag{3.4}$$

Moreover it satisfies Lipschitz condition:

$$\|\ \grave{B}(\varphi_{_1}) - \grave{B}(\varphi_{_2})\ \|\leq k(r)\ \|\ \varphi_{_1}-\varphi_{_2}\ \|, \tag{3.5}$$

for all $\varphi_{_1},\varphi_{_2}\in S(\varphi_{_0},r)$ *and* $o<r<R$.

Proof Let $\varphi(t)$be any fixed point in the space $0,<\delta<1$and $h(t)$be any arbitrary element in $H_{_\delta}$, then we obtain :

$$\begin{aligned} B(\varphi+h)-B(\varphi) &= h(t)-\lambda[G(t,Sg(t,\sigma,\varphi(\sigma)+h(\sigma)))-G(t,Sg(t,\sigma,\varphi(\sigma)))] \\ &= \grave{B}(\varphi)h+\eta(t,h), \end{aligned}$$

where $0\leq\xi\leq 1$ and

$$\begin{aligned} \eta(t,h) &= \lambda\int_0^1(1-\xi)[G_{v^2}(t,Sg(t,\sigma,\varphi(\sigma)+\xi h(\sigma)))(Sg_{_\varphi}(t,\sigma,\varphi(\sigma)+\xi h(\sigma))h(\sigma))^2 \\ &+ G_v(t,Sg(t,\sigma,\varphi(\sigma)+\xi h(\sigma)))Sg_{_{\varphi^2}}(t,\sigma,\varphi(\sigma)+\xi h(\sigma))h(\sigma)^2]d\xi. \end{aligned}$$

Now , we shall prove that

$$\lim_{\|h\|\to 0}\frac{\|\ \eta(t,h)\ \|}{\|\ h\ \|}=0.$$

Using the inequalities [9,13]

$$\left.\begin{aligned} &\|\int_a^b\frac{y(s)}{s-x}ds\ \|\leq\rho_{_0}\ \|\ y\ \|, where\ \rho_0\ is\ a\ positive\ constant \\ &\|\ uv\ \|\leq\|\ u\ \|\|\ v\ \|\ for\ all\ u,v\in H_{_\delta} \end{aligned}\right\}. \tag{3.6}$$

Now;

$$\begin{aligned} \|\ \eta(t,h)\ \| &\leq \|\ h(\sigma)^2\ \|\ \rho_{_0}[\ \|\ G_{v^2}(t,Sg(t,\sigma,\varphi(\sigma)))\ \|\|\ (g_{_\varphi}(t,\sigma,\varphi(\sigma)))^2\ \| \\ &+ \|\ G_v(t,Sg(t,\sigma,\varphi(\sigma)))\ \|\|\ g_{_{\varphi^2}}(t,\sigma,\varphi(\sigma))\ \|\ ]. \end{aligned}$$

Hence

$$\lim_{\|h\|\to 0}\frac{\|\ \eta(t,h)\ \|}{\|\ h\ \|}=0,$$

which prove the differentiability of $B(\varphi)$ in the sense of Frechet and its derivative is given by (3.4).
To prove the Frechet derivative $\grave{B}(\varphi)$ satisfies Lipschitz condition in the sphere

$$S(\varphi_0, R) = \{\varphi : \| \varphi - \varphi_0 \| < R\}.$$

We consider

$$\begin{aligned} \| \grave{B}(\varphi_1)h - \grave{B}(\varphi_2)h \| &= \| \lambda G_v(t, Sg(t,\sigma,\varphi_1(\sigma)))Sg_{\varphi}(t,\sigma,\varphi_1(\sigma))h(\sigma) \\ &- \lambda G_v(t, Sg(t,\sigma,\varphi_2(\sigma)))Sg_{\varphi}(t,\sigma,\varphi_2(\sigma))h(\sigma) \| \\ &\leq | \lambda | \| h \| [\ \| G_v(t, v_1(t)) \| \| Sg_{\varphi}(t,\sigma,\varphi_1(\sigma)) - Sg_{\varphi}(t,\sigma,\varphi_2(\sigma)) \| \\ &+ \| Sg_{\varphi}(t,\sigma,\varphi_2(\sigma)) \| \| G_v(t,v_1(t)) - G_v(t,v_2(t)) \| \ ] \\ &\leq \| h \| \, k(r) \| \varphi_1 - \varphi_2 \|, \end{aligned}$$

where $k(r) = | \lambda | \rho_0 [a_1(r)D + \| g_{\varphi} \| c_1(r)a_0(r)]$, and
$D = \max_t | \ G_v(t, Sg(t,\sigma,\varphi(\sigma))) \ |$ then the lemma is proved.

4. Solution of linear singular integral equation

To find the operator $\grave{B}(\varphi_0)^{-1}$, we investigate the solution of the equation

$$h(t) - \frac{\lambda G_v(t, v(t))}{2\pi} \int_0^{2\pi} g_{\varphi}(t,\sigma,\varphi(\sigma)) \ h(\sigma) \ cot \frac{\sigma - t}{2} \ d\sigma = f(t). \tag{4.1}$$

For this aim we introduce the following theorem:

Theorem 4.1 If the functions $G(t, v(t))$ *and* $g(t,\sigma,\varphi(\sigma))$ satisfy the conditions of lemma(3.2), then the linear operator defined by (3.4) has a bounded inverse $\grave{B}(\varphi_0)^{-1}$ for any fixed $\varphi_0 \in H_\delta, (0 < \delta < 1)$.

Proof
Let us transform the equation (4.1) by introducing new variables :

$$s = e^{it}, \tau = e^{i\sigma}, d\tau = ie^{i\sigma} d\sigma,$$

since

$$\frac{1}{2} \ cot \frac{\sigma - t}{2} d\sigma = (\frac{1}{\tau - s} - \frac{1}{2\tau}) d\tau,$$

then equation (4.1) has the form

$$h(s) - \frac{\lambda X_v(s, v(s))}{\pi i} \int_\gamma ik_{\varphi}(s,\tau,\varphi(\tau)) \ h(\tau) \ (\frac{1}{\tau - s} - \frac{1}{2\tau}) d\tau \ = f(s), \tag{4.2}$$

where γ is a unit circle , $G_v(t, v(t)) = X_v(s, v(s))$ and
$g_{\varphi}(t,\sigma,\varphi(\sigma)) = k_{\varphi}(s,\tau,\varphi(\tau))$.

We introduce the sectionally holomorphic function of variable z as follows:

$$H(z) = \frac{\lambda X_v(s, v(s))}{2\pi i} \int_\gamma \ \frac{ik_{\varphi}(s,\tau,\varphi(\tau))}{\tau - z} \ h(\tau) d\tau - C, \tag{4.3}$$

and

$$\begin{aligned} H(\infty) &= -C = \frac{-\lambda X_v(s,v(s))}{4\pi}\int_{\gamma} \frac{ik_{\varphi}(s,\tau,\varphi(\tau))}{\tau} h(\tau)d\tau \\ &= \frac{-i\lambda G_v(t,v(t))}{4\pi}\int_0^{2\pi} g_{\varphi}(t,\sigma,\varphi(\sigma))\, h(\sigma)d\sigma. \end{aligned}$$

According to Sokhotoski formulae[9], we have

$$\begin{aligned} H^{\pm}(s) &= \pm\frac{i\lambda X_v(s,v(s))}{2}k_{\varphi}(s,s,\varphi(s))h(s) \\ &+ \frac{\lambda X_v(s,v(s))}{2\pi i}\int_{\gamma} \frac{ik_{\varphi}(s,\tau,\varphi(\tau))}{\tau - s} h(\tau)d\tau - C. \end{aligned}$$

Therefore

$$\left.\begin{aligned} H^+(s) - H^-(s) = i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))h(s) \\ H^+(s) + H^-(s) = \tfrac{\lambda X_v(s,v(s))}{\pi i}\int_{\gamma} \tfrac{ik_{\varphi}(s,\tau,\varphi(\tau))}{\tau-s} h(\tau)d\tau - 2C \end{aligned}\right\}. \tag{4.4}$$

Substituting from equation (4.4) into equation (4.2)we have

$$h(s) - f(s) + 2C = H^+(s) + H^-(s) + 2C. \tag{4.5}$$

Hence we get

$$h(s) = H^+(s) + H^-(s) + f(s), \tag{4.6}$$

therefore from (4.4) and (4.6) we have,

$$h(s)[1 \pm i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))] = 2H^{\pm}(s) + f(s),$$

since$1 \pm i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s)) \neq 0$, then the last conditions equivalent to the following

$$\left.\begin{aligned} h(s) = \tfrac{2H^+(s)}{1+i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))} + \tfrac{f(s)}{1+i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}, \\ h(s) = \tfrac{2H^-(s)}{1-i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))} + \tfrac{f(s)}{1-i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))} \end{aligned}\right\}. \tag{4.7}$$

By equating the right hand side of equation (4.7) we get the Riemann boundary value problem

$$H^+(s) = \frac{1 + i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}H^-(s) + \frac{i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}f(s). \tag{4.8}$$

It is well known that the index of equation (4.8) is zero[8],then

$$\frac{1 + i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))} = \frac{X^+(s)}{X^-(s)},$$

where

$$X(z) = \frac{1}{2\pi}\int_{\gamma} ln\frac{1 + i\lambda X_v(s,v(s))ik_{\varphi}(s,\tau,\varphi(\tau))}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,\tau,\varphi(\tau))}\frac{d\tau}{\tau - z},$$

the problem (4.8)can be written in the form

$$\frac{H^+(s)}{X^+(s)} - \frac{H^-(s)}{X^-(s)} = \frac{i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))f(s)}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))X^+(s)}.$$

Hence ,from [8], the boundary value problem (4.8) has the solution

$$H(z) = X(z)[\frac{\lambda X_v(s,v(s))}{2\pi i}\int_{\gamma} \frac{ik_{\varphi}(s,\tau,\varphi(\tau))f(\tau)}{X^+(\tau)(1 - i\lambda X_v(s,v(s))k_{\varphi}(s,\tau,\varphi(\tau)))}\frac{d\tau}{\tau - s} - C].$$

By Sokhotski formulae

$$\begin{aligned} H^+(s) &= \frac{i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s))f(s)}{2(1 - i\lambda X_v(s,v(s))k_{\varphi}(s,s,\varphi(s)))} \\ &+ \frac{\lambda X_v(s,v(s))X^+(s)}{2\pi i}\int_{\gamma} \frac{ik_{\varphi}(s,\tau,\varphi(\tau))f(\tau)}{X^+(\tau)(1 - i\lambda X_v(s,v(s))k_{\varphi}(s,\tau,\varphi(\tau)))}\frac{d\tau}{\tau - s} \\ &- CX^+(s). \end{aligned} \tag{4.9}$$

Substituting from (4.9) into (4.7) we have,

$$\begin{aligned} h(s) &= \frac{f(s)}{u(s)} + \frac{z(s)\lambda X_v(s,v(s))}{u(s)\pi i}\int_{\gamma} \frac{ik_{\varphi}(s,\tau,\varphi(\tau))f(\tau)}{z(\tau)}\frac{d\tau}{\tau - s} \\ &- \frac{2Cz(s)}{u(s)}, \end{aligned} \tag{4.10}$$

where

$$u(s) = 1 + \lambda^2 X_v^2(s,v(s))k_{\varphi}^2(s,s,\varphi(s)),$$

$$z(s) = \sqrt{u(s)}e^{\Gamma(s)},$$

and

$$\Gamma(s) = \frac{1}{2\pi i}\int_{\gamma} ln\frac{1 + i\lambda X_v(s,v(s))ik_{\varphi}(s,\tau,\varphi(\tau))}{1 - i\lambda X_v(s,v(s))k_{\varphi}(s,\tau,\varphi(\tau))}\frac{d\tau}{\tau - s},$$

since

$$\frac{d\tau}{\tau - s} = \frac{1}{2}\ cot\frac{\sigma - t}{2} + \frac{i}{2}d\sigma.$$

Hence

$$\begin{aligned} z(e^{it}) = z(s) &= \sqrt{u(t)}exp\ \ (\frac{1}{4\pi}\int_0^{2\pi} ln\frac{1 + i\lambda G_v(t,v(t))g_{\varphi}(t,\sigma,\varphi(\sigma))}{1 - i\lambda G_v(t,v(t))g_{\varphi}(t,\sigma,\varphi(\sigma))}\ d\sigma \\ &\quad exp\ \ (\frac{1}{4\pi i}\int_0^{2\pi} ln\frac{1 + i\lambda G_v(t,v(t))g_{\varphi}(t,\sigma,\varphi(\sigma))}{1 - i\lambda G_v(t,v(t))g_{\varphi}(t,\sigma,\varphi(\sigma))}\quad cot\frac{\sigma - t}{2}\ d\sigma)). \end{aligned}$$

Now we determine the constant C as follows

$$\begin{aligned} C &= \frac{i\lambda G_v(t,v(t))}{4\pi}\int_0^{2\pi} g_{_\varphi}(t,\sigma,\varphi(\sigma))\ h(\sigma)\ \ d\sigma = \\ &= (1+\frac{iz(t)\lambda G_v(t,v(t))}{2\pi u(t)}\int_0^{2\pi} g_{_\varphi}(t,\sigma,\varphi(\sigma))d\sigma)^{-1} \\ &[\quad \frac{i\lambda G_v(t,v(t))}{4\pi}\int_0^{2\pi} g_{_\varphi}(t,\sigma,\varphi(\sigma))[\frac{f(t)}{u(t)} \\ &+ \frac{z(t)}{2\pi u(t)}\int_0^{2\pi}\frac{g_{_\varphi}(\xi,\sigma,\varphi(\sigma))f(\xi)}{z(\xi)}\ cot\frac{\xi-\sigma}{2}\ d\xi \\ &+ \frac{iz(t)}{2\pi u(t)}\int_0^{2\pi}\frac{\lambda G_v(\xi,v(\xi))g_{_\varphi}(\xi,\sigma,\varphi(\sigma))f(\xi)}{z(\xi)}\ d\xi]\ \ d\sigma.\} \end{aligned}$$

Then

$$\begin{aligned} h(t) &= \frac{f(t)}{u(t)}+\frac{\lambda G_v(t,v(t))z(t)}{2\pi u(t)}\int_0^{2\pi}\frac{g_{_\varphi}(t,\sigma,\varphi(\sigma))f(\sigma)}{z(\sigma)}\ cot\frac{\sigma-t}{2}\ d\sigma \\ &+ \frac{\lambda G_v(t,v(t))z(t)}{2\pi u(t)}\int_0^{2\pi}\frac{g_{_\varphi}(t,\sigma,\varphi(\sigma))f(\sigma)}{z(\sigma)}\ d\sigma - \frac{2Cz(t)}{u(t)} \\ &= \grave{B}(\varphi_{_0})^{-1}f(t). \end{aligned}$$

We shall prove that the operator $\grave{B}(\varphi_{_0})^{-1}$ is bounded.
It is easy to prove that $v(t),\ \Gamma(t)\ and\ z(t)\in H_{_\delta}$ therefore by using inequality (3.6) we get

$$\begin{aligned} \parallel \grave{B}(\varphi_{_0})^{-1}\parallel_\delta &\le \parallel\frac{1}{u}\parallel_\delta\{1+\rho_{_0}\mid\lambda\mid\parallel z\parallel_\delta\parallel G_v(t,v(t))\parallel_\delta\parallel g_{_\varphi}(t,t,\varphi(t))\parallel_\delta\parallel\frac{1}{z}\parallel_\delta \\ &+ \rho_{_1}\mid\lambda\mid\parallel z\parallel_\delta\parallel G_v(t,v(t))\parallel_\delta + 2\tilde{C}\parallel z\parallel_\delta\}, \end{aligned} \tag{4.11}$$

where

$$\rho_1=\frac{1}{2\pi}\int_0^{2\pi}\mid\frac{g_{_\varphi}(t,\sigma,\varphi(\sigma))}{z(\sigma)}\mid d\sigma$$

and

$$\begin{aligned} \tilde{C} &= (1+\frac{iz(t)\lambda G_v(t,v(t))}{2\pi u(t)}\int_0^{2\pi} g_{_\varphi}(t,\sigma,\varphi(\sigma))d\sigma)^{-1} \\ &[\quad \frac{i\lambda G_v(t,v(t))}{4\pi}\int_0^{2\pi} g_{_\varphi}(t,\sigma,\varphi(\sigma))[\frac{1}{u(t)} \\ &+ \frac{z(t)}{2\pi u(t)}\int_0^{2\pi}\frac{g_{_\varphi}(\xi,\sigma,\varphi(\sigma))}{z(\xi)}\ cot\frac{\xi-\sigma}{2}\ d\xi \\ &+ \frac{iz(t)}{2\pi u(t)}\int_0^{2\pi}\frac{\lambda G_v(\xi,v(\xi))g_{_\varphi}(\xi,\sigma,\varphi(\sigma))}{z(\xi)}\ d\xi]]\ \ d\sigma. \end{aligned}$$

We determine the norm of each term in right hand side of the above inequality.
From definition (3.1) we have

$$\parallel\frac{1}{u}\parallel_{_c}=\parallel\frac{1}{1+\lambda^2G_v^2(t,v(t))g_{_\varphi}^2(t,t,\varphi(t))}\parallel_c\le 1,$$

$$\begin{aligned}\left|\frac{1}{u(t_1)}-\frac{1}{u(t_2)}\right| &\leq |u(t_1)-u(t_2)| \leq |\lambda^2\,||\,G_v^2(t_1,v(t_1))g^2_{\varphi}(t_1,t_1,\varphi(t_1))\\ &- G_v^2(t_2,v(t_2))g^2_{\varphi}(t_2,t_2,\varphi(t_2))\,|\\ &\leq \lambda^2[|\,G_v(t_1,v(t_1))g_{\varphi}(t_1,t_1,\varphi(t_1))-G_v(t_2,v(t_2))g_{\varphi}(t_2,t_2,\varphi(t_2))\,|]\\ &[\quad 2\,\|\,G_v(t,v(t))\,\|_c\|\,g_{\varphi}(t,t,\varphi(t))\,\|_c],\end{aligned}$$

since

$$\|\,G_v(t,v(t))\,\|_c\leq c_1(r)\,\|\,v\,\|_c+\|\,G_v(t,0)\,\|_c,$$

similarly

$$\|\,g_{\varphi}(t,t,\varphi(t))\,\|_c\leq a_1(r)\,\|\,\varphi\,\|_c+\|\,g_{\varphi}(t,t,0)\,\|_c,$$

using conditions(3.2)and (3.3)we have

$$|\,g_{\varphi}(t_1,t_1,\varphi(t_1))-g_{\varphi}(t_2,t_2,\varphi(t_2))\,|\leq a_1(r)(2+H^{\delta}(\varphi))\,|\,t_1-t_2\,|^{\delta},$$

$$|\,G_v(t_1,v(t_1))-G_v(t_2,v(t_2))\,|\leq c_1(r)(1+H^{\delta}(v)))\,|\,t_1-t_2\,|^{\delta}\,.$$

and

$$|\,G_v(t_2,v(t_2))\,|\leq|\,G_v(t_2,0))\,|+c_1(r)(|\,v(t_2)\,|),$$

similarly

$$|\,g_{\varphi}(t_1,t_1,\varphi(t_1))\,|\leq a_1(r)\,|\,\varphi\,|+|\,g_{\varphi}(t_1,t_1,0)\,|\,.$$

Hence

$$\left|\frac{1}{u(t_1)}-\frac{1}{u(t_2)}\right|\leq\lambda^2\beta.$$

So

$$\left\|\frac{1}{u}\right\|_{\delta}\leq R_1, \tag{4.12}$$

where $R_1=1+\lambda^2\beta$ and

$$\begin{aligned}\beta &= [(|\,g_{\varphi}(t_1,t_1,0)\,|+a_1(r)\,|\,\varphi\,|)(c_1(r)(1+H^{\delta}(v))\,|\,t_1-t_2\,|^{\delta})\\ &+ (|\,G_v(t_2,0)\,|+c_1(r)\,|\,v\,|)(a_1(r)(2+H^{\delta}(\varphi))\,|\,t_1-t_2\,|^{\delta})]\\ &[\quad (c_1(r)\,\|\,v\,\|_c+\|\,G_v(t,0)\,\|_c)(a_1(r)\,\|\,\varphi\,\|_c+\|\,g_{\varphi}(t,t,0)\,\|_c)],\end{aligned}$$

To determine $\|\,z\,\|_{\delta}$ we get

$$\|\,z\,\|_{\delta}\leq\|\,\sqrt{u}\,\|_{\delta}\,(1+\|\,\Gamma\,\|_{\delta})e^{\|\Gamma\|_{\delta}}, \tag{4.13}$$

since

$$\|\,u\,\|_c\leq\sqrt{1+\lambda^2(c_1\,\|\,v\,\|_c+\|\,G_v(t,0)\,\|_c)^2(a_1\,\|\,\varphi\,\|_c+\|\,g_{\varphi}(t,t,0)\,\|_c)^2}.$$

By

applying Lagrange theorem:

$$\begin{aligned} \mid \sqrt{u(t_1)} - \sqrt{u(t_2)} \mid &= \mid \frac{1}{2}(1+\theta)^{-1/2}\ \lambda^2\ [G_v^2(t_1,v(t_1))\ g_{_\varphi}^2\,(t_1,t_1,\varphi(t_1)) \\ &- G_v^2(t_2,v(t_2))\ g_{_\varphi}^2\,(t_2,t_2,\varphi(t_2))] \mid \\ &\leq \lambda^2\beta, \end{aligned}$$

where θ between $\lambda G_v(t_1,v(t_1))\ g_{_\varphi}\,(t_1,t_1,\varphi(t_1))$ *and* $\lambda G_v(t_2,v(t_2))\ g_{_\varphi}\,(t_2,t_2,\varphi(t_2))$.
Then

$$\parallel \sqrt{u} \parallel_\delta \leq R_2, \tag{4.14}$$

where

$$R_2 = \sqrt{1 + (c_1 \parallel v \parallel_c + \parallel G_v(t,0) \parallel_c)^2 (a_1 \parallel \varphi \parallel_c + \parallel g_{_\varphi}\,(t,t,0) \parallel_c)^2} + \lambda^2\beta.$$

Also, we determine $\parallel \Gamma \parallel_\delta$, since

$$\Gamma(t) = \frac{1}{2\pi}\int_0^{2\pi} arctg\lambda\ G_v(t,v(t))\ g_{_\varphi}\,(t,\sigma,\varphi(\sigma))\ cot\ \frac{\sigma - t}{2} d\sigma + Q,$$

where

$$Q = \frac{1}{4\pi}\int_0^{2\pi} ln\frac{1 + i\lambda G_v(t,v(t)) g_{_\varphi}\,(t,\sigma,\varphi(\sigma))}{1 - i\lambda G_v(t,v(t)) g_{_\varphi}\,(t,\sigma,\varphi(\sigma))}\ d\sigma,$$

by using (3.6)we have

$$\parallel \Gamma \parallel_c \leq \rho_{_0} \parallel arctg\lambda\ G_v(t,v(t))\ g_{_\varphi}\,(t,t,\varphi(t)) \parallel_c + \mid Q \mid \leq \frac{\rho_{_0}\pi}{2} + \mid Q \mid,$$

$$\mid arctg\lambda\ G_v(t_1,v(t_1))\ g_{_\varphi}\,(t_1,t_1,\varphi(t_1)) - arctg\lambda\ G_v(t_2,v(t_2))\ g_{_\varphi}\,(t_2,t_2,\varphi(t_2)) \mid$$

$$\begin{aligned} &\leq \mid \frac{\lambda}{1+\theta_1^2}[G_v(t_1,v(t_1))\ g_{_\varphi}\,(t_1,t_1,\varphi(t_1)) - G_v(t_2,v(t_2))\ g_{_\varphi}\,(t_2,t_2,\varphi(t_2))] \mid \\ &\leq \mid \lambda \mid [(\mid g_{_\varphi}\,(t_1,t_1,0) \mid + a_1(r) \mid \varphi \mid)(c_1(r)(1 + H^\delta(v)) \mid t_1 - t_2 \mid^\delta) \\ &+ (\mid G_v(t_2,0) \mid + c_1(r) \mid v \mid)(a_1(r)(2 + H^\delta(\varphi)) \mid t_1 - t_2 \mid^\delta)], \end{aligned}$$

where θ_1 between $\lambda G_v(t_1,v(t_1))\ g_{_\varphi}\,(t_1,t_1,\varphi(t_1))$ *and* $\lambda G_v(t_2,v(t_2))\ g_{_\varphi}\,(t_2,t_2,\varphi(t_2))$. Therefore

$$\parallel \Gamma \parallel_\delta \leq R_3, \tag{4.15}$$

where

$$\begin{aligned} R_3 &= \frac{\rho_{_0}\pi}{2} + \mid Q \mid + \mid \lambda \mid [(\mid g_{_\varphi}\,(t_1,t_1,0) \mid \\ &+ a_1(r) \mid \varphi \mid (c_1(r)(1 + H^\delta(v)) \mid t_1 - t_2 \mid^\delta) \\ &+ (\mid G_v(t_2,0) \mid + c_1(r) \mid v \mid)(a_1(r)(2 + H^\delta(\varphi)) \mid t_1 - t_2 \mid^\delta)]. \end{aligned}$$

Substituting from (4.14) and (4.15) into (4.13) we have

$$\parallel z \parallel_\delta \leq R_2(1 + R_3)e^{R_3}. \tag{4.16}$$

From (4.14) we can determine $\| \frac{1}{z} \|_\delta$,

$$\| \frac{1}{z} \|_\delta \leq \frac{1}{\| \sqrt{u} \|_\delta}(1+ \| \Gamma \|_\delta) e^{\|\Gamma\|_\delta}.$$

But

$$\| \frac{1}{\sqrt{u}} \|_c \leq \| \frac{1}{\sqrt{1+\lambda^2 G_v^2(t_2, v(t_2))\ g_{\varphi}^2(t_2,t_2,\varphi(t_2))}} \|_c \leq 1$$

and

$$\mid \frac{1}{\sqrt{u(t_1)}} - \frac{1}{\sqrt{u(t_2)}} \mid \leq \mid \sqrt{u(t_1)} - \sqrt{u(t_2)} \mid \leq \lambda^2 \beta$$

then

$$\| \frac{1}{\sqrt{u}} \|_\delta \leq R_4,$$

where

$$R_4 = (1+\lambda^2\beta).$$

So that

$$\| \frac{1}{z} \|_\delta \leq R_4(1+R_3)e^{R_3}. \tag{4.17}$$

Then:

$$\| \grave{B}(\varphi_0)^{-1} \| \leq \alpha_0,$$

where

$$\begin{aligned} \alpha_0 &= R_1(1+\rho \mid \lambda \mid R_2(1+R_3)e^{R_3})(\| G_v(t,0) \|_c \\ &+ c_1(r)(1+ \| v \|)(a_1(r)(2+ \| \varphi \|) \\ &+ \| g_{\varphi}(t,t,0) \|_c)(R_4(1+R_3)e^{R_3}) \\ &+ \mid \rho_1 \mid\mid \lambda \mid R_2(1+R_3)e^{R_3})(\| G_v(t,0) \|_c + c_1(r)(1+ \| v \|) \\ &+ 2\tilde{C} R_2(1+R_3)e^{R_3}, \end{aligned}$$

Hence the theorem is proved.
Now ,we determine $\| \grave{B}(\varphi_0)^{-1} B(\varphi_0) \|$ as follows:

$$\| \grave{B}(\varphi_0)^{-1} B(\varphi_0) \| \leq \alpha_0 \| B(\varphi_0) \| \leq \mu_0, \tag{4.18}$$

where

$$\mu_0 = \alpha_0(\| \varphi_0 \| + \mid \lambda \mid c_0(r)(1+ \| v \|) + \| G(t,0) \|_c),$$

Since

$$a = \| \grave{B}(\varphi_0)^{-1} B(\varphi_0) \|,$$

hence

$$a \leq b[\| \varphi_0 \| + \mid \lambda \mid c_0(r)(1+ \| v \|) + \| G(t,0) \|_c],$$

and

$$b \leq \alpha_0$$

therefore , the following theorem is valid.

Theorem 4.2 Suppose that the equation (2.3)has a unique positive root r_* in $[0, R]$ and $\psi(R) \leq 0$. Then the equation$B(\varphi) = 0$has a unique solution φ_* in $S(\varphi_0, R)$ and the Newton-Kantorovich approximations:

$$\varphi_n = \varphi_{n-1} - \grave{B}(\varphi_{n-1})^{-1} B(\varphi_{n-1}), \quad n \in N,$$

are defined for all $n \in N$, belong to $S(\varphi_0, r_*)$ and converges to φ_*. Moreover , the following estimate holds

$$\| \varphi_{n+1} - \varphi_n \| \leq r_{n+1} - r_n, \quad \| \varphi_* - \varphi_n \| \leq r_* - r_n,$$

where the sequence $(r_n)_{n \in N}$converges to r_*, is defined by the recurrence formula

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{\psi(r_n)}{\grave{\psi}(r_n)}, \quad n \in N.$$

We will illustrate the theorem 4.2 by the following example. Consider the nonlinear function

$$f(u) = \frac{1}{6}u^3 + \frac{1}{6}u^2 - \frac{5}{6}u + \frac{1}{3},$$

with derivative

$$\grave{f}(u) = \frac{1}{2}u^2 + \frac{1}{3}u - \frac{5}{6},$$

it's clear that

$$\begin{aligned} \frac{\| \grave{f}(u_1) - \grave{f}(u_2) \|}{\| u_1 - u_2 \|} &\leq \frac{1}{6}[\| 3(u_1 + u_2) \| + 2] \\ &\leq r + \frac{1}{3}, \end{aligned}$$

therefore we get

$$k(r) = r + \frac{1}{3}.$$

Obviously, the scaler equation (2.3) takes the form

$$\psi(r) = a + \frac{b}{6}r^3 + \frac{b}{6}r^2 - r.$$

The equation

$$\psi(r) = 0, \tag{4.19}$$

has a unique positive solution r_* in$[0, R]$ if and only if

$$[\frac{q}{2}]^2 + [\frac{p}{3}]^3 > 0,$$

where ,

$$p = -\frac{1}{3} - \frac{6}{b} \quad and \quad q = \frac{2}{27} + \frac{2}{b} + \frac{6a}{b}.$$

Hence , the function $f(u) = 0$ has a unique solution u_* in $S(0, R)$ and the assumptions of theorem (4.2) are verified.

Received: October 2008. Revised: February 2009.

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