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Fischer decomposition by inframonogenic functions

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ABSTRACT

Let $\partial_{\underline{x}}$ denote the Dirac operator in $\mathbb{R}^m$. In this paper, we present a refinement of the biharmonic functions and at the same time an extension of the monogenic functions by considering the equation $\partial_{\underline{x}} f \partial_{\underline{x}} = 0$. The solutions of this "sandwich" equation, which we call inframonogenic functions, are used to obtain a new Fischer decomposition for homogeneous polynomials in $\mathbb{R}^m$.

RESUMEN

Denotemos por $\partial_{\underline{x}}$ el operador de Dirac en $\mathbb{R}^m$. En este artículo, nosotros presentamos un refinamiento de las funciones biarmónicas y al mismo tiempo una extensión de las funciones monogénicas considerando la ecuación $\partial_{\underline{x}} f \partial_{\underline{x}} = 0$. Las soluciones de esta ecuación tipo "sándwich", las cuales llamaremos inframonogénicas, son utilizadas para obtener una nueva descomposición de Fischer para polinomios homogéneos en $\mathbb{R}^m$.

Key words and phrases: *Inframonogenic functions; Fischer decomposition.*

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1 Introduction

Let $\mathbb{R}_{0,m}$ be the 2^m-dimensional real Clifford algebra constructed over the orthonormal basis $(e_1, \ldots, e_m)$ of the Euclidean space $\mathbb{R}^m$ (see [6]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations $e_j e_k + e_k e_j = -2\delta_{jk}$ and a general element of $\mathbb{R}_{0,m}$ is of the form $a = \sum_A a_A e_A$, $a_A \in \mathbb{R}$, where for $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$, $j_1 < \cdots < j_k$, $e_A = e_{j_1} \ldots e_{j_k}$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Notice that any $a \in \mathbb{R}_{0,m}$ may also be written as $a = \sum_{k=0}^m [a]_k$ where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m}^{(k)}$. Here $\mathbb{R}_{0,m}^{(k)}$ denotes the subspace of k-vectors defined by

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : \ a = \sum_{|A|=k} a_A e_A, \quad a_A \in \mathbb{R} \right\}.$$

In particular, $\mathbb{R}_{0,m}^{(1)}$ and $\mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}$ are called, respectively, the space of vectors and paravectors in $\mathbb{R}_{0,m}$. Observe that $\mathbb{R}^{m+1}$ may be naturally identified with $\mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}$ by associating to any element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ the paravector $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j$.

Conjugation in $\mathbb{R}_{0,m}$ is given by

$$\overline{a} = \sum_A a_A \overline{e}_A, \quad \overline{e}_A = (-1)^{\frac{|A|(|A|+1)}{2}} e_A.$$

One easily checks that $\overline{ab} = \overline{b}\,\overline{a}$ for any $a, b \in \mathbb{R}_{0,m}$. Moreover, by means of the conjugation a norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,m}$ by putting

$$|a|^2 = [a\overline{a}]_0 = \sum_A a_A^2.$$

The $\mathbb{R}_{0,m}$-valued solutions $f(\underline{x})$ of $\partial_{\underline{x}} f(\underline{x}) = 0$, with $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ being the Dirac operator, are called left monogenic functions (see [4, 8]). The same name is used for null-solutions of the operator $\partial_x = \partial_{x_0} + \partial_{\underline{x}}$ which is also called generalized Cauchy-Riemann operator.

In view of the non-commutativity of $\mathbb{R}_{0,m}$ a notion of right monogenicity may be defined in a similar way by letting act the Dirac operator or the generalized Cauchy-Riemann operator from the right. Functions that are both left and right monogenic are called two-sided monogenic.

One can also consider the null-solutions of $\partial_{\underline{x}}^k$ and ∂_x^k ($k \in \mathbb{N}$) which gives rise to the so-called k-monogenic functions (see e.g. [2, 3, 15]).

It is worth pointing out that $\partial_{\underline{x}}$ and ∂_x factorize the Laplace operator in the sense that

$$\Delta_{\underline{x}} = \sum_{j=1}^m \partial_{x_j}^2 = -\partial_{\underline{x}}^2, \quad \Delta_x = \partial_{x_0}^2 + \Delta_{\underline{x}} = \partial_x \overline{\partial}_x = \overline{\partial}_x \partial_x.$$

Let us now introduce the main object of this paper.

Definition 1.1. *Let Ω be an open set of $\mathbb{R}^m$ (resp. $\mathbb{R}^{m+1}$). An $\mathbb{R}_{0,m}$-valued function $f \in \mathcal{C}^2(\Omega)$ will be called an inframonogenic function in Ω if and only if it fulfills in Ω the "sandwich" equation*

$$\partial_{\underline{x}} f \partial_{\underline{x}} = 0 \quad (\textit{resp.}\ \partial_x f \partial_x = 0).$$

Here we list some motivations for studying these functions.

1. If a function f is inframonogenic in $\Omega \subset \mathbb{R}^m$ and takes values in $\mathbb{R}$, then f is harmonic in Ω.
2. The left and right monogenic functions are also inframonogenic.
3. If a function f is inframonogenic in $\Omega \subset \mathbb{R}^m$, then it satisfies in Ω the overdetermined system $\partial^3_{\underline{x}} f = 0 = f \partial^3_{\underline{x}}$. In other words, f is a two-sided 3-monogenic function.
4. Every inframonogenic function $f \in \mathcal{C}^4(\Omega)$ is biharmonic, i.e. it satisfies in Ω the equation $\Delta^2_x f = 0$ (see e.g. [1, 11, 13, 16]).

The aim of this paper is to present some simple facts about the inframonogenic functions (Section 2) and establish a Fischer decomposition in this setting (Section 3).

2 Inframonogenic functions: simple facts

It is clear that the product of two inframonogenic functions is in general not inframonogenic, even if one of the factors is a constant.

Proposition 2.1. *Assume that f is an inframonogenic function in $\Omega \subset \mathbb{R}^m$ such that $e_j f$ (resp. $f e_j$) is also inframonogenic in Ω for each $j = 1, \ldots, m$. Then f is of the form*

$$f(\underline{x}) = c\underline{x} + M(\underline{x}),$$

where c is a constant and M a right (resp. left) monogenic function in Ω.

Proof. The proposition easily follows from the equalities

$$\begin{aligned} \partial_{\underline{x}}\big(e_j f(\underline{x})\big)\partial_{\underline{x}} &= -2\partial_{x_j} f(\underline{x})\partial_{\underline{x}} - e_j\big(\partial_{\underline{x}} f(\underline{x})\partial_{\underline{x}}\big), \\ \partial_{\underline{x}}\big(f(\underline{x}) e_j\big)\partial_{\underline{x}} &= -2\partial_{x_j}\partial_{\underline{x}} f(\underline{x}) - \big(\partial_{\underline{x}} f(\underline{x})\partial_{\underline{x}}\big)e_j, \end{aligned} \tag{1}$$

$j = 1, \ldots, m.$ □

For a vector $\underline{x}$ and a k-vector Y_k, the inner and outer product between $\underline{x}$ and Y_k are defined by (see [8])

$$\underline{x} \bullet Y_k = \begin{cases} [\underline{x} Y_k]_{k-1} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases} \qquad \text{and} \qquad \underline{x} \wedge Y_k = [\underline{x} Y_k]_{k+1}.$$

In a similar way $Y_k \bullet \underline{x}$ and $Y_k \wedge \underline{x}$ are defined. We thus have that

$$\begin{aligned} \underline{x} Y_k &= \underline{x} \bullet Y_k + \underline{x} \wedge Y_k, \\ Y_k \underline{x} &= Y_k \bullet \underline{x} + Y_k \wedge \underline{x}, \end{aligned}$$

where also

$$\underline{x} \bullet Y_k = (-1)^{k-1} Y_k \bullet \underline{x},$$
$$\underline{x} \wedge Y_k = (-1)^{k} Y_k \wedge \underline{x}.$$

Let us now consider a k-vector valued function F_k which is inframonogenic in the open set $\Omega \subset \mathbb{R}^m$. This is equivalent to say that F_k satisfies in Ω the system

$$\begin{cases} \partial_{\underline{x}} \bullet (\partial_{\underline{x}} \bullet F_k) & = 0 \\ \partial_{\underline{x}} \wedge (\partial_{\underline{x}} \bullet F_k) - \partial_{\underline{x}} \bullet (\partial_{\underline{x}} \wedge F_k) & = 0 \\ \partial_{\underline{x}} \wedge (\partial_{\underline{x}} \wedge F_k) & = 0. \end{cases}$$

In particular, for $m = 2$ and $k = 1$, a vector-valued function $\underline{f} = f_1 e_1 + f_2 e_2$ is inframonogenic if and only if

$$\begin{cases} \partial_{x_1 x_1} f_1 - \partial_{x_2 x_2} f_1 + 2\partial_{x_1 x_2} f_2 = 0 \\ \partial_{x_1 x_1} f_2 - \partial_{x_2 x_2} f_2 - 2\partial_{x_1 x_2} f_1 = 0. \end{cases}$$

We now try to find particular solutions of the previous system of the form

$$f_1(x_1, x_2) = \alpha(x_1) \cos(n x_2),$$
$$f_2(x_1, x_2) = \beta(x_1) \sin(n x_2).$$

It easily follows that α and β must fulfill the system

$$\alpha'' + n^2 \alpha + 2n\beta' = 0$$
$$\beta'' + n^2 \beta + 2n\alpha' = 0.$$

Solving this system, we get

$$f_1(x_1, x_2) = \big((c_1 + c_2 x_1)\exp(n x_1) + (c_3 + c_4 x_1)\exp(-n x_1)\big)\cos(n x_2), \tag{2}$$
$$f_2(x_1, x_2) = \big((c_3 + c_4 x_1)\exp(-n x_1) - (c_1 + c_2 x_1)\exp(n x_1)\big)\sin(n x_2). \tag{3}$$

Therefore, we can assert that the vector-valued function

$$\underline{f}(x_1, x_2) = \big((c_1 + c_2 x_1)\exp(n x_1) + (c_3 + c_4 x_1)\exp(-n x_1)\big)\cos(n x_2) e_1 + \big((c_3 + c_4 x_1)\exp(-n x_1) - (c_1 + c_2 x_1)\exp(n x_1)\big)\sin(n x_2) e_2, \quad c_j, n \in \mathbb{R},$$

is inframonogenic in $\mathbb{R}^2$. Note that if $c_1 = c_3$ and $c_2 = c_4$, then

$$f_1(x_1, x_2) = 2(c_1 + c_2 x_1)\cosh(n x_1)\cos(n x_2),$$
$$f_2(x_1, x_2) = -2(c_1 + c_2 x_1)\sinh(n x_1)\sin(n x_2).$$

Since the functions (2) and (3) are harmonic in $\mathbb{R}^2$ if and only if $c_2 = c_4 = 0$, we can also claim that not every inframonogenic function is harmonic.

Here is a simple technique for constructing inframonogenic functions from two-sided monogenic functions.

Proposition 2.2. *Let $f(\underline{x})$ be a two-sided monogenic function in $\Omega \subset \mathbb{R}^m$. Then $\underline{x}f(\underline{x})$ and $f(\underline{x})\underline{x}$ are inframonogenic functions in Ω.*

Proof. It is easily seen that

$$\big(\underline{x}f(\underline{x})\big)\partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j}\big(\underline{x}f(\underline{x})\big)e_j = \underline{x}\big(f(\underline{x})\partial_{\underline{x}}\big) + \sum_{j=1}^{m} e_j f(\underline{x})e_j = \sum_{j=1}^{m} e_j f(\underline{x})e_j.$$

We thus get

$$\partial_{\underline{x}}\big(\underline{x}f(\underline{x})\big)\partial_{\underline{x}} = -\sum_{j=1}^{m} e_j\big(\partial_{\underline{x}}f(\underline{x})\big)e_j - 2f(\underline{x})\partial_{\underline{x}} = 0.$$

In the same fashion we can prove that $f(\underline{x})\underline{x}$ is inframonogenic. □

We must remark that the functions in the previous proposition are also harmonic. This may be proved using the following equalities

$$\Delta_{\underline{x}}\big(\underline{x}f(\underline{x})\big) = 2\partial_{\underline{x}}f(\underline{x}) + \underline{x}\big(\Delta_{\underline{x}}f(\underline{x})\big), \tag{4}$$

$$\Delta_{\underline{x}}\big(f(\underline{x})\underline{x}\big) = 2f(\underline{x})\partial_{\underline{x}} + \big(\Delta_{\underline{x}}f(\underline{x})\big)\underline{x}, \tag{5}$$

and the fact that every monogenic function is harmonic. At this point it is important to notice that an $\mathbb{R}_{0,m}$-valued harmonic function is in general not inframonogenic. Take for instance $h(\underline{x})e_j$, $h(\underline{x})$ being an $\mathbb{R}$-valued harmonic function. If we assume that $h(\underline{x})e_j$ is also inframonogenic, then from (1) it may be concluded that $\partial_{\underline{x}}h(\underline{x})$ does not depend on x_j. Clearly, this condition is not fulfilled for every harmonic function.

We can easily characterize the functions that are both harmonic and inframonogenic. Indeed, suppose that $h(\underline{x})$ is a harmonic function in a star-like domain $\Omega \subset \mathbb{R}^m$. By the Almansi decomposition (see [12, 15]), we have that $h(\underline{x})$ admits a decomposition of the form

$$h(\underline{x}) = f_1(\underline{x}) + \underline{x}f_2(\underline{x}),$$

where $f_1(\underline{x})$ and $f_2(\underline{x})$ are left monogenic functions in Ω. It is easy to check that

$$\partial_{\underline{x}}h(\underline{x}) = -mf_2(\underline{x}) - 2\mathrm{E}_{\underline{x}}f_2(\underline{x}),$$

$\mathrm{E}_{\underline{x}} = \sum_{j=1}^{m} x_j\partial_{x_j}$ being the Euler operator. Thus $h(\underline{x})$ is also inframonogenic in Ω if and only if $mf_2(\underline{x}) + 2\mathrm{E}_{\underline{x}}f_2(\underline{x})$ is right monogenic in Ω. In particular, if $h(\underline{x})$ is a harmonic and inframonogenic homogeneous polynomial of degree k, then $f_1(\underline{x})$ is a left monogenic homogeneous polynomial of degree k while $f_2(\underline{x})$ is a two-sided monogenic homogeneous polynomial of degree $k-1$.

The following proposition provides alternative characterizations for the case of k-vector valued functions.

Proposition 2.3. *Suppose that F_k is a harmonic (resp. inframonogenic) k-vector valued function in $\Omega \subset \mathbb{R}^m$ such that $2k \neq m$. Then F_k is also inframonogenic (resp. harmonic) if and only if one of the following assertions is satisfied:*

(i) *$F_k(\underline{x})\underline{x}$ is left 3-monogenic in Ω;*

(ii) $\underline{x}F_k(\underline{x})$ *is right* 3*-monogenic in* Ω*;*

(iii) $\underline{x}F_k(\underline{x})\underline{x}$ *is biharmonic in* Ω*.*

Proof. We first note that

$$e_je_Ae_j = \begin{cases} (-1)^{|A|}e_A & \text{for} \quad j \in A, \\ (-1)^{|A|+1}e_A & \text{for} \quad j \notin A, \end{cases}$$

which clearly yields $\sum_{j=1}^m e_je_Ae_j = (-1)^{|A|}(2|A|-m)e_A$. It thus follows that for every k-vector valued function F_k,

$$\sum_{j=1}^m e_jF_ke_j = (-1)^k(2k-m)F_k.$$

Using the previous equality together with (4) and (5), we obtain

$$\begin{aligned} \partial_{\underline{x}}\Delta_{\underline{x}}\big(F_k(\underline{x})\underline{x}\big) &= 2\partial_{\underline{x}}F_k(\underline{x})\partial_{\underline{x}} + \big(\partial_{\underline{x}}\Delta_{\underline{x}}F_k(\underline{x})\big)\underline{x} + (-1)^k(2k-m)\Delta_{\underline{x}}F_k, \\ \Delta_{\underline{x}}\big(\underline{x}F_k(\underline{x})\big)\partial_{\underline{x}} &= 2\partial_{\underline{x}}F_k(\underline{x})\partial_{\underline{x}} + \underline{x}\big(\Delta_{\underline{x}}F_k(\underline{x})\partial_{\underline{x}}\big) + (-1)^k(2k-m)\Delta_{\underline{x}}F_k, \\ \Delta^2_{\underline{x}}\big(\underline{x}F_k(\underline{x})\underline{x}\big) &= 4\Big(2\partial_{\underline{x}}F_k(\underline{x})\partial_{\underline{x}} + (-1)^k(2k-m)\Delta_{\underline{x}}F_k + \big(\partial_{\underline{x}}\Delta_{\underline{x}}F_k(\underline{x})\big)\underline{x} \\ &\quad + \underline{x}\big(\Delta_{\underline{x}}F_k(\underline{x})\partial_{\underline{x}}\big)\Big) + \underline{x}\big(\Delta^2_{\underline{x}}F_k(\underline{x})\big)\underline{x}. \end{aligned}$$

The proof now follows easily. □

Before ending the section, we would like to make two remarks. First, note that if m even, then a $m/2$-vector valued function $F_{m/2}(\underline{x})$ is inframonogenic if and only if $F_{m/2}(\underline{x})$ and $F_{m/2}(\underline{x})\underline{x}$ are left 3-monogenic, or equivalently, $F_{m/2}(\underline{x})$ and $\underline{x}F_{m/2}(\underline{x})$ are right 3-monogenic. Finally, for m odd the previous proposition remains valid for $\mathbb{R}_{0,m}$-valued functions.

3 Fischer decomposition

The classical Fischer decomposition provides a decomposition of arbitrary homogeneous polynomials in $\mathbb{R}^m$ in terms of harmonic homogeneous polynomials. In this section we will derive a similar decomposition but in terms of inframonogenic homogeneous polynomials. For other generalizations of the Fischer decomposition we refer the reader to [5, 7, 8, 9, 10, 12, 14, 17, 18].

Let $\mathsf{P}(k)$ ($k \in \mathbb{N}_0$) denote the set of all $\mathbb{R}_{0,m}$-valued homogeneous polynomials of degree k in $\mathbb{R}^m$. It contains the important subspace $\mathsf{I}(k)$ consisting of all inframonogenic homogeneous polynomials of degree k.

An an inner product may be defined in $\mathsf{P}(k)$ by setting

$$\langle P_k(\underline{x}), Q_k(\underline{x})\rangle_k = \Big[\overline{P_k(\partial_{\underline{x}})}\,Q_k(\underline{x})\Big]_0, \quad P_k(\underline{x}), Q_k(\underline{x}) \in \mathsf{P}(k),$$

$\overline{P_k(\partial_{\underline{x}})}$ is the differential operator obtained by replacing in $P_k(\underline{x})$ each variable x_j by ∂_{x_j} and taking conjugation.

From the obvious equalities

$$[\overline{e_j a}\, b]_0 = -[\overline{a} e_j b]_0,$$
$$[\overline{a e_j}\, b]_0 = -[\overline{a} b e_j]_0, \quad a, b \in \mathbb{R}_{0,m},$$

we easily obtain

$$\langle \underline{x} P_{k-1}(\underline{x}), Q_k(\underline{x}) \rangle_k = -\left\langle P_{k-1}(\underline{x}), \partial_{\underline{x}} Q_k(\underline{x}) \right\rangle_{k-1},$$
$$\langle P_{k-1}(\underline{x}) \underline{x}, Q_k(\underline{x}) \rangle_k = -\left\langle P_{k-1}(\underline{x}), Q_k(\underline{x}) \partial_{\underline{x}} \right\rangle_{k-1},$$

with $P_{k-1}(\underline{x}) \in \mathsf{P}(k-1)$ and $Q_k(\underline{x}) \in \mathsf{P}(k)$. Hence for $P_{k-2}(\underline{x}) \in \mathsf{P}(k-2)$ and $Q_k(\underline{x}) \in \mathsf{P}(k)$, we deduce that

$$\langle \underline{x} P_{k-2}(\underline{x}) \underline{x}, Q_k(\underline{x}) \rangle_k = \left\langle P_{k-2}(\underline{x}), \partial_{\underline{x}} Q_k(\underline{x}) \partial_{\underline{x}} \right\rangle_{k-2}. \tag{6}$$

Theorem 3.1 (Fischer decomposition). *For $k \geq 2$ the following decomposition holds:*

$$\mathsf{P}(k) = \mathsf{I}(k) \oplus \underline{x} \mathsf{P}(k-2) \underline{x}.$$

Moreover, the subspaces $\mathsf{I}(k)$ and $\underline{x}\mathsf{P}(k-2)\underline{x}$ are orthogonal w.r.t. the inner product $\langle\, , \rangle_k$.

Proof. The proof of this theorem will be carried out in a similar way to that given in [8] for the case of monogenic functions.

As $\mathsf{P}(k) = \underline{x}\mathsf{P}(k-2)\underline{x} \oplus (\underline{x}\mathsf{P}(k-2)\underline{x})^{\perp}$ it is sufficient to show that

$$\mathsf{I}(k) = (\underline{x}\mathsf{P}(k-2)\underline{x})^{\perp}.$$

Take $P_k(\underline{x}) \in (\underline{x}\mathsf{P}(k-2)\underline{x})^{\perp}$. Then for all $Q_{k-2}(\underline{x}) \in \mathsf{P}(k-2)$ it holds

$$\left\langle Q_{k-2}(\underline{x}), \partial_{\underline{x}} P_k(\underline{x}) \partial_{\underline{x}} \right\rangle_{k-2} = 0,$$

where we have used (6). In particular, for $Q_{k-2}(\underline{x}) = \partial_{\underline{x}} P_k(\underline{x}) \partial_{\underline{x}}$ we get that $\partial_{\underline{x}} P_k(\underline{x}) \partial_{\underline{x}} = 0$ or $P_k(\underline{x}) \in \mathsf{I}(k)$. Therefore $(\underline{x}\mathsf{P}(k-2)\underline{x})^{\perp} \subset \mathsf{I}(k)$.

Conversely, let $P_k(\underline{x}) \in \mathsf{I}(k)$. Then for each $Q_{k-2}(\underline{x}) \in \mathsf{P}(k-2)$,

$$\langle \underline{x} Q_{k-2}(\underline{x}) \underline{x}, P_k(\underline{x}) \rangle_k = \left\langle Q_{k-2}(\underline{x}), \partial_{\underline{x}} P_k(\underline{x}) \partial_{\underline{x}} \right\rangle_{k-2} = 0,$$

whence $P_k(\underline{x}) \in (\underline{x}\mathsf{P}(k-2)\underline{x})^{\perp}$. □

By recursive application of the previous theorem we get:

Corollary 3.1 (Complete Fischer decomposition). *If $k \geq 2$, then*

$$\mathsf{P}(k) = \bigoplus_{s=0}^{[k/2]} \underline{x}^s \mathsf{I}(k-2s) \underline{x}^s.$$

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