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The Maxwell problem and the Chapman projection[1]

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ABSTRACT

We study the large-time behavior of global smooth solutions to the Cauchy problem for hyperbolic regularization of conservation laws. An attracting manifold of special smooth global solutions is determined by the Chapman projection onto the phase space of consolidated variables. For small initial data we construct the Chapman projection and describe its properties in the case of the Cauchy problem for moment approximations of kinetic equations. The existence conditions for the Chapman projection are expressed in terms of the solvability of the Riccati matrix equations with parameter.

RESUMEN

Nosotros estudiamos el comportamiento temporal de soluciones globales suaves del problema de Cauchy para regularización hiperbólica de leyes de conservación. Una variedad atractora de soluciones globales suaves es determinada por la proyección de Chapman sobre el espacio de fase de las variables consolidadas. Para datos iniciales pequeños nosotros construimos la proyección de Chapman y descubrimos sus propiedades en el caso del problema de Cauchy para aproximación de momentos en ecuaciones kineticas. Las condiciones de existencia para la proyección de Chapman son expresadas en términos de la solubilidad de las ecuaciones matriciales de Riccati con parámetros.

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1 Introduction

1.1 The state equation. Closure

This paper is devoted to mathamatical aspects of the Maxwell problem [2] about the derivation of the Navier-Stokes equation from kinetics. Following [1] we study the behavior of solutions to the Cauchy problem for hyperbolic regularizations of conservation laws or (in another terminology) for systems of conservation laws with relaxation. Consider $m-$ conservation laws (1.1) and $N-m$ conservation laws with relaxation (1.2):

$$\partial_t u_i + \mathrm{div}_x\, f^i(u,v) = 0, \quad i = 1,\dots,m, \tag{1.1}$$

$$\partial_t v_k + \mathrm{div}_x\, g^k(u,v) + b^k(u)v = 0, \quad k = m+1,\dots,N. \tag{1.2}$$

then we have m conservative variables $u(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^m$ and $N-m$ co-called nonequilibrium variables $v(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{N-m}$, where $x \in \mathbb{R}^d$, b is the relaxation $(N-m)\times(N-m)$-matrix,

$$f^i(u,v) \in \mathbb{R}^d,\ i = 1,\dots,m; \quad g^k(u,v) \in \mathbb{R}^d,\ k = 1,\dots,N-m,$$

are currents. The leading part of the system (1.1) is nonstrictly hyperbolic in the sense of the following definition.

Definition 1.1. A system is nonstrictly hyperbolic if the characteristic matrix

$$\tau E + \xi \cdot \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix} \tag{1.3}$$

has only real (possibly multiple) roots $\tau = \tau_j(\xi,u,v),\ j = 1,\dots,N$.

The condition in Definition 1.1 is satisfied if the system (1.1) is simmetrizable. Examples of such systems are the following: - moment approximations of kinetic equations and the Dirac-Schwinger extension of the Maxwell equations [3]. Hyperbolic regularizations of conservation laws (or systems of conservation laws with relaxation) were considered by many authors. First of all, this concerns the study of the relaxation phenomenon, in particular, the stability and singular limit as the relaxation time tends to zero (cf., for example, [4]-[7]). The so-called "intermediate attractor" for (1.1-1.2) was studied in connection with the Maxwell problem (cf. [8,9]). To derive equations of hydrodynamics from the kinetic gas theory, it is important to find a simple functional dependence of the transport coefficients on the interaction potential and thereby to simplify the analysis of the equations under consideration. We intereste in the Chapman conjecture [1], [9], about the existence problem of state equation

$$v = \mathcal{Q}u, \tag{1.4}$$

(so-called the Chapman state equation or the Chapman projection) expressing the nonequillibrium variables in terms of the conservative variables (the projection into the phase space of conservative variables), where $\mathcal{Q}$ is an operator with respect to space variables x. This equation completes the system of conservation laws

$$\partial_t w + \partial_x f(w, \mathcal{Q}u(w)) = 0. \tag{1.5}$$

so that the solutions w to the Caushy problem of the corresponding closer (1.5) define the set of the special solutions $U_{ChEns} = \{u = w,\ v = \mathcal{Q}w\}$ to the Cauchy problem for the system (1.1-1.2) form an invariant attracting manifold $\mathcal{M}_{ChEns}$, called an intermediate attractor. In other words, for any solution $U = (u, v)$ to the Cauchy problem for the system (1.1-1.2) with the initial data $U|_{t=0} = (u^0, v^0)$ it is possible to choose initial data $w_0 = \mathcal{T}(u_0, v_0)$ for the closure (1.5) in such a way that some norm of the difference $U - U_{ChEns}$ between U and the special solution $U_{ChEns} = (w, \mathcal{Q}w)$ tends to zero as $t \to \infty$. Moreover, if, in the phase space of conservative variables,

$$w \to 0, \quad \text{when} \quad t \to \infty,$$

then U_H tends to zero faster than U_{ChEns}. We can say that in this case the influence of nonequilibrium veriables is inessential(we have the separation of dynamics) [9].

Now we can define the approximation of the state equation and corresponding closure(so-called Navier-Stokes approximation). Due to physical point of view [9] we assume that derivatives of nonequilibrium variables are small, then we find the following relation

$$v = -b^{-1} \operatorname{div}_x g(u, 0) \tag{1.6}$$

and the corresponding closure

$$\partial_t u + \partial_x f(u, -b^{-1} \operatorname{div}_x g(u, 0)) = 0 \tag{1.7}$$

(so-called Navier-Stokes approximation to (1.1-1.2)), where $\det b(u) \neq 0$. For thirteen-moment Grad system to the Boltzman kinetic equation the Navier-Stokes approximation (1.7) is the Navier-Stokes equations exactly.

Considering conservation laws with stiff relaxation

$$\partial_t u + \operatorname{div}_x\ f(u, v) = 0, \quad \partial_t v + \operatorname{div}_x\ g(u, v) + \frac{1}{\varepsilon} b(u) v = 0, \tag{1.8}$$

we find that the Navier-Stokes approximation

$$v = -\varepsilon b^{-1} \operatorname{div}_x g(u, 0), \quad \partial_t u + \operatorname{div}_x\ f(u, -\varepsilon b^{-1} \operatorname{div}_x g(u, 0)) = 0, \tag{1.9}$$

is the first approch to so-called local equilibrium approch(see [1])

$$\partial_t u + \operatorname{div}_x\ f(u, 0) = 0$$

2 Linear analysis. Reduction to a quadratic matrix equation.

2.1 Reduction to a quadratic matrix equation

We consider the Cauchy problem for the first order linear hyperbolic system with constant coefficients and with relaxation [1]

$$\partial_t u + \mathcal{A}_j \partial_{x_j} u + Bu = 0 \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^N$, $\mathcal{A}_j$ and B are constant matrices. In the case of the system (2.1), the Chapman conjecture [1,9] of the state equation existence asserts that

$$u = \Pi\, u_c = (u_c, \Pi_{21} u_c),$$

where $u_c = (u_1, .., u_m, 0, .., 0)^T$ and Π - is a zero order pseudodifferential matrix operator. Suppose Following to [1] that the matrix of operator Π corresponding to the Chapman-Enskog projection into m equations of the system (2.1) has the form

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$$

where $\Pi_{11} = E_m$ is the unit matrix of order m and $\Pi_{22} = 0_{N-m}$ is the zero square matrix of order $N-m$. We denote by $\Lambda(\xi)$ the resolvent matrix $\sum_{j=1}^n \mathcal{A} i\xi_j + B$ and represent it in the block form:

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$

Since Π is a projection,

$$\Pi\partial_t u_c + \mathcal{A}\Pi\partial_x u_c + B\Pi\, u_c = 0, \tag{2.2}$$

Since $\Pi^2 = \Pi$,

$$\Pi\partial_t u_c + \Pi\mathcal{A}\Pi\partial_x u_c + \Pi\, B\Pi\, u_c = 0. \tag{2.3}$$

Subtracting (2.3) from (2.2), we find

$$(E - \Pi)(\mathcal{A}\partial_x + B)\Pi\, u_c = 0$$

We denote by P the Fourier image of Π with respect to x. After the Fourier transform with respect to x, the last equality takes the form $(E-P)\Lambda\, Pv_c = 0$, i.e. $\Lambda\, Pv_c \in Ker(E-P)$. We note that for $\forall v \in Ker(E-P)$ admitting the representation $v^T = (v_m^T, v_{N-m}^T)$, with $v_k \in \mathbb{C}^k$ the following equality holds: $v_{N-m} = P_{21} v_m$. Hence we find the system of equations for P_{21} which completely determines the projection Π:

$$P_{21}(\Lambda_{11} + \Lambda_{12} P_{21}) = \Lambda_{21} + \Lambda_{22} P_{21}.$$

After transformations this equation takes the form

$$P_{21}\Lambda_{12}(\xi)P_{21} - \Lambda_{22}(\xi)P_{21} + P_{21}\Lambda_{11}(\xi) - \Lambda_{21}(\xi) = 0, \tag{2.4}$$

i.e., we obtain a Riccati type matrix equation. This object is nontrivial object. For example, we will consider two special 2×2 cases (2.4):

$$X^2 = 0, \quad X^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There are infinitely many such matrices in the first case, and they form two-dimensional cone in $C^4(\det X = 0,\ \mathbf{tr} X = 0)$. There are no solutions to the second equation, since a matrix has only the zero eigenvalue if the squared matrix possesses this property, i.e. X is nilpotent and the squared nilpotent matrix of second order vanishes.

Lemma 3.1. For any $\gamma \in \mathbb{R}$ the set of solutions to the matrix equation (2.4) with a matrix Λ coincides with the set of solutions to the same equation (2.4) with the matrix è $\Lambda + \gamma\, E$.

Proof. Indeed, with $\Lambda + \gamma\, E$ we associate the matrix equation

$$P_{21}\Lambda_{12}P_{21} - (\Lambda_{22} + \gamma\, E)P_{21} + P_{21}(\Lambda_{11} + \gamma\, E) - \Lambda_{21} = 0,$$

where the left-hand side differs from the left-hand side of (2.4) by $-\gamma\, EP_{21} + P_{21}\gamma\, E = 0$. It is obvious that the sets of solutions to these equations coincide.

Thus, to study the matrix equation (2.4), we can assume without loss of generality that $det(\Lambda) \neq 0$.

2.2 Solutions to the Quadratic Matrix Equation in the Case $|\Lambda| \neq 0$

This section is devoted to the solvability condition for the matrix equation.

Proposition 3.1. Assume that $|\Lambda| \neq 0$ and

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \tag{2.5}$$

where P_{11} is the unit matrix of order m, P_{22} is the zero square matrix of order $N-m$, and P_{12} is the zero matrix. Then the quadratic matrix equation (2.4) is solvable if and only if there exists a matrix P of the form (2.5) such that P is a solution to the quadratic matrix equation

$$(E-P)\Lambda\, P = 0. \tag{2.6}$$

Proof. We first assume that the matrix equation (2.4) is solvable. Taking P of the form (2.5) and representing the product $(E-P)\Lambda\, P$ in the block form, we see that P is a solution to the matrix equation (2.6). Conversely, let P of the form (2.5) be a solution to the matrix equation (2.6). Representing $M = (E-P)\Lambda\, P$ via blocks of the same size as the blocks of P, we see that the blocks M_{11}, M_{12}, and M_{22} are zero. The equation for M_{21} coincides with (2.4) up to a sign, i.e., the matrix equation (2.4) is solvable.

As a consequence it follows

Theorem 3.1. Let a matrix Π_{21} be a solution

$$\Pi_{21}\Lambda_{12}\Pi_{21} - \Lambda_{22}\Pi_{21} + \Pi_{21}\Lambda_{11} - \Lambda_{21} = 0 \tag{2.7}$$

and $X = \Lambda\,\Pi$, where $\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$ is a quadratic matrix of order N, Π_{11} is the identity matrix of order m, and Π_{12}, Π_{22} are zero matrices. Then X is a solution to the quadratic matrix equation

$$X^2 - \Lambda\, X = 0. \tag{2.8}$$

The matrix equation (2.8) is simpler than the general matrix equation and it's not difficulty to describe one completely. Solutions of the matrix equation (2.4) correspond to a part of the set of solutions for the equation (2.8) only. So that we must to define the selection rule.

Theorem 3.2. Let $|\Lambda| \neq 0$. Then the quadratic matrix equation (2.4) is solvable if and only if there are two solutions X_1 and X_2 to the quadratic matrix equation

$$X^2 - \Lambda X = 0, \tag{2.9}$$

such that
1. $X_1 e_j = 0$ for all $j > m$.
2. $e_j^T X_2 = e_j^T \Lambda$ for all $j \leq m$.
3. $\Lambda X_2 = X_1 \Lambda$.

Proof. Assume that the matrix equation (2.4) is solvable. Then the matrix equation (2.6) is also solvable. We note that a matrix P belongs to the above class if and only if $Pe_j = 0\ \forall j > m$ and $e_j^T P = e_j^T\ \forall j \leq m$. Multiplying the matrix equation (2.6) by Λ from the left and making the change of variables $X_1 = \Lambda P$, we see that the matrix X_1 is a solution to (2.9) and satisfies condition 1). Similarly, multiplying (2.6) by Λ from the right and making the change of variables $X_2 = P\Lambda$, we find that the matrix X_2 is a solution to (2.9) and satisfies condition 2). Since $X_1 = \Lambda P$ and $X_2 = P\Lambda$ condition 3) is also valid.

Assume that there exist two solutions X_1 and X_2 to the matrix equation (2.9) satisfying conditions 1)-3). We set $P = \Lambda^{-1} X_1 = X_2 \Lambda^{-1}$. Then the matrix P has the required form because of conditions 1) and 2). Substituting $X_1 = \Lambda P$ into (2.9) and multiplying by Λ^{-1} from the left, we find that P is a solution to (2.6).

Theorem 3.3. Let $|\Lambda| \neq 0$. Then the quadratic matrix equation (2.4) is solvable if and only if there is a solution X_1 to the quadratic matrix equation (2.9) such that
1. $X_1 e_j = 0$ for all $j > m$.
2. $e_j^T \Lambda^{-1} X_1 = e_j^T$ for all $j \leq m$.

Proof. We set $X_2 = \Lambda^{-1} X_1 \Lambda$. It is obvious that X_2 is a solution to the matrix equation (2.9). Furthermore, X_2 satisfies condition 2) of Theorem 3.2 because of condition 2) of Theorem 3.3. Since $X_2 = \Lambda^{-1} X_1 \Lambda$ we have $\Lambda X_2 = X_1 \Lambda$, i.e., condition 3) of Theorem 3.2 is also satisfied.

The proof details of next results look for in [14,17]

Lemma 3.1. Suppose that $det(\Lambda) \neq 0$, X is a solution to the matrix equation (2.9), and vectors $h_1, \dots, h_N$ form the Jordan basis for X. Then there exists $K \geq 0$ such that $h_1, \dots, h_K$ belong to the Jordan basis for Λ(moreover, if $X h_j = \lambda\, h_j + h_{j-1}$, then $\Lambda\, h_j = \lambda\, h_j + h_{j-1}$) and $h_{K+1}, \dots, h_N$ are the eigenvectors corresponding to the eigenvalue 0.

Lemma 3.2. Let $det(\Lambda) \neq 0$. For $K \geq 0$ we denote by X a matrix with the Jordan basis $h_1, \dots, h_N$, where the vectors $h_1, \dots, h_K$ form the Jordan basis for Λ. (listed in such as way that if $X h_j = \lambda\, h_j + h_{j-1}$, then $\Lambda\, h_j = \lambda\, h_j + h_{j-1}$) and $h_{K+1}, \dots, h_N$ are the eigenvectors corresponding to the eigenvalue 0. Then X is a solution to the matrix equation (2.9).

Bring one more the geometrical formulation of the necessary and sufficient conditions of the solvability of the quadratic matrix equation (2.7)

Theorem 3.4. Let $|\Lambda| \neq 0$, and let vectors $v_1, \dots, v_m$ satisfy the following conditions:
1. $V = \mathbf{Lin}\{v_j\}_1^m$ is an eigenspace of the matrix Λ, i.å. $\Lambda V = V$.
2. $v_1, .., v_m, e_{m+1}, .., e_N$ form a basis.
Then the quadratic matrix equation (2.4) is solvable. The inverse assertion is also true.

2.3 Explicit Formula

Now, we discuss a possible explicit formula for solutions to the Riccati matrix equation.

Theorem 4.1.

Suppose that vectors $v_1, \dots, v_m$ form a basis for a linear Λ-invariant subspace V and $v_1, \dots, v_m$, $e_{m+1}, \dots, e_n$ is a basis for R^n. We regard these vectors as columns of a matrix $\begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}$. Then the solution to the matrix equation (2.4), associated with these vectors listed in the above order, is represented in the form

$$P_{21} = C_{21} C_{11}^{-1} \tag{2.10}$$

Proof. Since we can assume that $det(\Lambda) \neq 0$, for the solution to the matrix equation (2.4) we have

$$\begin{pmatrix} E & 0 \\ P_{21} & 0 \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} C_{11} & 0 \\ C_{21} & E \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11}^{-1} & 0 \\ -C_{21}C_{11}^{-1} & E \end{pmatrix},$$

where J_1 is a block from the Jordan form of the matrix Λ corresponding to the space V . Hence

$$\begin{pmatrix} E & 0 \\ P_{21} & 0 \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} C_{11}J_1C_{11}^{-1} & 0 \\ C_{21}J_1C_{11}^{-1} & 0 \end{pmatrix}.$$

Multiplying both sides of the last equality by Λ from the left, we find

$$\begin{pmatrix} \Lambda_{11} + \Lambda_{12}P_{21} & 0 \\ \Lambda_{21} + \Lambda_{22}P_{21} & 0 \end{pmatrix} = \begin{pmatrix} C_{11}J_1C_{11}^{-1} & 0 \\ C_{21}J_1C_{11}^{-1} & 0 \end{pmatrix},$$

which implies $P_{21}C_{11}J_1C_{11}^{-1} - C_{21}J_1C_{11}^{-1} = 0$. in view of (2.4). Since Λ is invertible, the matrix J_1 is also invertible. Hence we can multiply the last equality by $C_{11}J_1^{-1}$ from the right. Then $P_{21}C_{11} = C_{21}$, which implies (2.10).

2.4 The Number of Solutions

Corollary 5.1. With every m-dimensional eigenspace V of the matrix Λ at most one solution to the matrix equation (2.4) is associated.

Proof. Indeed, either V does not provide any solution to (2.4) (if $\mathbf{Lin}\{v_1, \dots, v_m, e_{m+1}, \dots, e_n\} = R^n$, where $V = \mathbf{Lin}\{v_1, \dots, v_m\}$) or V can be associated with a solution to (2.4) by formula (2.10). In the second case, we show that the solution is independent of the choice of the basis for the space V. Let $w_1, \dots, w_m$ be another basis for V . We write the vectors $v_1, \dots, v_m$ as columns of a matrix W^0

and the vectors $w_1, \ldots, w_m$ as columns of a matrix W^1. Since these bases generate the same linear space V , there exists a nonsingular matrix K such that $W^1 = W^0 K$ or, in the block form,

$$\begin{pmatrix} W_1^1 \\ W_2^1 \end{pmatrix} = \begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix} K,$$

which implies $W_j^1 = W_j^0 K,\ j = 1, 2$. Hence the solution of the form (2.10) corresponding to the basis for W^1 can be written as

$$P_{21,W} = W_2^1 (W_1^1)^{-1} = W_2^0 K K^{-1} (W_1^0)^{-1} = P_{21,V}.$$

Thus, the solutions defined by the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ coincide.

Next results we bring for the information(details look in [19,22])

Theorem 5.2. Let the matrix equation (2.4) have infinite number of solutions. Then there exists $\lambda \in \mathbb{C}$ such that $dim(Ker(\Lambda - \lambda E)) \geq 2$.

Theorem 5.2.

The set of solutions to the matrix equation (2.4) is infinite then and only then if there exists the eigenspaces V and W of the matrix Λ, satisfying the following conditions:
1. V defines the solution to the equation (2.4).
2. W is a eigenspace of the matrix Λ, corresponding to a eigenvalue λ.
3. W contain two incollinear eigenvectors.
4. $V \cap W \neq \{0\}$.
5. $W \setminus V \neq \emptyset$.

2.5 Continuity of Solutions to the Quadratic Matrix Equation

In this section, we study the continuity of the constructed solutions with respect to the parameter ξ. We begin with auxiliary assertions(see [17]).

Lemma 6.1. Let a matrix $A(x)$ be a continuous function of the parameter x in some neighborhood $U(x_0)$ of a point x_0. Denote by $\lambda(x)$ an eigenvalue of $A(x)$ that continuously depends on x in $U(x_0)$ and is simple in a punctured neighborhood of x_0. Then the corresponding eigenvector $v_\lambda(x)$ is also continuous with respect to x in the same neighborhood.

Lemma 6.2. Suppose that a matrix-valued function $A(x)$ is continuous with respect to x in some neighborhood $U(x_0)$ of a point x_0 and its kernel has constant dimension k in $U(x_0)$. Let vectors $v_1(x), \ldots, v_m(x)$. $\mathbf{Ker}(A(x))$ be continuous in $U(x_0)$ and linearly independent in the corresponding punctured neighborhood. Then there exists a basis for the invariant subspace $V(x) = \mathbf{Lin}\{v_1(x), \ldots, v_m(x)\}$, i.e., $w_1(x), \ldots, w_m(x)$, that is continuous in $U(x_0)$ and is linearly independent in $U(x_0)$.

Theorem 6.1. Suppose that a matrix $A(x)$ continuously depends on x in some neighborhood $U(x_0)$ of a point x_0 and $\lambda_1(x), \ldots, \lambda_k(x)$ are continuous eigenvalues of $A(x)$ such that each of them is simple in a punctured neighborhood of x_0, $\lambda_1(x_0) = ... = \lambda_k(x_0) = \lambda_0$. Let $V(x)$ be the eigenspace of $A(x)$ corresponding to $\lambda_1(x), ..., \lambda_k(x)$. If there are no eigenvalue $\lambda(x)$ of $A(x)$, different from

$\lambda_j(x), j = 1, \ldots, k$, that is continuous and $\lambda(x_0) = \lambda_0$, then there exists a basis for $V(x)$ continuously depending on x in $U(x_0)$.

Proof. In a punctured neighborhood of x_0, for a basis for $V(x)$ we take the eigenvectors $v_j(x)$ corresponding to the eigenvalues $\lambda_j(x)$ of the matrix $A(x)$. By the above lemma, $v_j(x)$ are continuous in $U(x_0)$. Further, let us introduce the matrix $M(x) = \Pi_{j=1}^{k}(A(x) - \lambda_j(x)E)$.. Under the assumptions of the theorem, $M(x)$ is continuous in $U(x_0)$, $\dim(Ker(M(x))) = k$ and $V(x) = Ker(M(x))$. Hence the matrix $M(x)$ and subspace $V(x)$ satisfy the assumptions of Lemma 5.2 with $m = k$, which implies the required assertion. Theorems 5.1 and 4.1 lead to the following assertion concerning the continuity of solutions to the quadratic matrix equation (2.4) with respect to the parameter ξ

Theorem 6.2. Assume that a matrix Λ is continuously depends on the parameter ξ and is invertible for all $\xi \in \Xi_0$; moreover, the eigenvalues of Λ are simple for all $\xi \notin \Xi_*$, where the set Ξ_* is finite. Then the matrix equation (2.4) with the matrix Λ has a solution continuously depending on the parameter ξ if and only if there exists an m-dimensional eigenspace V satisfying the assumptions of Theorem 4.1 for all $\xi \in \Xi_0$.

Proof. Indeed, by Theorem 4.1 and its consequences, the subspace V determines a solution to the quadratic matrix equation (2.4) in the form $P_{21} = C_{21}C_{11}^{-1}$. Since the matrix Λ satisfies the assumptions of Theorem 5.1, the basis for the space V continuously depends on ξ. Therefore, the invertibility of C_{11} immediately implies the required assertion.

2.6 The Lyapunov Equation. Separation of Dynamics

The Lyapunov matrix equation

$$-M_{11}Q_{12} + Q_{12}M_{22} - M_{12} = 0 \tag{2.11}$$

is a special case of the quadratic matrix equation (2.4) with vanishing quadratic term. The following assertion is proved in [20].

Theorem 7.1. Suppose that $det(M_{11}) \neq 0$ and $det(M_{22}) \neq 0$. Assume that the matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

has no eigenvalues λ such that, in the block form, the corresponding eigenvector has the form $v_0 = \begin{pmatrix} v_{0,1} \\ 0 \end{pmatrix}$ and the corresponding associated eigenvector has the form $v_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix}$, where $v_{1,2} \neq 0$. Then there exists a solution Q_{12} to the Lyapunov matrix equation (2.11) with the matrices M_{11}, M_{12}, and M_{22}.

We construct the canonical form of (2.1).

Lemma 7.1. Suppose that a matrix S is invertible and can be written in the block form with blocks $S_{ij},\ i,j = 1,2$, where S_{11} and S_{22} are square matrices. Assume also that $FS = SF = E$, where the matrix F can be represented by blocks of the same size. In this case, if $S_{11} = E$ then the matrix F_{22} is invertible.

Proof. Assume the contrary. Since $FS = E$, we have

$$F_{21} + F_{22}S_{21} = 0. \tag{2.12}$$

Since F_{22} is noninvertible, there is a row $h \neq 0$ such that $hF_{22} = 0$. Using (2.12), we find $hF_{21} = 0$. But, in this case, the last rows of the matrix F are linearly dependent: there is a row v such that $v \neq 0$ and $vF = 0$. Thus, the matrix F is noninvertible. On the other hand, the matrix F is the inverse of S. We arrive at a contradiction.

Theorem 7.2. Suppose that Λ is divided into blocks $\Lambda_{ij}, i, j, = 1, 2$. Then the quadratic matrix equation (2.4) is solvable if and only if there exists a matrix S satisfying the following conditions:
1. S is invertible,
2. $S_{11} = E$.
3. $(S^{-1}\Lambda\, S)_{21} = 0$.

Proof. Assume that there exists a matrix S satisfying conditions 1) -3). For $F = S^{-1}$ we have

$$F_{21} + F_{22}S_{21} = 0, \quad F_{21}(\Lambda_{11} + \Lambda_{12}S_{21}) + F_{22}(\Lambda_{21} + \Lambda_{22}S_{21}) = 0.$$

Expressing F_{21} from the first equation and substituting into the second equation, we find

$$F_{22}(-S_{21}(\Lambda_{11} + \Lambda_{12}S_{21})) + F_{22}(\Lambda_{21} + \Lambda_{22}S_{21}) = 0.$$

We note that the matrix S satisfies the assumptions of Lemma 6.1. Hence

$$(-S_{21}(\Lambda_{11} + \Lambda_{12}S_{21})) + (\Lambda_{21} + \Lambda_{22}S_{21}) = 0,$$

i.e., the matrix S_{21} satisfies the quadratic matrix equation (2.4). Assume that the quadratic matrix equation (2.4) is solvable. We set $S_{11} = E,\ S_{12} = 0,\ S_{21} = P_{21},\ S_{22} = E$. It is easy to verify that the inverse matrix exists: $S^{-1} = 2E - S$. We see that the matrix S satisfies conditions 1) and 2). Computing $(S^{-1}\Lambda\, S)_{21}$, we find

$$(S^{-1}\Lambda\, S)_{21} = F_{21}(\Lambda_{11} + \Lambda_{12}S_{21}) + F_{22}(\Lambda_{21} + \Lambda_{22}S_{21}) =$$

$$= (-P_{21})(\Lambda_{11} + \Lambda_{12}P_{21}) + (\Lambda_{21} + \Lambda_{22}P_{21}) = 0,$$

since P_{21} is a solution to the quadratic matrix equation (2.4). Thus, the matrix S also satisfies condition 3). The theorem is proved. Thus, the existence of a Chapman-Enskog projection is equivalent to the possibility to represent the original system in the block form such that $(S^{-1}\Lambda\, S)_{21} = 0$, which allows us to separate dynamics. The following theorem (cf. the proof in [16]) provides us with conditions under which a matrix . can be reduced to the block-diagonal form.

Theorem 7.3. Assume that a matrix Λ is invertible and $v_1, \dots, v_m$ is a basis for its eigenspace V such that $\mathbf{Lin}\{v_1, \dots, v_m, e_{m+1}, \dots, e_N\} = R^N$. We also assume that V cannot be extended to an

$m+1$-dimensional eigenspace of the matrix Λ by extending the basis $v_1, \ldots, v_m$ with an associated eigenvector of Λ. Then there exist matrices P_{21} and Q_{12} such that

$$\begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix} \begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix} \Lambda \begin{pmatrix} E & 0 \\ P_{21} & E \end{pmatrix} \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}.$$

Now, we consider the representation of the solution as the sum of three terms and introduce the notion of the L_2-well-posedness in the sense of Chapman-Enskog.

Suppose that a matrix Λ satisfies the assumptions of Theorem 7.3. We make the change of variables $U = S^{-1}u$. Then a solution to the Cauchy problem (2.1) with the initial data

$$U|_{t=0} = \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix}$$

can be written in terms of the Fourier images as follows:

$$U = e^{-Mt} \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix},$$

where $M = S^{-1}\Lambda\, S$. By Theorem 7.3, the matrix M takes the form

$$M = \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix},$$

which implies

$$U = \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} exp\left(- \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} t \right) \begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix} \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix} = U_{Ch} + U_{Cor} + U_H,$$

where each of the terms is a solution to the system (2.1) with some initial data:

$$U_{Ch} = e^{-Mt} \begin{pmatrix} \mathcal{U}_0 \\ 0 \end{pmatrix}, \quad U_{Cor} = e^{-Mt} \begin{pmatrix} -Q_{12}\mathcal{V}_0 \\ 0 \end{pmatrix}, \quad U_H = \begin{pmatrix} Q_{12} e^{-M_{22}t} \mathcal{V}_0 \\ e^{-M_{22}t} \mathcal{V}_0 \end{pmatrix}.$$

The first term U_{Ch} corresponds to the projection onto the phase space of consolidated variables, the second term UCor is a corrector describing the influence of the initial data relative to nonequillibrium variables, and the third term UH is a remainder.

Definition 7.1. We say that a projection P satisfies the Chapman L_2-well-posedness condition for a class of initial data $\mathcal{H} = \{(\mathcal{U}_0, \mathcal{V}_0)\}$ if for any initial data $(\mathcal{U}_0, \mathcal{V}_0) \in \mathcal{H}$ there is a constant $T_0 > 0$ such that for all $t > \ T_0$

$$\frac{||U_H||(t)}{||U_{Ch}||(t)} \leq K e^{-\delta\, t}, \ t > T_0, \tag{2.13}$$

where K and $\delta > 0$ are constants.

2.7 Crack Condition and the Existence of an Attracting Manifold

We find conditions that guarantee the validity of the estimate

$$||U_H|| = o(||U_{Ch}||),\ t \to \infty,$$

where $\|f\|$ denotes the norm of f in the space L_2. For this purpose, we prove several technical auxiliary assertions(see [21]):

Lemma 8.1. Suppose that a matrix Λ polynomially depends on ξ and there exists $k_0 > 0$ such that for all $\xi : |\xi| > k_0$, all the eigenvalues $\lambda(\xi)$ of Λ are algebraically simple and $|\lambda(\xi)| \le C_1(1+|\xi|)^{d_1}$, where C_1 and d_1 are constants. Let v be an eigenvector of Λ. Then for $|\xi| > k_0$:

$$\frac{max\{|e_i^T v|\}}{\min\{|e_i^T v| \neq 0\}} \le C_2(1+|\xi|)^{d_2}, \tag{2.14}$$

where C_2 and d_2 are constants.

Lemma 8.2. Suppose that a matrix Λ is defined for all $\xi \in \mathbb{R}$ and satisfies the assumptions of Theorem 6.3 for all $\xi \in \Xi$, where $\Xi = \mathbb{R}\backslash\Xi_-$ and the set Ξ_- is finite. Then P_{21} and Q_{12} are defined on Ξ. Assume that the matrices $P_{21}(\xi)$ and $Q_{12}(\xi)$ can be defined by continuity on the set Ξ_-. We also assume that the matrix Λ. polynomially depends on ξ and there is $k_0 > 0$ such that for all $\xi : |\xi| > k_0$, all the eigenvalues of the matrix Λ are algebraically simple and satisfy the following estimate: $|\lambda(\xi)| \le C_1(1+|\xi|)^{d_1}$, where C_1 and d_1 are constants. Then there is $d \in \mathbb{N}$ such that for all $\xi \in \mathbb{R}$

$$|P_{21}| \le K_1(1+|\xi|)^d, \quad |Q_{12}| \le K_2(1+|\xi|)^{3d},$$

where K_1 and K_2 are constants and $|A|$ is the matrix norm of A in L_∞.

Notation 8.1. The minimal number $d \in \mathbb{N}$ satisfying the assumptions of Lemma 8.2 is denoted by d_Λ.

We also need a two-sided estimate for $|e^{-Mt}v|$, where $|.|$ — denotes the $L_\infty(\mathbb{R})$. For the sake of brevity, we introduce the following notation.

Notation 8.2. Suppose that a square matrix M continuously depends on the parameter ξ. Let $\lambda_j,\ j = 1, ..., s$, be eigenvalues of M. We denote by d_j the maximal size of the Jordan cell corresponding to the eigenvalue λ_j . Let the eigenvalues λ_j be listed in ascending order of the real part. Let $l(M)$ and $L(M)$ denote the minimal and maximal eigenvalues respectively, i.e.

$$l(M) = \operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots \le \operatorname{Re}\lambda_s = L(M).$$

We set $d(M) = d_1$.

We will use one technical lemma still(see [19]:

Lemma 8.3. Let a square matrix M continuously depend on the parameter ξ. Then for any $\varepsilon > 0$ there is $T_0 > 0$ such that for all $t > T_0$ the following estimate holds:

$$e^{-L(M)t}|v| \le |e^{-Mt}v| \le \frac{1+\varepsilon}{(d(M)-1)!}|M|^{d(M)-1}e^{-l(M)t}t^{d(M)-1}|v|, \tag{2.15}$$

where $|A|$ denotes the matrix norm of A in $L_\infty(\mathbb{R})$.

Notation 8.3. Let $\Gamma(\xi)$ be a finite set of continuous functions $\gamma_1(\xi), \dots, \gamma_s(\xi)$ of the parameter ξ. Introduce the notation $l(\xi, \Gamma(\xi)) = \inf_s\{\operatorname{Re}\gamma_s(\xi) \mid \gamma_s(\xi) \in \Gamma(\xi)\}$, $l_0(\Gamma) = \inf_\xi l(\xi, \Gamma(\xi))$, $L(\xi, \Gamma(\xi)) = \sup_s\{\operatorname{Re}\gamma_s(\xi) \mid \gamma_s(\xi) \in \Gamma(\xi)\}$, $L_0(\Gamma) = \sup_\xi L(\xi, \Gamma(\xi)))$.

Condition 8.1. A pair of sets $\Gamma_1(\xi)$ and $\Gamma_2(\xi)$ satisfy the strong crack condition if

$$\exists\gamma > 0 : \; l_0(\Gamma_2) - L_0(\Gamma_1) \geq \gamma. \tag{2.16}$$

Now, we formulate the conditions for the existence of an attracting manifold.

Theorem 8.1. Let the matrix Λ in the problem (2.1) satisfy the assumptions of Lemma 8.2. Suppose that Γ_1 is the set of all those eigenvalues of Λ that determine the separation of dynamics for the eigenspace V and Γ_2 is the set of all the remaining eigenvalues of Λ. Assume that Γ_1 and Γ_2 satisfy the strong crack condition. Let the Fourier images of initial data $(\mathcal{U}_0, \mathcal{V}_0)$ belong to the set

$$\mathcal{H} = \{(\mathcal{U}_0, \mathcal{V}_0) : \; ||\mathcal{U}_0|| \neq 0, \; (1 + |\xi|)^{3d_\Lambda}|M_{22}|^{d(M_{22})-1}\mathcal{V}_0 \in L_2(\mathbb{R})\},$$

Then the projection P corresponding to the separation of dynamics satisfies the Chapman-Enskog L_2-well-posedness condition (Definition 6.1) for the class of initial data $\mathcal{H}$ with constants K and δ such that

(i) K depends on $||\mathcal{U}_0||$, $||\mathcal{V}_0||$,

(ii) δ depends on δ and some properties of the matrix M.

Proof. Indeed,

$$||U_H(t)|| = \left(\int_{\mathbb{R}} \left| \begin{pmatrix} Q_{12}e^{-M_{22}t}\mathcal{V}_0 \\ e^{-M_{22}t}\mathcal{V}_0 \end{pmatrix} \right|^2 d\xi\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} |1 + |Q_{12}|^2||e^{-M_{22}t}\mathcal{V}_0|^2 d\xi\right)^{\frac{1}{2}}.$$

Using Lemmas 8.2 and 8.3, we find

$$||U_H(t)||^2 \leq \int_{\mathbb{R}} (1 + K_2^2(1 + |\xi|)^{10d_\Lambda})(\frac{1+\varepsilon}{(d(M_{22}) - 1)!})^2 |M_{22}|^{2d(M_{22})-2} e^{-2l(M_{22})t} t^{2d(M_{22})-2} |\mathcal{V}_0|^2 d\xi.$$

From (2.16) it follows that

$$e^{-l(M_{22})t} \leq e^{-l_0(\Gamma_2)t} \leq e^{-\gamma t}e^{-L_0(\Gamma_1)t}; \quad e^{-L(M_{11})t} \geq e^{-L_0(\Gamma_1)t}.$$

By Lemma 8.3,

$$||U_{Ch}(t)|| \geq \left(\int_{\mathbb{R}} e^{-2L(M_{11})t}|\mathcal{U}_0|^2 d\xi\right)^{\frac{1}{2}}.$$

Combining the last four inequalities, we find

$$\left(\frac{||U_H||(t)}{||U_{Ch}||(t)}\right)^2 \leq \frac{e^{-2\gamma t}t^{2d(M_{22})-2}\int_{\mathbb{R}} e^{-2L_0(\Gamma_1)t}(1 + K_2^2(1 + |\xi|)^{10d_\Lambda})\left(\frac{1+\varepsilon}{(d(M_{22})-1)!}\right)^2 |\mathcal{V}_0|^2 d\xi}{\int_{\mathbb{R}} e^{-2L_0(\Gamma_1)t}|\mathcal{U}_0|^2 d\xi},$$

which implies the required estimate (2.13) because $L_0(\Gamma_1)$ is independent of ξ.

3 Nonlinear analysis. Chapman projection

3.1 Statement of the Problem and Auxiliaries

We consider the nonlinear system of equations

$$\partial_t u + \sum_{j=1}^{n} \mathcal{A}_j \partial_{x_j} u + Bu = f(u), \tag{3.1}$$

with the initial condition $u|_{t=0} = \phi$, where u is an N-dimensional vector, A_j and B are constant matrices, $n \le 3$, and $f(u)$ is a vector-valued polynomial, i.e. $f(u) = \sum_{j=1}^{N} \left(\sum_{\sigma \in \Theta_j} K(j,\sigma) u^\sigma \right) e_j$, where $\sigma \in (\mathbb{N} \cup \{0\})^N$, $u^\sigma = \prod_{j=1}^{N} u_j^{\sigma_j}$. We set

$$\alpha = \min\{|\sigma| : \sigma \in \cup_{j=1}^{N} \Theta_j\}, \quad \alpha + \beta = \max\{|\sigma| : \sigma \in \cup_{j=1}^{N} \Theta_j\}.$$

Assume that $f(u)$ contains no terms of zero or first order, i.e. $\alpha \ge 2$. We denote by $|| \cdot ||$ the norm in L_2 with respect to the variable x. Let $|u| = \sqrt{u^T u}$ and let $|u|_0$ denote the norm of u in C. Following [10], we denote by ∂ the vector consisting of all first order derivatives and by ∂_x the vector consisting of first order derivatives with respect to the spatial variables, i.e. $\partial = (\partial_x,\ \partial_t)$. For the sake of brevity, we write ∂_j instead of ∂_{x_j} .

We begin with the following auxiliary assertion generalizing Lemma 8.3.

Lemma 9.1. Let M be a square matrix. Then there are constants $C_M \in \mathbb{R}$ and $d_M \in \mathbb{Z}$ such that for any vector v and a number $t \ge 0$

$$|e^{-Mt} v| \le C_M (1 + t^{d_M}) e^{-l(M)t} |v|. \tag{3.2}$$

The following assertion concerns estimates for the norms of $f(u)$ and its derivatives is tru:

Lemma 9.2. For a vector-valued function $u(x,t) \in C([0,T), H^2) \cap C^1([0,T), H^1)$ with $T > 0$ and $s \in \{1,2\}$, $j \in \{1,2,3\}$ the following estimates hold:

$$||f(u)|| \le C_{0,0} |u|_0^{\alpha-1} (1 + |u|_0^{\beta}) ||u||, \tag{3.3}$$

$$||\partial_j^s f(u)|| \le C_{s,0} |u|_0^{\alpha-1} (1 + |u|_0^{\beta}) ||\partial_j^s u||, \tag{3.4}$$

The following assertion concerning the norm of a vector-valued polynomial is a consequence of the above lemma.

Lemma 9.3. Consider a vector-valued function $u(x,t) \in C([0,T), H^2) \cap C^1([0,T), H^1)$ with $T > 0$, $x \in \mathbb{R}^n$, $n \le 3$. Let $g(u)$ be a vector-valued polynomial with $\alpha \ge 1$. Then there are $\kappa \in (0,1)$ and $C_G > 0$ such that for all $u(x,t)$ such that

$$||g(u)||_{H^2} \le 2C_G ||u||_{H^2}^{\alpha}. \tag{3.5}$$

Proof. Indeed, from the inequalities (3.3) and (3.4) it follows that

$$||g(u)||_{H^2} \leq \text{const}\, |u|_0^{\alpha-1}(1+|u|_0^{\beta})||u||_{H^2}.$$

By the embedding theorem,

$$||g(u)||_{H^2} \leq C_G||u||^{\alpha}_{H^2}(1+||u||^{\beta}_{H^2}).$$

Hence the required inequality (3.5) holds for sufficiently small κ.

Whence we obtain

Lemma 9.4. Let $u(x,t)$ and $v(x,t)$ be vector-valued functions such that $u(x,t), v(x,t) \in C([0,T),H^2)\cap C^1([0,T),H^1)$ for some $T>0$. Assume that $x\in\mathbb{R}^n$ and $n\leq 3$, $f(u)$ is a vector-valued polynomial with $\alpha\geq 2$. Then there are $\kappa\in(0,1)$ and $C_*>0$ such that for all $u(x,t), v(x,t)$ the inequalities $||u||_{H^2}<\kappa$, $||v||_{H^2}<\kappa$ imply the inequality

$$||f(u)-f(v)||_{H^2} \leq C_*(||u||^{\alpha-1}_{H^2}+||v||^{\alpha-1}_{H^2})||u-v||_{H^2}. \tag{3.6}$$

Lemma 9.5. Suppose that $t>0$ and $P(\tau)$ is a continuous function such that the inequality $P(\tau)\geq 0$ for all $\tau\in[0,t]$. Let $d>0$. Then there is a constant $C_P>0$ such that

$$\int_0^t (1+(t-\tau)^d)^2 P(\tau)d\tau \leq C_P(1+t^d)^2\int_0^t (1+\tau^d)^2P(\tau)d\tau. \tag{3.7}$$

Proof. We have

$$\int_0^t (1+(t-\tau)^d)^2P(\tau)d\tau \leq C_1\int_0^t (1+\tau^{2d}+t^{2d})P(\tau)d\tau \leq$$

$$\leq C_1\left(t^{2d}\int_0^t P(\tau)d\tau + \int_0^t(1+\tau^{2d})P(\tau)d\tau\right) \leq C_1(1+t^{2d})\int_0^t (1+\tau^{2d})P(\tau)d\tau \leq$$

$$\leq C_1(1+t^d)^2\int_0^t(1+\tau^d)^2P(\tau)d\tau.$$

3.2 Method of Successive Approximations

We look for a solution to the system (3.1) with the initial data $u|_{t=0}=\phi(x)$ for small $\phi(x)$ by the method of successive approximations. We set $u_0=0$,

$$\partial_t u_k + \sum_{j=1}^n \mathcal{A}_j\partial_j u_k + Bu_k = f(u_{k-1}), \quad u_k|_{t=0}=\phi(x). \tag{3.8}$$

Introduce the notation $\Lambda=\sum_{j=1}^n \mathcal{A}_j i\xi_j + B$, $l_1=\inf_\xi \min_{\lambda\in\sigma(\Lambda)} \operatorname{Re}\lambda$. We denote by $\mathcal{F}(\cdot)$ the Fourier transform with respect to the spatial variables. We estimate from above the solution u_k to the problem (3.8).

Lemma 10.1. Let $l_1 > 0$. Then there exist constants $\kappa \in (0,1)$ and $C_1^* > 0$, $C_2^* > 0$ such that the solution uk to the problem (3.8) with the initial data ϕ such that $||\phi||_{H^2} < \kappa$ satisfies the following inequality for any $t \geq 0$:

$$||u_k||_{H^2} \leq C_1^*(1+t^{d_\Lambda})e^{-l_1 t}(||\phi||_{H^2} + C_2^*\sqrt{t}||\phi||^\alpha_{H^2}). \tag{3.9}$$

Proof. We first prove that for sufficiently small initial data

$$||u_k||_{H^2} \leq C_\Lambda(1+t^{d_\Lambda})e^{-l_1 t}(||\phi||_{H^2} + C_k\sqrt{t}||\phi||^\alpha_{H^2}), \tag{3.10}$$

where the constants C_k depend on k. We write an explicit expression for C_k. For this purpose, we use the method of mathematical induction. Let $k = 1$. Then the problem (3.8) takes the form

$$\partial_t u_1 + \sum_{j=1}^n \mathcal{A}_j \partial_j u_1 + Bu_1 = 0, \ u_1|_{t=0} = \phi.$$

The solution to this problem is written in terms of the Fourier images as follows: $\mathcal{F}(u_1) = e^{-\Lambda t}\mathcal{F}(\phi)$. By Lemma 10.1

$$||u_1||^2 = ||\mathcal{F}(u_1)||^2 = \int_{\mathbb{R}^n} |e^{-\Lambda t}\mathcal{F}(\phi)|^2 d\xi \leq C_\Lambda^2 \int_{\mathbb{R}^n} (1+t^{d_\Lambda})^2 e^{-2l_1 t}|\mathcal{F}(\phi)|^2 d\xi =$$

$$= C_\Lambda^2(1+t^{d_\Lambda})^2 e^{-2l_1 t}||\mathcal{F}(\phi)||^2 = C_\Lambda^2(1+t^{d_\Lambda})^2 e^{-2l_1 t}||\phi||^2.$$

A similar inequality holds for the derivatives of u_1. Thus,

$$||u_1||_{H^2} \leq C_\Lambda(1+t^{d_\Lambda})e^{-l_1 t}||\phi||_{H^2},$$

and the inequality (3.10) is true with $C_1 = 0$. Now, we write an explicit formula for the solution to the problem (3.8) in terms of the Fourier images:

$$\mathcal{F}(u_k) = e^{-\Lambda t}\mathcal{F}(\phi) + \int_0^t e^{\Lambda(\tau - t)}\mathcal{F}(f(u_{k-1}(\tau)))d\tau. \tag{3.11}$$

We set $I_{\sigma,k} = (i\xi)^\sigma \int_0^t e^{\Lambda(\tau-t)}\mathcal{F}(f(u_{k-1}(\tau)))d\tau$ and find

$$||I_{\sigma,k}|| \leq C_\Lambda C_k \sqrt{t}(1+t^{d_\Lambda})e^{-l_1 t}||\phi||^\alpha_{H^2},$$

where the constant C_k is independent of σ. The proof of this assertion is similar to that of the inequality (3.10) for all k. Indeed, we have the auxiliary estimates

$$||I_{\sigma,k}||^2 \leq t\int_0^t ||(i\xi)^\sigma e^{\Lambda(\tau-t)}\mathcal{F}(f(u_{k-1}(\tau)))||^2 d\tau \leq$$

$$\leq tC_\Lambda^2 \int_0^t (1+(t-\tau)^{d_\Lambda})^2 e^{2l_1(\tau-t)} \int_{\mathbb{R}^n} |(i\xi)^\sigma \mathcal{F}(f(u_{k-1}(\tau)))|^2 d\xi d\tau \leq$$

$$\leq tC_\Lambda^2 \int_0^t (1+(t-\tau)^{d_\Lambda})^2 e^{2l_1(\tau-t)}||f(u_{k-1}(\tau))||^2_{H^2} d\tau.$$

Using Lemmas 9.3 and 9.5, we find

$$||I_{\sigma,k}||^2 \leq 4C_\Lambda^2 C_F^2 C_P t(1+t^{d_\Lambda})^2 e^{-2l_1 t}\int_0^t (1+\tau^{d_\Lambda})^2 e^{2l_1\tau}||u_{k-1}(\tau)||_{H^2}^{2\alpha}d\tau. \tag{3.12}$$

Let $k=2$. Using the estimate (3.10) for $k=1$, we find

$$||I_{\sigma,2}||^2 \leq 4C_\Lambda^{2+2\alpha}C_F^2 C_P t(1+t^{d_\Lambda})^2 e^{-2l_1 t}\int_0^t e^{2l_1(1-\alpha)\tau}(1+\tau^{d_\Lambda})^{2+2\alpha}||\phi||_{H^2}^{2\alpha}d\tau \leq$$

$$\leq 4C_\Lambda^{2+2\alpha}C_F^2 C_P t(1+t^{d_\Lambda})^2 e^{-2l_1 t}\int_0^{+\infty} e^{2l_1(1-\alpha)\tau}(1+\tau^{d_\Lambda})^{2+2\alpha}||\phi||_{H^2}^{2\alpha}d\tau.$$

Note that for $\alpha \geq 2$ the integral is convergent. Setting

$$C_2^2 = 4C_\Lambda^{2\alpha}C_F^2 C_P\int_0^{+\infty} e^{2l_1(1-\alpha)\tau}(1+\tau^{d_\Lambda})^{2+2\alpha}d\tau,$$

we obtain an inequality of the required form for $||I_{\sigma,2}||$.

Assume that the inequality (3.10) is valid for all $k \leq r$, where $r \geq 2$. Then for $k = r+1$, by the inequality (3.12)

$$||I_{\sigma,r+1}||^2 \leq C_\Lambda^2 t(1+t^{d_\Lambda})^2 e^{-2l_1 t}||\phi||_{H^2}^{2\alpha}\Big(C'C_2^2 +$$

$$+4C_\Lambda^{2\alpha}C_F^2 C_P C' C_r^{2\alpha}\int_0^{+\infty} e^{2l_1(1-\alpha)\tau}(1+\tau^{d_\Lambda})^{2+2\alpha}\tau^\alpha||\phi||_{H^2}^{2\alpha(\alpha-1)}d\tau\Big).$$

Setting

$$J = \int_0^{+\infty} e^{2l_1(1-\alpha)\tau}(1+\tau^{d_\Lambda})^{2+2\alpha}\tau^\alpha d\tau),$$

we obtain the required estimate (3.10) with

$$C_{r+1}^2 = C_2^2 C' + 4C_\Lambda^{2\alpha}C_F^2 C_P C' J||\phi||_{H^2}^{2\alpha(\alpha-1)}C_r^{2\alpha}$$

We note that $C' > 1$. We choose $\kappa > 0$ such that for $||\phi|| < \kappa$

$$4C_\Lambda^{2\alpha}C_F^2 C_P C' J||\phi||_{H^2}^{2\alpha(\alpha-1)} < \frac{1}{(C_2^2 C'+1)^\alpha}.$$

Let $q_r = C_r^2$. Then $q_2 < C_2^2 C' + 1$. We note that for κ, as above, $q_r < C_2^2 C' + 1$ for all $r \geq 2$. Indeed,

$$q_{r+1} = C_2^2 C' + 4C_\Lambda^{2\alpha}C_F^2 C_P C' J||\phi||_{H^2}^{2\alpha(\alpha-1)}q_r^\alpha < C_2^2 C' + \frac{1}{(C_2^2 C'+1)^\alpha}q_r^\alpha < C_2^2 + 1.$$

Thus, $C_r < \sqrt{C_2^2 C' + 1}$ and the inequality (3.10) with small κ implies (3.9).

From this result it follows

Lemma 10.2. Suppose that $l_1 > 0$ and $||\phi||_{H^2} < \kappa$ in (3.8) with sufficiently small κ. Then the solutions u_k to the system (3.8) converge in $C((0,+\infty);H^2)$.

3.3 Construction of a Nonlinear Chapman Projection

11.1. Weak nonlinearity. We consider the system

$$\begin{aligned}&\partial_t u + \mathcal{A}_{11}\partial_x u + \mathcal{A}_{12}\partial_x v + B_{11}u + B_{12}v = 0,\\ &\partial_t v + \mathcal{A}_{21}\partial_x u + \mathcal{A}_{22}\partial_x v + B_{21}u + B_{22}v = G(u)v\end{aligned} \tag{3.13}$$

with the initial data $u|_{t=0} = \phi_1(x)$, $v|_{t=0} = \phi_2(x)$. We set $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Suppose that $u(x,t)\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^m$ and $v(x,t)\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^{N-m}$. Assume that the data of the problem (3.13) satisfy all the assumptions of Lemma 10.2. We also assume that the following condition is satisfied.

Condition 11.1. The linearized part of the problem (3.1) and the initial data satisfy all the assumptions of Theorem 7.1. Moreover, $l_j = \inf_\xi \min_{\lambda\in\Gamma_j} \operatorname{Re}\lambda$ and $l_1 > 0$, $l_2 - \alpha l_1 < 0$. We denote by P_{21} the symbol of the Chapman-Enskog projection for the linearized problem (3.13). If the initial data ϕ are sufficiently smooth and

$$||\phi||^2_{H^2} + ||P_{21}(\partial_x)\phi||^2_{H^2} < \kappa \ll 1$$

then, according to the method of successive approximations, there exists a solution $\begin{pmatrix} w \\ z \end{pmatrix}$ to the problem (3.13) with the initial data $w|_{t=0} = \Upsilon(\phi_1,\phi_2)$, $z|_{t=0} = P_{21}(\partial_x)\Upsilon(\phi_1,\phi_2)$, where Υ is the operator of the initial data corresponding to the sum of U_{Ch} and U_{Cor} in the linear case. The goal of this section is to construct a nonlinear operator $\mathcal{P}_{21}(w,\partial_x)$ such that $z = \mathcal{P}_{21}(w,\partial_x)w$.

Let $M = S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix}.$$

We write the system (3.13) in terms of the Fourier images and use the fact that P_{21} is the symbol of the Chapman projection for the linearized problem. Then

$$\partial_t\mathcal{F}(w) + M_{11}\mathcal{F}(w) + M_{12}\mathcal{F}(v') = 0,$$

$$\partial_t\mathcal{F}(v') + M_{22}\mathcal{F}(v') = \mathcal{F}(G(w)z),$$

where $z = P_{21}w + v'$. Based on this fact, we look for z in the form

$$z = P_{21}w + \sum_{j=1}^{\infty} v_j, \tag{3.14}$$

where v_j is a solution to the equation

$$\partial_t\mathcal{F}(v_j) + M_{22}\mathcal{F}(v_j) = \mathcal{F}(G(w)v_{j-1}) \tag{3.15}$$

with the initial data $v_j|_{t=0} = 0$ and $v_0 = P_{21}w$. Using the method of variation of constants, we find

$$v_j = \mathcal{F}^{-1}\left(e^{-M_{22}t}\int_0^t e^{M_{22}\tau}\mathcal{F}(G(w(\tau))v_{j-1}(\tau))d\tau\right).$$

This representation shows that $v_j = \Pi_j(w, \partial_x)w$. It remains to prove that for small ϕ the series (3.14) is convergent.

From Lemma 10.1 and the method of successive approximations it follows that

$$||w||_{H^2} \leq C_0 e^{-l_1 t}(1+t^d)(||\phi||_{H^2} + ||P_{21}\phi||_{H^2}).$$

Furthermore, for $P_{21}w$ we have the similar estimate

$$||P_{21}w|| \leq C_1 e^{-l_1 t}(1+t^d)||\phi||_{H^s}$$

with some s. Based on these two inequalities and the embedding theorem, we find the following estimate for v_1:

$$||v_1||^2 \leq t\int_0^t ||e^{M_{22}(\tau-t)}\mathcal{F}(G(w(\tau))P_{21}w(\tau))||^2 d\tau \leq te^{-2l_2 t}\int_0^t e^{2l_2\tau}q_0(t,\tau)|w|_\infty^{2\alpha-2}||P_{21}w(\tau)||^2 d\tau,$$

where $q_0(t,\tau)$ is a polynomial depending only on the structure of M. Further,

$$||v_1||^2 \leq C_2 te^{-2l_2 t}\int_0^t e^{2(l_2-\alpha l_1)\tau}q_0(t,\tau)(1+\tau^d)^{2\alpha}(||\phi||_{H^2} + ||P_{21}\phi||_{H^2})^{2\alpha-2}||\phi||_{H^s}^2 d\tau.$$

Hence, under the above conditions on the system (3.13), there are constants $d_1 \geq 0$ and $K_1 > 0$ such that

$$||v_1||^2 \leq K_1 t(1+t^{d_1})e^{-2l_2 t}||\phi||_{H^s}^{2\alpha}. \tag{3.16}$$

Moreover, d_1 depends only on the structure of the matrix M and K_1 is independent of ϕ.

Similarly, for v_2 we find

$$||v_2||^2 \leq t\int_0^t ||e^{M_{22}(\tau-t)}\mathcal{F}(G(w(\tau))v_1(\tau))||^2 d\tau \leq$$

$$\leq C_3 te^{-2l_2 t}\int_0^t e^{2l_2\tau}q_0(t,\tau)(1+\tau^d)^{2\alpha-2}e^{-(2\alpha-2)l_1\tau}||v_1(\tau)||^2(||\phi||_{H^2} + ||P_{21}\phi||_{H^2})^{2\alpha-2}d\tau.$$

Using the above estimate for v_1, we find

$$||v_2||^2 \leq K_2 te^{-2l_2 t}(1+t^{d_1}))||\phi||_{H^s}^{4\alpha-2},$$

where the constant K_2 is independent of ϕ, because $\int_0^t \tau^r e^{-\gamma\tau}d\tau \leq \int_0^\infty \tau^r e^{-\gamma\tau}d\tau = \text{const}$.

Arguing in the same way, it is easy to obtain the inequality

$$||v_j||^2 \leq K_j t(1+t^{d_1})e^{-2l_2 t}||\phi||_{H^s}^{2j\alpha-2j+2},$$

where the constants K_j are independent of ϕ and $K_j \leq K_0^j$, $K_0 = \text{const}$. Hence for sufficiently small ϕ the series (3.14) is convergent.

We note that the smallness of the norm of the initial data in some space H^s and the estimate

$$|G(w)|_\infty \leq C_G|w|_\infty^{\alpha-1}(1+|w|_\infty^\beta)$$

imply

$$|G(w)|_\infty \leq C'_G|w|_\infty^{\alpha-1},$$

where $C'_G > C_G$. Furthermore, d_1 is the degree of the polynomial $q_0(t,\tau)$ in the variable t. Since q_0 is a polynomial, there exist constants I_1 and I_2 such that

$$\int_0^{+\infty} e^{2(l_2-\alpha l_1)\tau} q_0(t,\tau)(1+\tau^d)^{2\alpha} d\tau \leq I_1(1+t^{d_1}),$$

and

$$\int_0^{+\infty} e^{-(2\alpha-2)l_1\tau} q_0(t,\tau)(1+\tau^d)^{2\alpha-2}(1+\tau^{d_1})\tau d\tau \leq I_2(1+t^{d_1}).$$

Indeed, both integrals on the left-hand sides of these inequalities are polynomials in t of degree d_1, which implies the required estimates.

To prove the assertions concerning the constants K_j , we need the following lemma.

Lemma 11.1. Assume that all the assumptions of Lemma 10.2 and Condition 11.1 are satisfied. Then the solution v_j to the problem (3.15) with the initial data $v_j|_{t=0} = 0$ for $j \geq 1$ satisfies the inequality

$$||v_j||^2 \leq K_j t(1+t_1^d)e^{-2l_2 t}||\phi||_{H^s}^{2j\alpha-2j+2},$$

where $K_j \leq (C'_G)^j C_W^{2j\alpha-2j+2} I_2^{j-1} I_1$ and $C_W = \max\{C_0, C_1\}$.

Proof. We use the method of mathematical induction. As was already shown, the required estimate is valid for $||v_1||^2$. Furthermore, it is easy to see that

$$K_1(1+t^{d_1}) = C_2 \int_0^{+\infty} e^{2(l_2-\alpha l_1)\tau} q_0(t,\tau)(1+\tau^d)^{2\alpha} d\tau \leq C_2 I_1(1+t^{d_1}),$$

where $C_2 = C'_G C_W^{2\alpha}$. Thus, the corresponding inequality for K_1 also holds.

Assume that the assertion holds for $j \leq k$. Then for $j = k+1$ we have

$$||v_{k+1}||^2 \leq t \int_0^t ||e^{M_{22}(\tau-t)} \mathcal{F}(G(w(\tau))v_k(\tau))||^2 d\tau \leq$$

$$\leq te^{-2l_2 t} \int_0^t e^{2l_2\tau} q_0(t,\tau)|w(\tau)|_\infty^{2\alpha-2}||v_k(\tau)||^2 d\tau \leq K_{k+1}t(1+t^{d_1})e^{-2l_2 t}||\phi||_{H^s}^{2(k+1)\alpha-2k},$$

where $C_3 = C'_G C_W^{2\alpha-2}$, and from the inequality

$$\int_0^t q_0(t,\tau)(1+\tau^d)^{2\alpha-2} e^{-(2\alpha-2)l_1\tau}(1+\tau^{d_1})\tau d\tau \leq$$

$$\leq \int_0^{+\infty} q_0(t,\tau)(1+\tau^d)^{2\alpha-2} e^{-(2\alpha-2)l_1\tau}(1+\tau^{d_1})\tau d\tau \leq I_2(1+t^{d_1})$$

we obtain the required estimate for K_{k+1}.

11.2. General case. We consider the system

$$\partial_t u + \mathcal{A}_{11}\partial_x u + \mathcal{A}_{12}\partial_x v + B_{11}u + B_{12}v = G_{11}(u)u,$$
$$\partial_t v + \mathcal{A}_{21}\partial_x u + \mathcal{A}_{22}\partial_x v + B_{21}u + B_{22}v = G_{21}(u)u + G_{22}(u)v \tag{3.17}$$

with the same initial data, as above. Then we construct a nonlinear operator $\mathcal{P}_{21}(\partial_x, w)$ that determines the solution $\begin{pmatrix} w \\ z \end{pmatrix}$. We look for z in the form (3.14), where v_j are solutions to the problem

$$\partial_t \mathcal{F}(v_1) + M_{22}\mathcal{F}(v_1) = \mathcal{F}(P_{21}(G_{11}(w)w) + G_{21}(w)w + G_{22}(w)P_{21}w),\ \mathcal{F}(v_1)|_{t=0} = 0,$$

$$\partial_t \mathcal{F}(v_j) + M_{22}\mathcal{F}(v_j) = \mathcal{F}(G_{22}(w)v_{j-1}),\ \mathcal{F}(v_j)|_{t=0} = 0,\ j \geq 2.$$

Then we can estimate $||v_1||$ as follows:

$$||v_1||^2 \leq t\int_0^t C_M^2(1+(t-\tau)^{d_M})^2 e^{2l_2(\tau-t)}||P_{21}(G_{11}(w)w) + G_{21}(w)w + G_{22}(w)P_{21}w||^2(\tau)d\tau \leq$$

$$\leq \mathrm{const}\, t(1+t^{d_M})^2 e^{-2l_2 t}||\phi||_{H^2}^{2\alpha}\int_0^{+\infty} e^{2(l_2-\alpha l_1)\tau}(1+\tau^{d_M})^2(1+\tau^{d_\Lambda})^{2\alpha}(1+\sqrt{\tau})^{2\alpha}d\tau =$$

$$= K_1' t(1+t^{d_M})^2 e^{-2l_2 t}||\phi||_{H^2}^{2\alpha}.$$

Thus, for v_1 we have an estimate of the form (3.16). We note that the equation for $v_j,\ j>1$, is the same as in the previous subsection. Furthermore, for estimating from above $v_j,\ j>1$, we used the estimate (3.16), but not an explicit form of v_1. Consequently, Lemma 11.1 remains valid. Therefore, the series (3.14) converges in the L_2-norm for small initial data, which means the existence of a nonlinear projection $\mathcal{P}_{21}$.

3.4 Properties of Nonlinear Projections

We study properties of the nonlinear operator $\mathcal{P}_{21}$ constructed in the previous section.

Lemma 12.1. Let the data of the problem (3.17) satisfy all the assumptions of Lemma 10.2 and Condition 11.1. Assume that $\phi \in H^3$, $|P_{21}|_0 \leq \mathrm{const}(1+|\xi|^s)$, $s \leq 2$. Then for every term v_j of the series (3.14) the following inequality holds:

$$||v_j||_{H^1}^2 \leq K_0^j t(1+t^{d_M})^2 e^{-2l_2 t}||\phi||_{H^3}^{2j\alpha-2j+2}. \tag{3.18}$$

Proof. We estimate each $||\partial_k v_j||$. For this purpose, we note that $||\partial_k v_j||$ satisfies the problem

$$\partial_t \mathcal{F}(\partial_k v_j) + M_{22}\mathcal{F}(\partial_k v_j) = \mathcal{F}(\partial_k(F_j(w, v_{j-1}))),\quad \mathcal{F}(\partial_k v_j)|_{t=0} = 0,$$

where

$$F_1(w, v_0) = F_1(w) = -P_{21}(G_{11}(w)w) + G_{12}(w)w + G_{22}(w)P_{21}w,\ F_j(w, v_{j-1}) = G_{22}(w)v_{j-1},\ j \geq 2.$$

Using Lemmas 9.1 and 9.5, we find

$$||\partial_k v_1||^2 \leq \mathrm{const}\, t(1+t^{d_M})^2 e^{-2l_2 t}\int_0^{+\infty} e^{2l_2\tau}(1+\tau^{d_M})^2||F_1(w)||_{H^1}^2 d\tau,$$

Using Lemma 9.2 and the inequality $|P_{21}| \leq \mathrm{const}(1+|\xi|^s)$, $s \leq 2$ we finally find

$$||\partial_k v_1||^2 \leq \mathrm{const}\, t(1+t^{d_M})^2 e^{-2l_2 t}||\phi||_{H^3}^{2\alpha}.$$

Futher

$$||\partial_k v_j||^2 \leq C_M t(1+t^{d_M})^2 e^{-2l_2 t} \int_0^{+\infty} e^{2l_2\tau}(1+\tau^{d_M})^2 ||\partial_k(G_{22}(w)v_{j-1})||^2 d\tau,$$

Taking into account that G_{22} is a matrix polynomial and arguing as in Lemma 11.1, we obtain the required estimates.

For the sake of brevity, we introduce the notation $L_1 = \sup_\xi \max_{\lambda\in\Gamma_1} \operatorname{Re}\lambda$.

Theorem 12.1. Let $\begin{pmatrix} w \\ z \end{pmatrix}$ be a solution to the system (3.17) with the initial data $w|_{t=0} = \phi_1$, $z|_{t=0} = P_{21}\phi_1$. Let $\phi \in H^3$, and let all the assumptions of Lemma 12.1 be satisfied. Denote by $\begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$ the solution to the linearized problem (3.17) with the same initial data. If $\alpha > \frac{L_1}{l_1}$, $||\phi||_{H^3} < \kappa \ll 1$, then the following estimate holds:

$$||e^{M_{11}t}\mathcal{F}(w-w_0)||^2 \leq \frac{1}{\gamma}\operatorname{const}(||\phi||^{2\alpha}_{H^2} + ||\phi||^{\alpha+1}_{H^3}),$$

where

$$0 < \gamma < \min\{\frac{l_2 - L_1}{2}, 2\alpha l_1 - 2L_1\}.$$

Proof. We note that the Fourier images satisfy the equality

$$\partial_t\mathcal{F}(w) + M_{11}\mathcal{F}(w) + M_{12}\mathcal{F}(z - P_{21}w) = \mathcal{F}(G_{11}(w)w).$$

Hence

$$\mathcal{F}(w) = e^{-M_{11}t}\left(\mathcal{F}(\phi_1) + \int_0^t e^{M_{11}\tau}(\mathcal{F}(G_{11}(w)w) - M_{12}\mathcal{F}(z-P_{21}w))d\tau\right).$$

Thus,

$$||e^{M_{11}t}(w-w_0)|| \leq ||\int_0^t e^{M_{11}\tau}\mathcal{F}(G_{11}(w)w)d\tau|| + ||\int_0^t e^{M_{11}\tau}M_{12}\mathcal{F}(z-P_{21}w)d\tau||.$$

Further,

$$||\int_0^t e^{M_{11}\tau}\mathcal{F}(G_{11}(w)w)d\tau||^2 \leq \frac{\operatorname{const}}{\gamma}\int_0^{+\infty} e^{(2L_1+\gamma)\tau}(1+\tau^d)^2 ||w||^{2\alpha}_{H^2} d\tau.$$

Using Lemma 10.1, we find

$$||\int_0^t e^{M_{11}\tau}\mathcal{F}(G_{11}(w)w)d\tau||^2 \leq \frac{\operatorname{const}}{\gamma}\int_0^{+\infty} e^{(2L_1+\gamma-2\alpha l_1)\tau}(1+\tau^d)^2(1+\tau^{d_\Lambda})^{2\alpha}(||\phi||_{H^2}+||\phi||^{\alpha}_{H^2}\sqrt{\tau})^{2\alpha}d\tau,$$

By the conditions on α, γ, and ϕ, it follows that $||\int_0^t e^{M_{11}\tau}\mathcal{F}(G_{11}(w)w)d\tau||^2 \leq \frac{\operatorname{const}}{\gamma}||\phi||^{2\alpha}_{H^2}$. Estimating the second term, we find

$$||\int_0^t e^{M_{11}\tau}M_{12}\mathcal{F}(z-P_{21}w)d\tau||^2 \leq \int_0^{+\infty} e^{-\gamma\tau}d\tau \int_0^{+\infty} e^{(2L_1+\gamma)\tau}(1+\tau^d)^2||M_{12}\mathcal{F}(z-P_{21}w)||^2 d\tau.$$

We note that $M_{12} = \Lambda_{12}$. Thus, $||M_{12}\mathcal{F}(z - P_{21}w)|| = ||z - P_{21}w||_{H^1}$. Using Lemma 14.1 and taking ϕ with sufficiently small H^3-norm, we find

$$|| \int_0^t e^{M_{11}\tau} M_{12}\mathcal{F}(z - P_{21}w)d\tau||^2 \leq \frac{\text{const}}{\gamma}||\phi||_{H^3}^{\alpha+1}.$$

which implies the required assertion.

Remark 12.1. Applications of the obtained results to models of continuum mechanics can be found in [11, 13, 14, 16] Acknowledgments.

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