

Examples of a complex hyperpolar action without singular orbit

NAOYUKI KOIKE

*Department of Mathematics, Faculty of Science,
Tokyo University of Science,
1-3 Kagurazaka Shinjuku-ku,
Tokyo 162-8601, Japan
email: koike@ma.kagu.tus.ac.jp*

ABSTRACT

The notion of a complex hyperpolar action on a symmetric space of non-compact type has recently been introduced as a counterpart to the hyperpolar action on a symmetric space of compact type. As examples of a complex hyperpolar action, we have Hermann type actions, which admit a totally geodesic singular orbit (or a fixed point) except for one example. All principal orbits of Hermann type actions are curvature-adapted and proper complex equifocal. In this paper, we give some examples of a complex hyperpolar action without singular orbit as solvable group free actions and find complex hyperpolar actions all of whose orbits are non-curvature-adapted or non-proper complex equifocal among the examples. Also, we show that some of the examples possess the only minimal orbit.

RESUMEN

La noción de una acción hiperpolar compleja sobre un espacio simétrico de tipo no compacto fue recientemente introducida como el análogo de la acción hiperpolar sobre un espacio simétrico de tipo compacto. Como ejemplos de una acción hiperpolar compleja, nosotros tenemos acciones de tipo Hermann, las cuales admiten una órbita (o un punto fijo) singular totalmente geodésica excepto para un ejemplo. Todas las órbitas principales de acciones de tipo Hermann son curvatura-adaptadas y unifocales complejas propias. En

este artículo, nosotros damos algunos ejemplos de una acción hiperpolar compleja sin orbitas singulares como grupo soluble de acciones libres y encontramos acciones complejas hiperpolares cuyas orbitas son no curvatura-adaptadas o no propias unifocales complejas. También, mostramos que algunos de los ejemplos poseen solamente orbitas minimales.

Key words and phrases: *symmetric space, complex hyperpolar action, complex equifocal submanifold.*

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1 Introduction

In symmetric spaces, the notion of an equifocal submanifold was introduced in [30]. This notion is defined as a compact submanifold with globally flat and abelian normal bundle such that the focal radius functions for each parallel normal vector field are constant. However, this conditions of the equifocality is rather weak in the case where the symmetric spaces are of non-compact type and the submanifold is non-compact. So we [13, 14] have recently introduced the notion of a complex equifocal submanifold in a symmetric space G/K of non-compact type. This notion is defined by imposing the constancy of the complex focal radius functions instead of focal radius functions. Here we note that the complex focal radii are the quantities indicating the positions of the focal points of the extrinsic complexification of the submanifold, where the submanifold needs to be assumed to be complete and of class C^ω (i.e., real analytic). On the other hand, Heintze-Liu-Olmos [10] has recently defined the notion of an isoparametric submanifold with flat section in a general Riemannian manifold as a submanifold such that the normal holonomy group is trivial, its sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction and that the image of the normal space at each point by the normal exponential map is flat and totally geodesic. We [14] showed the following fact:

All isoparametric submanifolds with flat section in a symmetric space G/K of non-compact type are complex equifocal and that conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric ones with flat section.

Here the curvature-adaptedness means that, for each normal vector v of the submanifold, the Jacobi operator $R(\cdot, v)v$ preserves the tangent space of the submanifold invariantly and the restriction of $R(\cdot, v)v$ to the tangent space commutes with the shape operator A_v , where R is the curvature tensor of G/K . Furthermore, as a subclass of the class of complex equifocal submanifolds, we [15] defined that of the proper complex equifocal submanifolds in G/K as a complex equifocal submanifold whose lifted submanifold to $H^0([0, 1], \mathfrak{g})$ ($\mathfrak{g} := \text{Lie } G$) through some pseudo-Riemannian submersion of $H^0([0, 1], \mathfrak{g})$ onto G/K is proper complex isoparametric in the sense of [13], where we note that $H^0([0, 1], \mathfrak{g})$ is a pseudo-Hilbert space consisting of certain kind of paths in the Lie algebra \mathfrak{g} of G . Let G/K be a symmetric space of non-compact type and H be a closed subgroup of G which admits an embedded complete flat submanifold meeting all H -orbits orthogonally. Then the H -action on G/K is called a *complex hyperpolar action*. This action was named thus because this action has not necessarily a singular orbit (which should be called a pole of this action) but the complexified action

has a singular orbit. Note that all cohomogeneity one actions are complex hyperpolar. We [14] showed that principal orbits of a complex hyperpolar actions are isoparametric submanifolds with flat section and hence they are complex equifocal. Conversely we [17] have recently showed that all homogeneous complex equifocal submanifolds occurs as principal orbits of complex hyperpolar actions. Let H' be a symmetric subgroup of G (i.e., there exists an involution σ of G with $(\text{Fix } \sigma)_0 \subset H' \subset \text{Fix } \sigma$), where $\text{Fix } \sigma$ is the fixed point group of σ and $(\text{Fix } \sigma)_0$ is the identity component of $\text{Fix } \sigma$. Then the H' -action on G/K is called a *Hermann type action*. A Hermann type action admits a totally geodesic orbit or a fixed point. Except for one example, the totally geodesic orbit is singular (see Theorem E of [17]). We [15] showed that principal orbits of a Hermann type action are proper complex equifocal and curvature-adapted. We [17] have recently showed that all complex hyperpolar actions of cohomogeneity greater than one on G/K admitting a totally geodesic orbit and all complex hyperpolar actions of cohomogeneity one on G/K admitting reflective orbit are orbit equivalent to Hermann type actions (see Theorems B, C and Remark 1.1 in [17]). Let G/K be a symmetric space of non-compact type, $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ ($\mathfrak{f} := \text{Lie } K$) be the Cartan decomposition associated with (G, K) , \mathfrak{a} be the maximal abelian subspace of \mathfrak{p} , $\tilde{\mathfrak{a}}$ be the Cartan subalgebra of \mathfrak{g} containing \mathfrak{a} and $\mathfrak{g} = \mathfrak{f} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa's decomposition. Let A, \tilde{A} and N be the connected Lie subgroups of G having $\mathfrak{a}, \tilde{\mathfrak{a}}$ and \mathfrak{n} as their Lie algebras, respectively. Let $\pi : G \rightarrow G/K$ be the natural projection. The symmetric space G/K is identified with the solvable group AN with a left-invariant metric through $\pi|_{AN}$. In this paper, we first prove the following fact for a complex hyperpolar action without singular orbit.

Theorem A. *Any complex hyperpolar action on $G/K (= AN)$ without singular orbit is orbit equivalent to the free action of some solvable group contained in $\tilde{A}N$.*

Next we give some examples of a complex hyperpolar action without singular orbit as the free actions of solvable groups contained in AN (see Examples 1 and 2 of Section 3), which contain examples of cohomogeneity one actions without singular orbit constructed by J. Berndt and H. Tamaru [3] as special cases (see also [1]). Among these examples, we find complex hyperpolar actions all of whose orbits are non-proper complex equifocal or non-curvature-adapted. As its result, we have the following facts.

Theorem B. (i) *For any symmetric space G/K of non-compact type and any positive integer r with $r \leq \text{rank}(G/K)$, there exists a complex hyperpolar action without singular orbit such that the cohomogeneity is equal to r and that any of the orbits is not proper complex equifocal.*

(ii) *Let G/K be one of $SU(p, q)/S(U(p) \times U(q))$ ($p < q$), $Sp(p, q)/Sp(p) \times Sp(q)$ ($p < q$), $SO^*(2n)/U(n)$ ($n : \text{odd}$), $E_6^{-14}/Spin(10) \cdot U(1)$ or $F_4^{-20}/Spin(9)$. Then, for any positive integer r with $r \leq \text{rank}(G/K)$, there exists a complex hyperpolar action without singular orbit such that the cohomogeneity is equal to r and that any of the orbits is not curvature-adapted.*

Also, among those examples, we find complex hyperpolar actions possessing the only minimal orbit. As its result, we have the following fact.

Theorem C. For any irreducible symmetric space G/K of non-compact type and any positive integer $r \leq [\frac{1}{2}(\text{rank}(G/K) + 1)]$, there exists a complex hyperpolar action without singular orbit such that the cohomogeneity is equal to r and that the only orbit is minimal.

2 Complex equifocal submanifolds

In this section, we recall the notions of a complex equifocal submanifold and a proper complex equifocal submanifold. We first recall the notion of a complex equifocal submanifold. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M . Let $v \in T_x^\perp M$ and $X \in T_x M$ ($x = gK$). Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The strongly M -Jacobi field Y along γ_v with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D_{sv}^{co} - sD_{sv}^{si} \circ A_v))(X),$$

where $Y'(0) = \tilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$ and D_{sv}^{co} (resp. D_{sv}^{si}) is given by

$$\begin{aligned} D_{sv}^{co} &= g_* \circ \cos(\sqrt{-1}\text{ad}(sg_*^{-1}v)) \circ g_*^{-1} \\ \left(\text{resp. } D_{sv}^{si} &= g_* \circ \frac{\sin(\sqrt{-1}\text{ad}(sg_*^{-1}v))}{\sqrt{-1}\text{ad}(sg_*^{-1}v)} \circ g_*^{-1} \right). \end{aligned}$$

Here ad is the adjoint representation of the Lie algebra \mathfrak{g} of G . All focal radii of M along γ_v are obtained as real numbers s_0 with $\text{Ker}(D_{s_0 v}^{co} - s_0 D_{s_0 v}^{si} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\text{Ker}(D_{z_0 v}^{co} - z_0 D_{z_0 v}^{si} \circ A_v^c) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \text{Ker}(D_{z_0 v}^{co} - z_0 D_{z_0 v}^{si} \circ A_v^c)$ the *multiplicity* of the complex focal radius z_0 , where A_v^c is the complexification of A_v and $D_{z_0 v}^{co}$ (resp. $D_{z_0 v}^{si}$) is a \mathbf{C} -linear transformation of $(T_x N)^c$ defined by

$$\begin{aligned} D_{z_0 v}^{co} &= g_*^c \circ \cos(\sqrt{-1}\text{ad}^c(z_0 g_*^{-1}v)) \circ (g_*^c)^{-1} \\ \left(\text{resp. } D_{z_0 v}^{si} &= g_*^c \circ \frac{\sin(\sqrt{-1}\text{ad}^c(z_0 g_*^{-1}v))}{\sqrt{-1}\text{ad}^c(z_0 g_*^{-1}v)} \circ (g_*^c)^{-1} \right), \end{aligned}$$

where g_*^c (resp. ad^c) is the complexification of g_* (resp. ad). Here we note that, in the case where M is of class C^ω , complex focal radii along γ_v indicate the positions of focal points of the extrinsic complexification $M^c(\hookrightarrow G^c/K^c)$ of M along the complexified geodesic $\gamma_{\iota_* v}^c$, where G^c/K^c is the anti-Kaehlerian symmetric space associated with G/K and ι is the natural immersion of G/K into G^c/K^c . See Section 4 of [14] about the definitions of G^c/K^c , $M^c(\hookrightarrow G^c/K^c)$ and $\gamma_{\iota_* v}^c$. Also, for a complex focal radius z_0 of M along γ_v , we call $z_0 v \in (T_x^\perp M)^c$ a *complex focal normal vector of M at x* . Furthermore, assume that M has globally flat normal bundle, that is, the normal holonomy group of M is trivial. Let \tilde{v} be a parallel unit normal vector field of M . Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \dots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or " $|r_{i,x}| = |r_{i+1,x}|$ & $\text{Re } r_{i,x} > \text{Re } r_{i+1,x}$ " or " $|r_{i,x}| = |r_{i+1,x}|$ & $\text{Re } r_{i,x} = \text{Re } r_{i+1,x}$ & $\text{Im } r_{i,x} = -\text{Im } r_{i+1,x} < 0$ ". Let r_i ($i = 1, 2, \dots$) be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i ($i = 1, 2, \dots$) *complex focal radius functions for \tilde{v}* . We call $r_i \tilde{v}$ a *complex focal normal vector field for \tilde{v}* . If, for each parallel unit normal vector field \tilde{v} of M , the number of distinct complex focal radii

along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for \tilde{v} is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*.

Let $N = G/K$ be a symmetric space of non-compact type and π be the natural projection of G onto G/K . Let (\mathfrak{g}, θ) be the orthogonal symmetric Lie algebra of G/K , $\mathfrak{f} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$, which is identified with the tangent space $T_{eK}N$. Let $\langle \cdot, \cdot \rangle$ be the $\text{Ad}(G)$ -invariant non-degenerate symmetric bilinear form of \mathfrak{g} inducing the Riemannian metric of N . Note that $\langle \cdot, \cdot \rangle|_{\mathfrak{f} \times \mathfrak{f}}$ (resp. $\langle \cdot, \cdot \rangle|_{\mathfrak{p} \times \mathfrak{p}}$) is negative (resp. positive) definite. Denote by the same symbol $\langle \cdot, \cdot \rangle$ the bi-invariant pseudo-Riemannian metric of G induced from $\langle \cdot, \cdot \rangle$ and the Riemannian metric of N . Set $\mathfrak{g}_+ := \mathfrak{p}$, $\mathfrak{g}_- := \mathfrak{f}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}_\pm} := -\pi_{\mathfrak{g}_-}^* \langle \cdot, \cdot \rangle + \pi_{\mathfrak{g}_+}^* \langle \cdot, \cdot \rangle$, where $\pi_{\mathfrak{g}_-}$ (resp. $\pi_{\mathfrak{g}_+}$) is the projection of \mathfrak{g} onto \mathfrak{g}_- (resp. \mathfrak{g}_+). Let $H^0([0, 1], \mathfrak{g})$ be the space of all L^2 -integrable paths $u : [0, 1] \rightarrow \mathfrak{g}$ (with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}_\pm}$). Let $H^0([0, 1], \mathfrak{g}_-)$ (resp. $H^0([0, 1], \mathfrak{g}_+)$) be the space of all L^2 -integrable paths $u : [0, 1] \rightarrow \mathfrak{g}_-$ (resp. $u : [0, 1] \rightarrow \mathfrak{g}_+$) with respect to $-\langle \cdot, \cdot \rangle|_{\mathfrak{g}_- \times \mathfrak{g}_-}$ (resp. $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_+ \times \mathfrak{g}_+}$). It is clear that $H^0([0, 1], \mathfrak{g}) = H^0([0, 1], \mathfrak{g}_-) \oplus H^0([0, 1], \mathfrak{g}_+)$. Define a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_0$ of $H^0([0, 1], \mathfrak{g})$ by $\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle dt$. It is easy to show that the decomposition $H^0([0, 1], \mathfrak{g}) = H^0([0, 1], \mathfrak{g}_-) \oplus H^0([0, 1], \mathfrak{g}_+)$ is an orthogonal time-space decomposition with respect to $\langle \cdot, \cdot \rangle_0$. For simplicity, set $H_\pm^0 := H^0([0, 1], \mathfrak{g}_\pm)$ and $\langle \cdot, \cdot \rangle_{0, H_\pm^0} := -\pi_{H_-^0}^* \langle \cdot, \cdot \rangle_0 + \pi_{H_+^0}^* \langle \cdot, \cdot \rangle_0$, where $\pi_{H_-^0}$ (resp. $\pi_{H_+^0}$) is the projection of $H^0([0, 1], \mathfrak{g})$ onto H_-^0 (resp. H_+^0). It is clear that $\langle u, v \rangle_{0, H_\pm^0} = \int_0^1 \langle u(t), v(t) \rangle_{\mathfrak{g}_\pm} dt$ ($u, v \in H^0([0, 1], \mathfrak{g})$). Hence $(H^0([0, 1], \mathfrak{g}), \langle \cdot, \cdot \rangle_{0, H_\pm^0})$ is a Hilbert space, that is, $(H^0([0, 1], \mathfrak{g}), \langle \cdot, \cdot \rangle_0)$ is a pseudo-Hilbert space. Let $H^1([0, 1], G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0, 1] \rightarrow G$ such that the weak derivative g' of g is squared integrable (with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}_\pm}$), that is, $g_*^{-1}g' \in H^0([0, 1], \mathfrak{g})$. Define a map $\phi : H^0([0, 1], \mathfrak{g}) \rightarrow G$ by $\phi(u) = g_u(1)$ ($u \in H^0([0, 1], \mathfrak{g})$), where g_u is the element of $H^1([0, 1], G)$ satisfying $g_u(0) = e$ and $g_{u*}^{-1}g'_u = u$. We call this map the *parallel transport map* (from 0 to 1). This submersion ϕ is a pseudo-Riemannian submersion of $(H^0([0, 1], \mathfrak{g}), \langle \cdot, \cdot \rangle_0)$ onto $(G, \langle \cdot, \cdot \rangle)$. Let $\pi : G \rightarrow G/K$ be the natural projection. It follows from Theorem A of [13] (resp. Theorem 1 of [14]) that, in the case where M is curvature adapted (resp. of class C^ω), M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. See [13] about the definition of a complex isoparametric submanifold in a pseudo-Hilbert space. In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ are proper complex isoparametric, that is, for each normal vector v , there exists a pseudo-orthonormal base of the complexified tangent sapce consisting of the eigenvectors of the complexified shape operator for v , then we call M a *proper complex equifocal submanifold*.

Next we recall the notion of an infinite dimensional anti-Kaehlerian isoparametric submanifold. Let M be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space V and A be the shape tensor of M . See [14] about the definitions of an infinite dimensional anti-Kaehlerian space and anti-Kaehlerian Fredholm submanifold in the space. Denote by the same symbol J the complex structures of M and V . Fix a unit normal vector v of M . If there exists $X (\neq 0) \in TM$ with $A_v X = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *J-eigenvalue of A_v* (or a *complex principal curvature of direction v*) and call X a *J-eigenvector for $a + b\sqrt{-1}$* . Also, we call the space of all *J*-eigenvectors for $a + b\sqrt{-1}$ a *J-eigenspace for $a + b\sqrt{-1}$* . The *J*-eigenspaces are orthogonal to one another and each *J*-eigenspace is *J*-invariant. We call the set of all *J*-eigenvalues of A_v the *J-spectrum of A_v* and denote it by $\text{Spec}_J A_v$. The set $\text{Spec}_J A_v \setminus \{0\}$ is described as follows:

$$\text{Spec}_J A_v \setminus \{0\} = \{\lambda_i \mid i = 1, 2, \dots\}$$

$$\left(\begin{array}{l} |\lambda_i| > |\lambda_{i+1}| \text{ or } "|\lambda_i| = |\lambda_{i+1}| \ \& \ \text{Re } \lambda_i > \text{Re } \lambda_{i+1}" \\ \text{or } "|\lambda_i| = |\lambda_{i+1}| \ \& \ \text{Re } \lambda_i = \text{Re } \lambda_{i+1} \ \& \ \text{Im } \lambda_i = -\text{Im } \lambda_{i+1} > 0" \end{array} \right).$$

Also, the J -eigenspace for each J -eigenvalue of A_v other than 0 is of finite dimension. We call the J -eigenvalue λ_i the i -th complex principal curvature of direction v . Assume that M has globally flat normal bundle. Fix a parallel normal vector field \tilde{v} of M . Assume that the number (which may be ∞) of distinct complex principal curvatures of direction \tilde{v}_x is independent of the choice of $x \in M$. Then we can define functions $\tilde{\lambda}_i$ ($i = 1, 2, \dots$) on M by assigning the i -th complex principal curvature of direction \tilde{v}_x to each $x \in M$. We call this function $\tilde{\lambda}_i$ the i -th complex principal curvature function of direction \tilde{v} . If M satisfies the following condition (AKI), then we call M an anti-Kaehlerian isoparametric submanifold:

(AKI) For each parallel normal vector field \tilde{v} , the number of distinct complex principal curvatures of direction \tilde{v}_x is independent of the choice of $x \in M$, each complex principal curvature function of direction \tilde{v} is constant on M and it has constant multiplicity.

Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of T_xM . If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal base of T_xM , then we call $\{e_i\}_{i=1}^\infty$ a J -orthonormal base. If there exists a J -orthonormal base consisting of J -eigenvectors of A_v , then A_v is said to be diagonalized with respect to the J -orthonormal base. If M is anti-Kaehlerian isoparametric and, for each $v \in T^\perp M$, the shape operator A_v is diagonalized with respect to a J -orthonormal base, then we call M a proper anti-Kaehlerian isoparametric submanifold. For arbitrary two unit normal vector v_1 and v_2 of a proper anti-Kaehlerian isoparametric submanifold, the shape operators A_{v_1} and A_{v_2} are simultaneously diagonalized with respect to a J -orthonormal base. Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space V . Let $\{E_i | i \in I\}$ be the family of distributions on M such that, for each $x \in M$, $\{E_i(x) | i \in I\}$ is the set of all common J -eigenspaces of A_v 's ($v \in T_x^\perp M$). The relation $T_xM = \bigoplus_{i \in I} E_i$ holds. Let λ_i ($i \in I$) be the section of $(T^\perp M)^* \otimes \mathbf{C}$ such that $A_v = \text{Re}\lambda_i(v)\text{id} + \text{Im}\lambda_i(v)J$ on $E_i(\pi(v))$ for each $v \in T^\perp M$, where π is the bundle projection of $T^\perp M$. We call λ_i ($i \in I$) complex principal curvatures of M and call distributions E_i ($i \in I$) complex curvature distributions of M .

In the case where M is a real analytic submanifold in a symmetric space G/K of non-compact type, it is shown that M is complex equifocal if and only if $(\pi^c \circ \phi^c)^{-1}(M^c)$ is anti-Kaehlerian isoparametric, where π^c is the natural projection of G^c onto G^c/K^c and ϕ^c is the parallel transport map for G^c (which is defined in similar to the above ϕ). Also, it is shown that M is proper complex equifocal if and only if $(\pi^c \circ \phi^c)^{-1}(M^c)$ is proper anti-Kaehlerian isoparametric.

3 Proof of Theorems A and B

In this section, we first prove Theorem A.

Proof of Theorem A. Let H be a complex hyperpolar action on $G/K(= AN)$ without singular orbit, $H = LR$ (L : semi-simple, R : solvable) be the Levi decomposition of H and $L = K_L A_L N_L$ (K_L : compact, A_L : abelian, N_L : nilpotent) be the Iwasawa decomposition of L . Since K_L is compact, it has a fixed point p_0 by the Cartan's fixed point theorem. Suppose that $K_L \cdot p \not\subset A_L N_L R \cdot p$ for some

$p \in G/K$. Then we have $\dim H \cdot p_0 < \dim H \cdot p$, which implies that $H \cdot p_0$ is a singular orbit. This contradicts the fact that the H -action has no singular orbit. Hence it follows that $K_L \cdot p \subset A_L N_L R \cdot p$ for any $p \in G/K$. Therefore we can show that the $A_L N_L R$ -action has the same orbits as the H -action. The group $A_L N_L R$ is decomposed into the product of some compact subgroup T' and some solvable normal subgroup S' admitting a maximal compact normal subgroup S'_K contained in the center of S' such that S'/S'_K is simply connected (see Theorem 6 of [19]). Since T' is compact, it is shown by the same argument as above that the S' -action has the same orbit as the $A_L N_L R$ -action (hence the H -action). Take any $p \in G/K$ and any $g \in S'$ with $g \neq e$. Since S' acts on G/K effectively, there exists $p_1 \in G/K$ with $g(p_1) \neq p_1$. The section Σ_{p_1} through p_1 is mapped into the section $\Sigma_{g(p_1)}$ through $g(p_1)$ by g . Since the S' -action has no singular orbit, we have $\Sigma_{p_1} \cap \Sigma_{g(p_1)} = \emptyset$. Let q be the intersection of $H \cdot p$ with Σ_{p_1} . Then $g(q)$ is the intersection of $H \cdot p$ with $\Sigma_{g(p_1)}$. Hence we have $g(q) \neq q$. Therefore S' acts on each H -orbit effectively. Since the isotropy group S'_p of S' at any $p \in G/K$ is compact, it is contained in a conjugate of S'_K (see Theorem 4 of [19]). Hence S'_p is contained in the center of S' . Therefore, since the S'_p -action has a fixed point p and it is effective, it is trivial. Thus the S' -action is free. Let $\mathfrak{s}' := \text{Lie } S'$ (the Lie algebra of S'), $\tilde{\mathfrak{s}}'$ be a maximal solvable subalgebra of \mathfrak{g} containing \mathfrak{s}' and \tilde{S}' be the connected subgroup of G with $\text{Lie } \tilde{S}' = \tilde{\mathfrak{s}}'$. Since \mathfrak{g} is a real semi-simple Lie algebra and $\tilde{\mathfrak{s}}'$ is a maximal solvable subalgebra of \mathfrak{g} , $\tilde{\mathfrak{s}}'$ contains a Cartan subalgebra $\tilde{\mathfrak{a}}'$ of \mathfrak{g} . Let \mathfrak{t}' (resp. \mathfrak{a}') be the toroidal part (resp. the vector part) of $\tilde{\mathfrak{a}}'$. There exists a Cartan decomposition $\mathfrak{g} = \mathfrak{f}' + \mathfrak{p}'$ of \mathfrak{g} with $\mathfrak{t}' \subset \mathfrak{f}'$ and $\mathfrak{a}' \subset \mathfrak{p}'$. Let $\mathfrak{g} = \mathfrak{g}'_0 + \sum_{\lambda \in \Delta'} \mathfrak{g}'_\lambda$ be the root space decomposition with respect to \mathfrak{a}' (i.e., \mathfrak{g}'_0 is the centralizer of \mathfrak{a}' in \mathfrak{g} and $\mathfrak{g}'_\lambda = \{X \in \mathfrak{g} \mid \text{ad}(a)(X) = \lambda(a)X \text{ for all } a \in \mathfrak{a}'\}$ and $\Delta' = \{\lambda \in (\mathfrak{a}')^* \setminus \{0\} \mid \mathfrak{g}'_\lambda \neq \{0\}\}$). Let $\mathfrak{n}' := \sum_{\lambda \in \Delta'_+} \mathfrak{g}'_\lambda$, where Δ'_+ is the positive root system with respect to some lexicographic ordering of \mathfrak{a}' . The algebra $\tilde{\mathfrak{a}}' + \mathfrak{n}'$ is a maximal solvable subalgebra of \mathfrak{g} . According to a result of [21], we may assume that $\tilde{\mathfrak{s}}' = \tilde{\mathfrak{a}}' + \mathfrak{n}'$ by retaking $\tilde{\mathfrak{a}}'$ if necessary. By imitating the proof of Lemma 5.1 of [3], it is shown that \mathfrak{a}' is a maximal abelian subspace of \mathfrak{p}' because the S' -action has flat section. There exists $g \in G$ satisfying $\text{Ad}(g)(\mathfrak{f}') = \mathfrak{f}$, $\text{Ad}(g)(\mathfrak{p}') = \mathfrak{p}$, $\text{Ad}(g)(\mathfrak{a}') = \mathfrak{a}$ and $\text{Ad}(g)(\tilde{\mathfrak{a}}') = \tilde{\mathfrak{a}}$, where Ad is the adjoint representation of G , \mathfrak{a} and $\tilde{\mathfrak{a}}$ are as in Introduction. Let $\mathfrak{s} := \text{Ad}(g)(\mathfrak{s}')$ and S be the connected subgroup of G with $\text{Lie } S = \mathfrak{s}$. Since the S -action is conjugate to the S' -action and $S \subset \tilde{AN}$, we obtain the statement of Theorem A. q.e.d.

Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Fix a lexicographic ordering of \mathfrak{a} . Let $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$, $\mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Delta_+} \mathfrak{p}_\lambda$ and $\mathfrak{f} = \mathfrak{f}_0 + \sum_{\lambda \in \Delta_+} \mathfrak{f}_\lambda$ be the root space decompositions of \mathfrak{g} , \mathfrak{p} and \mathfrak{f} with respect to \mathfrak{a} , where we note that

$$\begin{aligned} \mathfrak{g}_\lambda &= \{X \in \mathfrak{g} \mid \text{ad}(a)X = \lambda(a)X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \Delta), \\ \mathfrak{p}_\lambda &= \{X \in \mathfrak{p} \mid \text{ad}(a)^2 X = \lambda(a)^2 X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \Delta_+), \\ \mathfrak{f}_\lambda &= \{X \in \mathfrak{f} \mid \text{ad}(a)^2 X = \lambda(a)^2 X \text{ for all } a \in \mathfrak{a}\} \quad (\lambda \in \Delta_+ \cup \{0\}). \end{aligned}$$

Also, let $\mathfrak{g} = \mathfrak{f} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa decomposition of \mathfrak{g} and $G = KAN$ be the corresponding Iwasawa decomposition of G , where we note that $\mathfrak{n} = \sum_{\lambda \in \Delta_+} \mathfrak{g}_\lambda$. Now we shall give examples of a solvable group contained in AN whose action on $G/K (= AN)$ is complex hyperpolar. Denote by π the natural projection of G onto G/K . Since G/K is of non-compact type, π gives a diffeomorphism of AN onto G/K . Denote by $\langle \cdot, \cdot \rangle$ the left-invariant metric of AN induced from that of G/K by $\pi|_{AN}$. Also,

denote by $\langle \cdot, \cdot \rangle^G$ the bi-invariant metric of G inducing that of G/K . Note that $\langle \cdot, \cdot \rangle \neq \iota^* \langle \cdot, \cdot \rangle^G$, where ι is the inclusion map of AN into G . Let l be a r -dimensional subspace of $\mathfrak{a} + \mathfrak{n}$ and set $\mathfrak{s} := (\mathfrak{a} + \mathfrak{n}) \ominus l$, where $(\mathfrak{a} + \mathfrak{n}) \ominus l$ denotes the orthogonal complement of l in $\mathfrak{a} + \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_e$, where e is the identity element of G . If \mathfrak{s} is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ and $l_{\mathfrak{p}} := \text{pr}_{\mathfrak{p}}(l)$ ($\text{pr}_{\mathfrak{p}} : \text{the orthogonal projection of } \mathfrak{g} \text{ onto } \mathfrak{p}$) is abelian, then the S -action ($S := \exp_G(\mathfrak{s})$) is a complex hyperpolar action without singular orbit. We shall give examples of such a subalgebra \mathfrak{s} of $\mathfrak{a} + \mathfrak{n}$ and investigate the structure of the S -orbit.

Example 1. Let \mathfrak{b} be a $r (\geq 1)$ -dimensional subspace of \mathfrak{a} and $\mathfrak{s}_{\mathfrak{b}} := (\mathfrak{a} + \mathfrak{n}) \ominus \mathfrak{b}$. It is clear that $\mathfrak{b}_{\mathfrak{p}} (= \mathfrak{b})$ is abelian and that $\mathfrak{s}_{\mathfrak{b}}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Hence the $S_{\mathfrak{b}}$ -action ($S_{\mathfrak{b}} := \exp_G(\mathfrak{s}_{\mathfrak{b}})$) on G/K is a complex hyperpolar action without singular orbit.

Example 2. Let $\{\lambda_1, \dots, \lambda_k\}$ be a subset of a simple root system Π of Δ such that $H_{\lambda_1}, \dots, H_{\lambda_k}$ are mutually orthogonal, \mathfrak{b} be a subspace of $\mathfrak{a} \ominus \text{Span}\{H_{\lambda_1}, \dots, H_{\lambda_k}\}$ (where \mathfrak{b} may be $\{0\}$) and l_i ($i = 1, \dots, k$) be a one-dimensional subspace of $\mathbf{R}H_{\lambda_i} + \mathfrak{g}_{\lambda_i}$ with $l_i \neq \mathbf{R}H_{\lambda_i}$, where H_{λ_i} is the element of \mathfrak{a} defined by $\langle H_{\lambda_i}, \cdot \rangle \lambda_i(\cdot)$ and $\mathbf{R}H_{\lambda_i}$ is the subspace of \mathfrak{a} spanned by H_{λ_i} . Set $l := \mathfrak{b} + \sum_{i=1}^k l_i$. Then, it follows from Lemma 3.1 (see the below) that $l_{\mathfrak{p}}$ is abelian and that $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k} := (\mathfrak{a} + \mathfrak{n}) \ominus l$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Hence the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action ($S_{\mathfrak{b}, l_1, \dots, l_k} := \exp_G(\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k})$) on G/K is a complex hyperpolar action without singular orbit.

Lemma 3.1. *Let l and $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ be as in Example 2. Then $l_{\mathfrak{p}}$ is abelian and $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$.*

Proof. Let $H \in \mathfrak{b}$ and $X_i \in l_i$ ($i = 1, \dots, k$). Since $\lambda_i(H) = 0$ and $(X_i)_{\mathfrak{p}} \in \mathbf{R}H_{\lambda_i} \oplus \mathfrak{p}_{\lambda_i}$, we have $[H, (X_i)_{\mathfrak{p}}] = 0$. Fix $i, j \in \{1, \dots, k\}$ ($i \neq j$). Since λ_i and λ_j are simple roots and $\langle H_{\lambda_i}, H_{\lambda_j} \rangle = 0$, we have $[(X_i)_{\mathfrak{p}}, (X_j)_{\mathfrak{p}}] = 0$. Thus $l_{\mathfrak{p}}$ is abelian. Let $V, W \in \mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$. Since $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k} = (\mathfrak{a} \ominus (\mathfrak{b} + \sum_{i=1}^k \mathbf{R}H_{\lambda_i})) \oplus (\sum_{\lambda \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} \mathfrak{g}_{\lambda}) \oplus (\sum_{i=1}^k ((\mathbf{R}H_{\lambda_i} + \mathfrak{g}_{\lambda_i}) \ominus l_i))$, V and W are described as $V = V_0 + \sum_{\lambda \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} V_{\lambda} + \sum_{i=1}^k V_i$ and $W = W_0 + \sum_{\lambda \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} W_{\lambda} + \sum_{i=1}^k W_i$, respectively, where $V_0, W_0 \in \mathfrak{a} \ominus (\mathfrak{b} + \sum_{i=1}^k \mathbf{R}H_{\lambda_i})$, $V_{\lambda}, W_{\lambda} \in \mathfrak{g}_{\lambda}$ and $V_i, W_i \in (\mathbf{R}H_{\lambda_i} + \mathfrak{g}_{\lambda_i}) \ominus l_i$. Easily we have

$$\begin{aligned}
 [V, W] \equiv & \sum_{\lambda, \mu \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} [V_{\lambda}, W_{\mu}] \\
 & + \sum_{\lambda \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} \sum_{i=1}^k ([V_{\lambda}, W_i] + [V_i, W_{\lambda}]) + \sum_{i=1}^k \sum_{j=1}^k [V_i, W_j] \pmod{\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}}.
 \end{aligned}$$

Since $\lambda_1, \dots, \lambda_k$ are simple roots, $[V_{\lambda}, W_{\mu}]$, $[V_{\lambda}, W_i]$, $[V_i, W_{\lambda}]$ and $[V_i, W_j]$ ($\lambda, \mu \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}$, $1 \leq i, j \leq k$) belong to $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$. Therefore we have $[V, W] \in \mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$. Thus $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. q.e.d.

For the orbit $S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e$, we have the following facts.

Lemma 3.2. *Let $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ be as in Example 2, $\xi_0 \in \mathfrak{b}$, $\xi_{t_i}^i := \frac{1}{\cosh(|\lambda_i|t_i)}\xi^i - \frac{1}{|\lambda_i|} \tanh(|\lambda_i|t_i)H_{\lambda_i}$ be a unit vector of l_i ($i = 1, \dots, k$), where ξ^i is a unit vector of \mathfrak{g}_{λ_i} . Denote by A the shape tensor of the orbit $S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e (\subset AN)$. Then, for A_{ξ_0} and $A_{\xi_{t_i}^i}$, the following statements (i) ~ (vii) hold:*

- (i) For $X \in \mathfrak{a} \oplus (\mathfrak{b} + \sum_{i=1}^k \mathbf{R}H_{\lambda_i})$, we have $A_{\xi_0}X = A_{\xi_{t_i}^i}X = 0$ ($i = 1, \dots, k$).
- (ii) For $X \in \text{Ker}(\text{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \oplus \mathbf{R}\xi^i$, we have $A_{\xi_0}X = 0$ and $A_{\xi_{t_i}^i}X = -|\lambda_i| \tanh(|\lambda_i|t_i)X$.
- (iii) Assume that $2\lambda_i \in \Delta_+$. For $X \in \mathfrak{g}_{2\lambda_i}$, we have $A_{\xi_0}([\theta\xi^i, X]) = 0$ and

$$A_{\xi_{t_i}^i}X = -2|\lambda_i| \tanh(|\lambda_i|t_i)X - \frac{1}{2 \cosh(|\lambda_i|t_i)}[\theta\xi^i, X],$$

$$A_{\xi_{t_i}^i}([\theta\xi^i, X]) = -\frac{|\lambda_i|^2}{\cosh(|\lambda_i|t_i)}X - |\lambda_i| \tanh(|\lambda_i|t_i)[\theta\xi^i, X],$$

where θ is the Cartan involution of \mathfrak{g} with $\text{Fix } \theta = \mathfrak{f}$.

- (iv) For $X \in (\mathbf{R}\xi^i + \mathbf{R}H_{\lambda_i}) \oplus l_i$, we have $A_{\xi_0}X = 0$ and $A_{\xi_{t_i}^i}X = -|\lambda_i| \tanh(|\lambda_i|t_i)X$.
- (v) For $X \in (\mathfrak{g}_{\lambda_j} \oplus \mathbf{R}\xi^j) + ((\mathbf{R}\xi^j + \mathbf{R}H_{\lambda_j}) \oplus l_j) + \mathfrak{g}_{2\lambda_j}$ ($j \neq i$), we have $A_{\xi_0}X = A_{\xi_{t_i}^i}X = 0$.
- (vi) For $X \in \mathfrak{g}_\mu$ ($\mu \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}$), we have $A_{\xi_0}X = \mu(\xi_0)X$.
- (vii) Let $k_i := \exp\left(\frac{\pi}{\sqrt{2}|\lambda_i|}(\xi^i + \theta\xi^i)\right)$, where \exp is the exponential map of G . Then $\text{Ad}(k_i) \circ$

$A_{\xi_{t_i}^i} = -A_{\xi_{t_i}^i} \circ \text{Ad}(k_i)$ holds over $\mathfrak{n} \oplus \sum_{i=1}^k (\mathfrak{g}_{\lambda_i} + \mathfrak{g}_{2\lambda_i})$, where Ad is the adjoint representation of G .

Proof. Let $\text{pr}_{\mathfrak{a}+\mathfrak{n}}^1$ (resp. $\text{pr}_{\mathfrak{a}+\mathfrak{n}}^2$) be the projection of \mathfrak{g} onto $\mathfrak{a} + \mathfrak{n}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{f} + (\mathfrak{a} + \mathfrak{n})$ (resp. $\mathfrak{g} = (\mathfrak{f}_0 + \sum_{\lambda \in \Delta_+} \mathfrak{p}_\lambda) + (\mathfrak{a} + \mathfrak{n})$), $\text{pr}_{\mathfrak{f}}$ (resp. $\text{pr}_{\mathfrak{p}}$) be the projection of \mathfrak{g} onto \mathfrak{f} (resp. \mathfrak{p}) with respect to the decomposition $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ and $\text{pr}_{\mathfrak{f}_0}$ be the projection of \mathfrak{g} onto \mathfrak{f}_0 with respect to the decomposition $\mathfrak{g} = \mathfrak{f}_0 + (\mathfrak{a} + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda)$. Then we have

$$(3.1) \quad \text{pr}_{\mathfrak{p}} \circ \text{pr}_{\mathfrak{a}+\mathfrak{n}}^1 = \text{pr}_{\mathfrak{p}} \quad \text{and} \quad \text{pr}_{\mathfrak{f}} \circ \text{pr}_{\mathfrak{a}+\mathfrak{n}}^2 = \text{pr}_{\mathfrak{f}} - \text{pr}_{\mathfrak{f}_0}.$$

Let $H \in \mathfrak{a}$, $N_1, N_2 \in \mathfrak{n}$ and $E \in \mathfrak{g}_\lambda$ ($\lambda \in \Delta_+$). Denote by $\text{ad}(H)^*$ (resp. $\text{ad}(E)^*$) the adjoint operator of $\text{ad}(H)$ (resp. $\text{ad}(E)$) : $\mathfrak{a} + \mathfrak{n} \rightarrow \mathfrak{a} + \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_e$. Easily we can show

$$(3.2) \quad \text{ad}(H)^* = \text{ad}(H).$$

For simplicity, we denote $\text{pr}_{\mathfrak{f}}(\cdot)$ (resp. $\text{pr}_{\mathfrak{p}}(\cdot)$) by $(\cdot)_{\mathfrak{f}}$ (resp. $(\cdot)_{\mathfrak{p}}$). From (3.1) and the skew-symmetricness of $\text{ad}(\cdot)$ with respect to $\langle \cdot, \cdot \rangle_e^G$, we have

$$\begin{aligned} \langle \text{ad}(E)N_1, N_2 \rangle_e &= \langle \text{ad}(E_{\mathfrak{f}})((N_1)_{\mathfrak{p}}) + \text{ad}(E_{\mathfrak{p}})((N_1)_{\mathfrak{f}}), (N_2)_{\mathfrak{p}} \rangle_e^G \\ &= -\langle (N_1)_{\mathfrak{p}}, \text{ad}(E_{\mathfrak{f}})((N_2)_{\mathfrak{p}}) \rangle_e^G - \langle (N_1)_{\mathfrak{f}}, \text{ad}(E_{\mathfrak{p}})((N_2)_{\mathfrak{p}}) \rangle_e^G \\ &= -\langle (N_1)_{\mathfrak{p}}, (\text{pr}_{\mathfrak{a}+\mathfrak{n}}^1(\text{ad}(E_{\mathfrak{f}})N_2))_{\mathfrak{p}} \rangle_e^G \\ &\quad - \langle (N_1)_{\mathfrak{f}}, (\text{pr}_{\mathfrak{a}+\mathfrak{n}}^2(\text{ad}(E_{\mathfrak{p}})N_2))_{\mathfrak{f}} + \text{pr}_{\mathfrak{f}_0}(\text{ad}(E_{\mathfrak{p}})N_2) \rangle_e^G \\ &= -\langle N_1, \text{pr}_{\mathfrak{a}+\mathfrak{n}}^1(\text{ad}(E_{\mathfrak{f}})N_2) \rangle_e + \langle N_1, \text{pr}_{\mathfrak{a}+\mathfrak{n}}^2(\text{ad}(E_{\mathfrak{p}})N_2) \rangle_e \end{aligned}$$

and hence

$$\mathrm{pr}_{\mathfrak{n}}(\mathrm{ad}(E)^*N_2) = \mathrm{pr}_{\mathfrak{n}}(-\mathrm{pr}_{\mathfrak{a}+\mathfrak{n}}^1(\mathrm{ad}(E_{\mathfrak{f}})N_2) + \mathrm{pr}_{\mathfrak{a}+\mathfrak{n}}^2(\mathrm{ad}(E_{\mathfrak{p}})N_2)),$$

where $\mathrm{pr}_{\mathfrak{n}}$ is the projection of $\mathfrak{a} + \mathfrak{n}$ onto \mathfrak{n} . Also, we have

$$\langle \mathrm{ad}(E)H, N_2 \rangle_e = -\lambda(H)\langle E, N_2 \rangle_e = -\langle H, \langle E, N_2 \rangle_e H \lambda \rangle_e$$

and hence $\mathrm{pr}_{\mathfrak{a}}(\mathrm{ad}(E)^*N_2) = -\langle E, N_2 \rangle_e H \lambda$, where $\mathrm{pr}_{\mathfrak{a}}$ is the projection of $\mathfrak{a} + \mathfrak{n}$ onto \mathfrak{a} . Also, we can show $\mathrm{ad}(E)^*H = 0$. Therefore, we have

$$(3.3) \quad \mathrm{ad}(E)^* = \begin{cases} 0 & \text{on } \mathfrak{a} \\ -\langle E, \cdot \rangle_e \otimes H \lambda - \mathrm{pr}_{\mathfrak{n}} \circ \mathrm{pr}_{\mathfrak{a}+\mathfrak{n}}^1 \circ \mathrm{ad}(E_{\mathfrak{f}}) & \text{on } \mathfrak{n} \\ +\mathrm{pr}_{\mathfrak{n}} \circ \mathrm{pr}_{\mathfrak{a}+\mathfrak{n}}^2 \circ \mathrm{ad}(E_{\mathfrak{p}}) & \end{cases}$$

On the other hand, according to the Koszul's formula, we have

$$\begin{aligned} \langle A_{\xi}X, Y \rangle_e &= \frac{1}{2} (\langle [X, Y], \xi \rangle_e - \langle [Y, \xi], X \rangle_e + \langle [\xi, X], Y \rangle_e) \\ &= \frac{1}{2} \langle (\mathrm{ad}(\xi) + \mathrm{ad}(\xi)^*)X, Y \rangle_e \end{aligned}$$

for any $X, Y \in T_e(S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e) \mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ and any $\xi \in T_e^{\perp}(S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e) = \mathfrak{b} + \sum_{i=1}^k l_i$. That is, we have

$$(3.4) \quad A_{\xi} = \frac{1}{2} \mathrm{pr}_T \circ (\mathrm{ad}(\xi) + \mathrm{ad}(\xi)^*),$$

where pr_T is the orthogonal projection of $\mathfrak{a} + \mathfrak{n}$ onto $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$. From (3.2) and (3.4), we have

$$A_{\xi_0}X = \begin{cases} 0 & (X \in \mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k} \ominus \sum_{\mu \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}} \mathfrak{g}_{\lambda}) \\ \mu(\xi_0)X & (X \in \mathfrak{g}_{\mu}), \end{cases}$$

where $\mu \in \Delta_+ \setminus \{\lambda_1, \dots, \lambda_k\}$. From (3.3) and (3.4), we have

$$A_{\xi_{t_i}^i}X = 0 \quad (X \in \mathfrak{a} \ominus (\mathfrak{b} + \sum_{i=1}^k \mathbf{R}H_{\lambda_i})).$$

Set $\mathfrak{g}_{\lambda_j}^K := \mathrm{Ker}(\mathrm{ad}(\xi^j)|_{\mathfrak{g}_{\lambda_j}})$ and $\mathfrak{g}_{\lambda_j}^I := \mathrm{Im}(\mathrm{ad}(\theta\xi^j)|_{\mathfrak{g}_{2\lambda_j}})$ ($j = 1, \dots, k$). Then we have $\mathfrak{g}_{\lambda_j} = \mathfrak{g}_{\lambda_j}^K \oplus \mathfrak{g}_{\lambda_j}^I$. By simple calculations, it is shown that this decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle_e$. If $X \in \mathfrak{g}_{\lambda_j}^K \ominus \mathbf{R}\xi^j$, then it follows from (3.2), (3.3), (3.4), $\lambda_i, \lambda_j \in \Pi$ and $\langle H_{\lambda_i}, H_{\lambda_j} \rangle = 0$ (when $i \neq j$) that

$$A_{\xi_{t_i}^i}X = \begin{cases} -|\lambda_i| \tanh(|\lambda_i|t_i)X & (i = j) \\ 0 & (i \neq j). \end{cases}$$

If $X \in \mathfrak{g}_{2\lambda_j}$, then it follows from (3.2), (3.3), (3.4), $\lambda_i, \lambda_j \in \Pi$ and $\langle H_{\lambda_i}, H_{\lambda_j} \rangle = 0$ (when $i \neq j$) that

$$A_{\xi_{t_i}^i}X = \begin{cases} -2|\lambda_i| \tanh(|\lambda_i|t_i)X - \frac{1}{2 \cosh(|\lambda_i|t_i)} [\theta\xi^i, X] & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Also, we have

$$A_{\xi_{t_i}^i}([\theta\xi^i, X]) = -\frac{|\lambda_i|^2}{\cosh(|\lambda_i|t_i)}X - |\lambda_i| \tanh(|\lambda_i|t_i)[\theta\xi^i, X].$$

Let $X := \tanh(|\lambda_j|t_j)\xi^j + \frac{1}{|\lambda_j|\cosh(|\lambda_j|t_j)}H_{\lambda_j}$, which is a unit vector of $(\mathbf{R}\xi^j + \mathbf{R}H_{\lambda_j}) \ominus l_j$. From (3.2), (3.3), (3.4), $\lambda_i, \lambda_j \in \Pi$ and $\langle H_{\lambda_i}, H_{\lambda_j} \rangle = 0$ (when $i \neq j$), we have

$$\begin{aligned} A_{\xi_{t_i}^i} X &= -\frac{1}{2}|\lambda_i| \tanh(|\lambda_i|t_i)X + \frac{1}{2\cosh(|\lambda_i|t_i)}\text{pr}_T(\text{ad}(\xi^i)^* X) \\ &\quad - \frac{1}{2|\lambda_i|} \tanh(|\lambda_i|t_i)\text{pr}_T(\text{ad}(H_{\lambda_i})^* X) \\ &= \begin{cases} -|\lambda_i| \tanh(|\lambda_i|t_i)X & (i = j) \\ 0 & (i \neq j). \end{cases} \end{aligned}$$

This completes the proof of (i) \sim (vi). Finally we shall show the statement (vii). Let $X \in \mathfrak{n} \ominus \sum_{i=1}^k (\mathfrak{g}_{\lambda_i} + \mathfrak{g}_{2\lambda_i})$ and k_i be as in the statement (vii). From (3.2), (3.3), (3.4), $\lambda_j \in \Pi$ ($j = 1, \dots, k$) and $\langle H_{\lambda_i}, H_{\lambda_j} \rangle = 0$ (when $i \neq j$), we have

$$A_{\xi_{t_i}^i} X = \frac{1}{\cosh(|\lambda_i|t_i)}[\xi_{\mathfrak{p}}^i, X] - \frac{1}{|\lambda_i|} \tanh(|\lambda_i|t_i)[H_{\lambda_i}, X].$$

By operating $\text{Ad}(k_i)$ to both sides of this relation, we have

$$\text{Ad}(k_i)(A_{\xi_{t_i}^i} X) = -A_{\xi_{t_i}^i}(\text{Ad}(k_i)X),$$

where we use $\text{Ad}(k_i)(\xi_{\mathfrak{p}}^i) = -\xi_{\mathfrak{p}}^i$ and $\text{Ad}(k_i)(H_{\lambda_i}) = -H_{\lambda_i}$. Thus the statement (vii) is shown.

q.e.d.

Also, we have the following fact.

Lemma 3.3. *Let $\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k}$ be as in Example 2 and \bar{l}_i be the orthogonal projection of l_i onto \mathfrak{g}_{λ_i} . Set $\mathfrak{s}_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} := (\mathfrak{a} + \mathfrak{n}) \ominus (\mathfrak{b} + \sum_{i=1}^k \bar{l}_i)$ and $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} := \exp_G(\mathfrak{s}_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k})$. Then the $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k}$ -action is conjugate to the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action.*

Proof. Denote by ∇ the Levi-Civita connection of the left-invariant metric of AN . Let H be a vector of \mathfrak{b} , ξ^i be a unit vector of \bar{l}_i ($i = 1, \dots, k$) and γ_{ξ^i} be the geodesic in AN with $\dot{\gamma}_{\xi^i}(0) = \xi^i$. Let t_i be a real number with $\frac{1}{\cosh(|\lambda_i|t_i)}\xi^i - \tanh(|\lambda_i|t_i)H_{\lambda_i} \in l_i$ ($i = 1, \dots, k$). Denote by the same symbols H , ξ^i and H_{λ_i} the left-invariant vector fields arising from H , ξ^i and H_{λ_i} , respectively. By using the relation (5.4) of Section 5 of [20] (arising the Koszul formula for the left-invariant vector fields), we can show

$$\begin{aligned} \nabla_{\xi^1}\xi^1 &= |\lambda_1|H_{\lambda_1}, \quad \nabla_{\xi^1}H_{\lambda_1} = -|\lambda_1|\xi^1 \\ \nabla_{\xi^1}\xi^i &= \nabla_{\xi^1}H = \nabla_{H_{\lambda_1}}\xi^1 = \nabla_{H_{\lambda_1}}\xi^i = \nabla_{H_{\lambda_1}}H_{\lambda_1} = \nabla_{H_{\lambda_1}}H = 0, \end{aligned}$$

where $i = 2, \dots, k$. From $\nabla_{\xi^1}\xi^1 = |\lambda_1|H_{\lambda_1}$, $\nabla_{\xi^1}H_{\lambda_1} = -|\lambda_1|\xi^1$, $\nabla_{H_{\lambda_1}}\xi^1 = \nabla_{H_{\lambda_1}}H_{\lambda_1} = 0$, it follows that $\exp \mathbf{R}\{\xi^1, H_{\lambda_1}\}$ is a totally geodesic subgroup of AN . Hence $\dot{\gamma}_{\xi^1}(t)$ is expressed as $\dot{\gamma}_{\xi^1}(t) = a(t)(H_{\lambda_1})_{\gamma_{\xi^1}(t)} + b(t)(\xi^1)_{\gamma_{\xi^1}(t)}$. Furthermore, we have $\nabla_{\dot{\gamma}_{\xi^1}}\dot{\gamma}_{\xi^1} = (a' + |\lambda_1|b^2)H_{\lambda_1} + (b' - |\lambda_1|ab)\xi^1 = 0$, that is, $a' = -|\lambda_1|b^2$ and $b' = |\lambda_1|ab$. By solving this differential equation under the initial conditions $a(0) = 0$ and $b(0) = 1$, we have $a(t) = -\tanh(|\lambda_1|t)$ and $b(t) = \frac{1}{\cosh(|\lambda_1|t)}$. Hence we obtain

$\hat{\gamma}_{\xi^1}(t) = \frac{1}{\cosh(|\lambda_1|t)}(\xi^1)_{\gamma_{\xi^1}(t)} - \tanh(|\lambda_1|t)(H_{\lambda_1})_{\gamma_{\xi^1}(t)}$. From $\nabla_{\xi^1}\xi^i = \nabla_{\xi^1}H = \nabla_{H_{\lambda_1}}\xi^i = \nabla_{H_{\lambda_1}}H = 0$ ($i = 2, \dots, k$), it follows that ξ^i ($i = 2, \dots, k$) and H are parallel along γ_{ξ^1} (with respect to ∇). Denote by $P_{\gamma_{\xi^1}|_{[0,t]}}$ the parallel translation along $\gamma_{\xi^1}|_{[0,t]}$ (with respect to ∇) and $L_{\gamma_{\xi^1}(t)}$ the left translation by $\gamma_{\xi^1}(t)$. From the above facts, we have

$$\begin{aligned} T_{\gamma_{\xi^1}(t_1)}^\perp(S_{\mathbf{b}, \bar{l}_1, \dots, \bar{l}_k}) &= P_{\gamma_{\xi^1}|_{[0,t_1]}}(\mathbf{b} + \sum_{i=1}^k \bar{l}_i) = (L_{\gamma_{\xi^1}(t_1)})_*(\mathbf{b} + \sum_{i=2}^k \bar{l}_i + l_1) \\ &= (L_{\gamma_{\xi^1}(t_1)})_*(T_e^\perp S_{\mathbf{b}, l_1, \bar{l}_2, \dots, \bar{l}_k}), \end{aligned}$$

which implies $\gamma_{\xi^1}(t_1)^{-1}S_{\mathbf{b}, \bar{l}_1, \dots, \bar{l}_k}\gamma_{\xi^1}(t_1) = S_{\mathbf{b}, l_1, \bar{l}_2, \dots, \bar{l}_k}$. By repeating the same discussion, we obtain

$$(\gamma_{\xi^1}(t_1) \cdots \gamma_{\xi^k}(t_k))^{-1}S_{\mathbf{b}, \bar{l}_1, \dots, \bar{l}_k}(\gamma_{\xi^1}(t_1) \cdots \gamma_{\xi^k}(t_k)) = S_{\mathbf{b}, l_1, \dots, l_k}.$$

Thus the $S_{\mathbf{b}, \bar{l}_1, \dots, \bar{l}_k}$ -action is conjugate to the $S_{\mathbf{b}, l_1, \dots, l_k}$ -action. q.e.d.

For parallel submanifolds of a proper complex equifocal submanifold and a curvature-adapted complex equifocal submanifold, we have the following facts.

Lemma 3.4. (i) *All parallel submanifolds of a proper complex equifocal submanifold are proper complex equifocal.*

(ii) *All parallel submanifolds of a curvature-adapted complex equifocal submanifold are curvature-adapted and complex equifocal.*

Proof. First we shall show the statement (i). Let M be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type and \tilde{v} be the parallel normal vector field of M which is not a focal normal vector field. Denote by $\eta_{\tilde{v}}$ the end-point map for \tilde{v} and $M_{\tilde{v}} := \eta_{\tilde{v}}(M)$, which is a parallel submanifold of M . The vector field \tilde{v} is regarded as a parallel normal vector field of the complexification M^c along M . Let \tilde{v}^L be the horizontal lift of \tilde{v} to $H^0([0, 1], \mathfrak{g}^c)$ by the anti-Kaehlerian submersion $\pi^c \circ \phi^c : H^0([0, 1], \mathfrak{g}^c) \rightarrow G^c/K^c$, which is a parallel normal vector field of $\widetilde{M}^c (:= (\pi^c \circ \phi^c)^{-1}(M^c))$. Set $\widetilde{M}^c_{\tilde{v}^L} := \eta_{\tilde{v}^L}(\widetilde{M}^c)$, where $\eta_{\tilde{v}^L}$ is the end-point map for \tilde{v}^L . Note that $\widetilde{M}^c_{\tilde{v}^L} = (\pi^c \circ \phi^c)^{-1}((M_{\tilde{v}})^c)$. Denote by \widetilde{A} and $\widetilde{A}^{\tilde{v}^L}$ the shape tensors of \widetilde{M}^c and $\widetilde{M}^c_{\tilde{v}^L}$, respectively. Let $\{\lambda_i \mid i \in I\}$ be the set of all complex principal curvatures of \widetilde{M}^c and E_i be the complex curvature distribution for λ_i . Then, according to Lemma 3.2 of [16], we have

$$(3.5) \quad \widetilde{A}_w^{\tilde{v}^L} |_{(E_i)_u} = \frac{(\lambda_i)_u(w)}{1 - (\lambda_i)_u(\tilde{v}_u^L)} \text{id} \quad (i \in I, u \in \widetilde{M}^c_{\tilde{v}^L}),$$

where we note that $T_{\eta_{\tilde{v}^L}(u)}\widetilde{M}^c_{\tilde{v}^L} = T_u\widetilde{M}^c (= \overline{\bigoplus_{i \in I} (E_i)_u})$. This implies that $\widetilde{M}^c_{\tilde{v}^L}$ is proper anti-Kaehlerian isoparametric, that is, $M_{\tilde{v}}$ is proper complex equifocal. Thus the statement (i) is shown. Next we shall show the statement (ii). Let M be a curvature-adapted complex equifocal submanifold in G/K and \tilde{v} be the parallel normal vector field of M . Set $M_{\tilde{v}} := \eta_{\tilde{v}}(M)$. Denote by A and $A^{\tilde{v}}$ the shape tensors of M and $M_{\tilde{v}}$, respectively. Let $w \in T_x^\perp M$. Without loss of generality, we may assume that $x = eK$. Let \mathfrak{a} be a maximal abelian subspace of $\mathfrak{p} := T_{eK}(G/K)$ containing $T_{eK}^\perp M$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to \mathfrak{a} . Let $X \in \text{Ker}(A_w - \lambda \text{id}) \cap \text{Ker}(A_w - \mu \text{id}) \cap \mathfrak{p}_\alpha$

($\lambda \in \text{Spec } A_v$, $\mu \in \text{Spec } A_w$, $\alpha \in \Delta_+$). Let \tilde{w} be the parallel tangent vector field on the (flat) section Σ of M through eK with $\tilde{w}_{eK} = w$. Since $M_{\tilde{v}}$ is regarded as a partial tube over M , it follows from (ii) of Corollary 3.2 in [15] that

$$(3.6) \quad (A^{\tilde{v}})_{\tilde{w}_{\eta_{\tilde{v}}}(eK)}((\eta_{\tilde{v}})_*X) = \frac{1}{\alpha(v) - \lambda \tanh \alpha(v)} \{-\alpha(v)\alpha(w) \tanh \alpha(v) + \lambda \left(1 - \frac{\tanh \alpha(v)}{\alpha(v)}\right) \alpha(w) + \mu \tanh \alpha(v)\}(\eta_{\tilde{v}})_*X.$$

Let Z be the element of \mathfrak{p} with $\exp_G(Z)K = \eta_{\tilde{v}}(eK)$. For simplicity, set $g := \exp_G(Z)$. Since $g_* : \mathfrak{p} \rightarrow T_{\eta_{\tilde{v}}(eK)}(G/K)$ is the parallel translation along the normal geodesic $\gamma_Z \stackrel{\text{def}}{=} \gamma_Z(t) := \exp_G(tZ)K$, it follows from (3.1) of [15] that

$$\begin{aligned} (\eta_{\tilde{v}})_*X &= g_*(D_v^{co}(X) - D_v^{si}(A_v X)) \\ &= \left(\cosh \alpha(v) - \lambda \frac{\sinh \alpha(v)}{\alpha(v)} \right) g_*X \in g_*\mathfrak{p}_\alpha. \end{aligned}$$

Also, we have $g_*^{-1}(T_{\eta_{\tilde{v}}(eK)}^\perp M_{\tilde{v}}) = T_{eK}^\perp M \subset \mathfrak{a}$. Hence we have $R((\eta_{\tilde{v}})_*X, \tilde{w}_{\eta_{\tilde{v}}}(eK))\tilde{w}_{\eta_{\tilde{v}}}(eK) = -\alpha(w)^2(\eta_{\tilde{v}})_*X$, which together with (3.6) implies

$$[(A^{\tilde{v}})_{\tilde{w}_{\eta_{\tilde{v}}}(eK)}, R(\cdot, \tilde{w}_{\eta_{\tilde{v}}}(eK))\tilde{w}_{\eta_{\tilde{v}}}(eK)]((\eta_{\tilde{v}})_*X) = 0.$$

Therefore, it follows from the arbitrariness of X that $[(A^{\tilde{v}})_{\tilde{w}_{\eta_{\tilde{v}}}(eK)}, R(\cdot, \tilde{w}_{\eta_{\tilde{v}}}(eK))\tilde{w}_{\eta_{\tilde{v}}}(eK)]$ vanishes over $(\eta_{\tilde{v}})_*(\text{Ker}(A_v - \lambda \text{id}) \cap \text{Ker}(A_w - \mu \text{id}) \cap \mathfrak{p}_\alpha)$. Since M is curvature-adapted, we have

$$\bigoplus_{\lambda \in \text{Spec } A_v} \bigoplus_{\mu \in \text{Spec } A_w} \bigoplus_{\alpha \in \Delta_+} (\eta_{\tilde{v}})_*(\text{Ker}(A_v - \lambda \text{id}) \cap \text{Ker}(A_w - \mu \text{id}) \cap \mathfrak{p}_\alpha) = T_{\eta_{\tilde{v}}(eK)} M_{\tilde{v}}.$$

Hence we have $[(A^{\tilde{v}})_{\tilde{w}_{\eta_{\tilde{v}}}(eK)}, R(\cdot, \tilde{w}_{\eta_{\tilde{v}}}(eK))\tilde{w}_{\eta_{\tilde{v}}}(eK)] = 0$. Therefore, it follows from the arbitrariness of w that $M_{\tilde{v}}$ is curvature-adapted. It is clear that $M_{\tilde{v}}$ is complex equifocal. Thus the statement (ii) is shown. q.e.d.

For the $S_{\mathfrak{b}}$ -action and the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action, we have the following facts.

Proposition 3.5. (i) *All orbits of the $S_{\mathfrak{b}}$ -action are curvature-adapted but they are not proper complex equifocal.*

(ii) *Let $\lambda_1, \dots, \lambda_k (\in \Delta_+)$ be as in Example 2. If the root system Δ of G/K is non-reduced and $2\lambda_{i_0} \in \Delta_+$ for some $i_0 \in \{1, \dots, k\}$, then all orbits of the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action are not curvature-adapted. Also, if $\mathfrak{b} \neq \{0\}$, then they are not proper complex equifocal.*

Proof. First we shall show the statement (i). The group $S_{\mathfrak{b}}$ acts isometrically on $(AN, \langle \cdot, \cdot \rangle)$. Denote by A the shape tensor of the orbit $S_{\mathfrak{b}} \cdot e$ in AN . Since $\langle \cdot, \cdot \rangle$ is left-invariant, it follows from the Koszul formula that $\langle A_v X, Y \rangle = \langle \text{ad}(v)X, Y \rangle$ for any $v \in \mathfrak{l} = T_e^\perp(S_{\mathfrak{b}} \cdot e)$ and $X, Y \in \mathfrak{s} = T_e(S_{\mathfrak{b}} \cdot e)$. Hence we have $A_v|_{\mathfrak{a} \cap \mathfrak{l}} = 0$ and $A_v|_{\mathfrak{g}_\lambda} = \lambda(v)\text{id}$ ($\lambda \in \Delta_+$), where $v \in T_e^\perp(S_{\mathfrak{b}} \cdot e) = \mathfrak{l}(\subset \mathfrak{p})$. Therefore, the orbit $S_{\mathfrak{b}} \cdot e$ is curvature-adapted but it is not proper complex equifocal by (ii) of Theorem 1 of [14]. Hence so are all orbits of the $S_{\mathfrak{b}}$ -action by Lemma 3.3.

Next we shall show the statement (ii). Assume that the root system Δ of G/K is non-reduced. Denote by A the shape tensor of the orbit $S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e \subset AN$. Also, let $\xi_0 \in \mathfrak{b}$ and $\xi_{t_i}^i : \frac{1}{\cosh(|\lambda_i|t_i)}\xi^i - \frac{1}{|\lambda_i|} \tanh(|\lambda_i|t_i)H_{\lambda_i}$ ($\xi^i \in \mathfrak{g}_{\lambda_i}$) be a unit (tangent) vector of l_i . Then, according to Lemma 3.2, we see that

$$(3.7) \quad \begin{aligned} A_{\xi_0}|_{\mathfrak{s}_{\mathfrak{b}, l_1, \dots, l_k} \cap (\mathfrak{a} + \sum_{i=1}^k \mathfrak{g}_{\lambda_i})} &= 0, \\ A_{\xi_0}|_{\mathfrak{g}_{\mu}} &= \mu(\xi_0)\text{id} \quad (\mu \in \Delta_+ \setminus \bigcup_{i=1}^k \{\lambda_i\}), \\ A_{\xi_{t_i}^i}|_{\mathfrak{a} \ominus (\mathfrak{b} + \sum_{j=1}^k \mathbf{R}H_{\lambda_j})} &= 0, \\ A_{\xi_{t_i}^i}|_{\text{Ker}(\text{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \ominus \mathbf{R}\xi^i} &= -|\lambda_i| \tanh(|\lambda_i|t_i)\text{id} \\ A_{\xi_{t_i}^i}|_{(\mathbf{R}\xi^i + \mathbf{R}H_{\lambda_i}) \ominus l_i} &= -|\lambda_i| \tanh(|\lambda_i|t_i)\text{id} \end{aligned}$$

and that, in case of $2\lambda_i \in \Delta_+$, $A_{\xi_{t_i}^i}|_{\text{Im}(\text{ad}(\theta\xi^i)|_{\mathfrak{g}_{2\lambda_i}}) + \mathfrak{g}_{2\lambda_i}}$ has two eigenvalues

$$\mu_i^+ := -\frac{3}{2}|\lambda_i| \tanh(|\lambda_i|t_i) + \frac{1}{2}|\lambda_i| \sqrt{2 - \tanh^2(|\lambda_i|t_i)}$$

and

$$\mu_i^- := -\frac{3}{2}|\lambda_i| \tanh(|\lambda_i|t_i) - \frac{1}{2}|\lambda_i| \sqrt{2 - \tanh^2(|\lambda_i|t_i)}$$

with the same multiplicity. Note that $\mathfrak{g}_{\lambda_i} = \text{Ker}(\text{ad}(\xi^i)|_{\mathfrak{g}_{\lambda_i}}) \oplus \text{Im}(\text{ad}(\theta\xi^i)|_{\mathfrak{g}_{2\lambda_i}})$. The eigenspace for μ_i^+ (resp. μ_i^-) is spanned by

$$\begin{aligned} Z_{\xi^i, Y}^+ &:= [\theta\xi^i, Y] + |\lambda_i| \left(\sinh(|\lambda_i|t_i) - \sqrt{\sinh^2(|\lambda_i|t_i) + 2} \right) Y' \text{s} \quad (Y \in \mathfrak{g}_{2\lambda_i}) \\ (\text{resp. } Z_{\xi^i, Y}^- &:= [\theta\xi^i, Y] + |\lambda_i| \left(\sinh(|\lambda_i|t_i) + \sqrt{\sinh^2(|\lambda_i|t_i) + 2} \right) Y' \text{s} \quad (Y \in \mathfrak{g}_{2\lambda_i})). \end{aligned}$$

Denote by R the curvature tensor of $\langle \cdot, \cdot \rangle$. Also, denote by $X_{\mathfrak{f}}$ (resp. $X_{\mathfrak{p}}$) the \mathfrak{f} -component (resp. the \mathfrak{p} -component) of $X \in \mathfrak{g}$. Then we have

$$(3.8) \quad \begin{aligned} \left(R(Z_{\xi^i, Y}^{\pm}, \xi_{t_i}^i)\xi_{t_i}^i \right)_{\mathfrak{p}} &= -a[[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{p}}, (\xi_{t_i}^i)_{\mathfrak{p}}], (\xi_{t_i}^i)_{\mathfrak{p}}] \\ &= a(-[[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{p}}, \xi_{t_i}^i], \xi_{t_i}^i]_{\mathfrak{p}} + [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{f}}, (\xi_{t_i}^i)_{\mathfrak{f}}], (\xi_{t_i}^i)_{\mathfrak{p}}] \\ &\quad + [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{f}}, (\xi_{t_i}^i)_{\mathfrak{p}}], (\xi_{t_i}^i)_{\mathfrak{f}}] + [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{p}}, (\xi_{t_i}^i)_{\mathfrak{f}}], (\xi_{t_i}^i)_{\mathfrak{f}}]) \end{aligned}$$

for some non-zero constant a , where we note that $a = 1$ if the metric of G/K is induced from the restriction of the Killing form of \mathfrak{g} to \mathfrak{p} . Also we have

$$(3.9) \quad [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{p}}, (\xi_{t_i}^i)_{\mathfrak{f}}], (\xi_{t_i}^i)_{\mathfrak{f}}] = 0,$$

$$(3.10) \quad [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{f}}, (\xi_{t_i}^i)_{\mathfrak{f}}], (\xi_{t_i}^i)_{\mathfrak{p}}] = -\frac{\tanh(|\lambda_i|t_i)}{|\lambda_i| \cosh(|\lambda_i|t_i)} [[[\theta\xi^i, Y]_{\mathfrak{f}}, \xi_{t_i}^i], H_{\lambda_i}]$$

and

$$(3.11) \quad [[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{f}}, (\xi_{t_i}^i)_{\mathfrak{p}}], (\xi_{t_i}^i)_{\mathfrak{f}}] = \frac{|\lambda_i| \tanh(|\lambda_i|t_i)}{\cosh(|\lambda_i|t_i)} [[[\theta\xi^i, Y]_{\mathfrak{p}}, \xi_{t_i}^i].$$

Let η (resp. $\bar{\eta}$) be the element of $\mathfrak{a} + \mathfrak{n}$ with $\eta_{\mathfrak{f}} = [[[\theta\xi^i, Y]_{\mathfrak{f}}, \xi_{t_i}^i]$ (resp. $\bar{\eta}_{\mathfrak{p}} = [[[\theta\xi^i, Y]_{\mathfrak{p}}, \xi_{t_i}^i]$). Then it follows from (3.8) \sim (3.11) that

$$(R(Z_{\xi^i, Y}^{\pm}, \xi_{t_i}^i)\xi_{t_i}^i)_{\mathfrak{p}} = -a[[(Z_{\xi^i, Y}^{\pm})_{\mathfrak{p}}, \xi_{t_i}^i], \xi_{t_i}^i]_{\mathfrak{p}} + \frac{a|\lambda_i| \tanh(|\lambda_i|t_i)}{\cosh(|\lambda_i|t_i)} (2\eta_{\mathfrak{p}} + \bar{\eta}_{\mathfrak{p}}),$$

that is,

$$(3.12) \quad R(Z_{\xi^i, Y}^\pm, \xi_{t_i}^i) \xi_{t_i}^i = -a[[Z_{\xi^i, Y}^\pm, \xi_{t_i}^i], \xi_{t_i}^i] + \frac{a|\lambda_i| \tanh(|\lambda_i|t_i)}{\cosh(|\lambda_i|t_i)}(2\eta + \bar{\eta}).$$

We have $[\xi^i, \theta\xi^i] = bH_{\lambda_i}$ for some non-zero constant b . By simple calculation, we have

$$(3.13) \quad \begin{aligned} & [[Z_{\xi^i, Y}^\pm, \xi_{t_i}^i], \xi_{t_i}^i] \\ &= 2|\lambda_i| \tanh^2(|\lambda_i|t_i) \left(-\frac{3b|\lambda_i|^2}{\sinh(|\lambda_i|t_i)} + \sinh(|\lambda_i|t_i) \mp \sqrt{\sinh^2(|\lambda_i|t_i) + 2} \right) Y \\ & \quad + \tanh^2(|\lambda_i|t_i)[\theta\xi^i, Y]. \end{aligned}$$

From (3.12) and (3.13), it follows that $R(Z_{\xi^i, Y}^\pm, \xi_{t_i}^i) \xi_{t_i}^i$ belongs to $\text{Im ad}(\theta\xi^i) \oplus \mathfrak{g}_{2\lambda_i}$. Hence $R(\cdot, \xi_{t_i}^i) \xi_{t_i}^i$ preserves $\text{Im ad}(\theta\xi^i) \oplus \mathfrak{g}_{2\lambda_i}$ invariantly. It is clear that so is also $A_{\xi_{t_i}^i}$. From (3.12) and (3.13), we have $[R(\cdot, \xi_{t_i}^i) \xi_{t_i}^i, A_{\xi_{t_i}^i}]|_{\text{Im ad}(\theta\xi^i) \oplus \mathfrak{g}_{2\lambda_i}} \neq 0$, under a suitable choice of t_i . Therefore, $S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e$ is not curvature-adapted under suitable choices of l_1, \dots, l_k . Then, so are all orbits of the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action by Lemma 3.4. Furthermore, it follows from Lemma 3.3 that all orbits of the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action are not curvature-adapted under arbitrary choices of l_1, \dots, l_k . Also, it follows from the second relation of (3.7) that $S_{\mathfrak{b}, l_1, \dots, l_k} \cdot e$ (hence all orbits of the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action) is not proper complex equifocal in case of $\mathfrak{b} \neq \{0\}$. q.e.d.

From this proposition, we obtain the statements of Theorem B. Also, we have the following fact.

Proposition 3.6. *If $\mathfrak{b} = \{0\}$, then the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action possesses the only minimal orbit.*

Proof. According to Lemma 3.3, the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action is conjugate to $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k}$ -action, where \bar{l}_i is the orthogonal projection of l_i onto \mathfrak{g}_{λ_i} . Hence they are orbit equivalent to each other. Hence we suffice to show that the statement of this proposition holds for the $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k}$ -action. Let ξ^i be a unit vector of \bar{l}_i . Take $p \in AN$. We can express as $p = \gamma_{\xi^1}(t_1) \cdots \gamma_{\xi^k}(t_k)$ for some $t_1, \dots, t_k \in \mathbf{R}$, where γ_{ξ^i} is the geodesic with $\dot{\gamma}_{\xi^i}(0) = \xi^i$. Set $\hat{l}_i := \mathbf{R}\{\frac{1}{\cosh(|\lambda_i|t_i)}\xi^i - \frac{1}{|\lambda_i|} \tanh(|\lambda_i|t_i)H_{\lambda_i}\}$ ($i = 1, \dots, k$). For simplicity, set $\hat{\xi}_{t_i}^i := \frac{1}{\cosh(|\lambda_i|t_i)}\xi^i - \frac{1}{|\lambda_i|} \tanh(|\lambda_i|t_i)H_{\lambda_i}$. According to the proof of Lemma 3.3, we have

$$(\gamma_{\xi^1}(t_1) \cdots \gamma_{\xi^k}(t_k))^{-1} S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} (\gamma_{\xi^1}(t_1) \cdots \gamma_{\xi^k}(t_k)) = S_{\mathfrak{b}, \hat{l}_1, \dots, \hat{l}_k}.$$

Hence the orbit $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} \cdot p$ is congruent to the orbit $S_{\mathfrak{b}, \hat{l}_1, \dots, \hat{l}_k} \cdot e$. Denote by A the shape tensor of $S_{\mathfrak{b}, \hat{l}_1, \dots, \hat{l}_k} \cdot e$. According to Lemma 3.2, we have

$$\text{Tr } A_{\hat{\xi}_{t_i}^i} = -|\lambda_i| \tanh(|\lambda_i|t_i) \times (\dim \mathfrak{g}_{\lambda_i} + 2\dim \mathfrak{g}_{2\lambda_i}) \quad (i = 1, \dots, k).$$

Hence the orbit $S_{\mathfrak{b}, \hat{l}_1, \dots, \hat{l}_k} \cdot e$ is minimal if and only if $t_1 = \dots = t_k = 0$, where we note that $T_e^\perp(S_{\mathfrak{b}, \hat{l}_1, \dots, \hat{l}_k} \cdot e) = \mathbf{R}\{\xi_{t_1}^1, \dots, \xi_{t_k}^k\}$ because of $\mathfrak{b} = \{0\}$. That is, the orbit $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} \cdot p$ is minimal if and only if $p = e$. Thus the orbit $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k}$ -action possesses the only minimal orbit $S_{\mathfrak{b}, \bar{l}_1, \dots, \bar{l}_k} \cdot e$. This completes the proof.

q.e.d.

From this proposition, we obtain the statement of Theorem C. At the end of this paper, we propose the following question.

Question. *Is any complex hyperpolar action without singular orbit on a symmetric space of non-compact type orbit equivalent to either the $S_{\mathfrak{b}}$ -action ($\mathfrak{b} \subset \mathfrak{a}$) as in Example 1 or the $S_{\mathfrak{b}, l_1, \dots, l_k}$ -action ($l_i : a$ one dimensional subspace of \mathfrak{g}_{λ_i} ($i = 1, \dots, k$), $\mathfrak{b} \subset \mathfrak{a} \ominus \text{Span}\{H_{\lambda_i} \mid i = 1, \dots, k\}$) as in Example 2?*

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