CUBO A Mathematical Journal
Vol.12, Nº01, (59–66). March 2010

Weakly Picard Pairs of Multifunctions

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ABSTRACT

The purpose of this paper is to present a general answer for the following problem:
Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multifunctions. Determine the metric conditions which imply that (T_1, T_2) is a weakly Picard pair of multifunctions and T_1, T_2 are weakly Picard multifunctions , for multifunctions satisfying an implicit contractive condition, generalizing some results from [6] and [7].

RESUMEN

El proposito de este artículo es presentar una respuesta general para el siguiente problema: Sea (X, d) un espacio métrico y $T_1, T_2 : X \to P(X)$ dos multifunciones. Determine los condiciones metricas para las cuales (T_1, T_2) sea un par de multifunciones de Picard debil y T_1, T_2 sean multifunciones satisfaziendo una condición contractiva implícita, generalizando algunos resultados de [6] y [7].

Key words and phrases: *Multifunction, fixed point, implicit relation, weakly Picard multifunction, weakly Picard pair of multifunctions.*

Math. Subj. Class.: *47H10, 54H25.*

1 Introduction and Preliminaries

Let X be a nonempty set. We denote P(X) the set of all nonempty subsets of X, i.e. $P(X) = \{Y : \Phi \neq Y \subset X\}$. Let $T : X \to P(X)$ a multifunction. We denote by F_T the set of fixed points of T, i.e. $F_T = \{x \in X : x \in T(x)\}$.
Let (X,d) be a metric space. We denote by Pcl(X) the set of all nonempty and closed sets of X.
We also recall the functional
$D : P(X) \times P(X) \to R_+$, defined by
$D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$ for each $A, B \in P(X)$ and generalized Hansdorff-Pompeiu metric
$H : P(X) \times P(X) \to R_+ \cup \{+\infty\}$ defined by
$H(A, B) = max\,\{sup[D(a, B), a \in A], sup[D(A, b), b \in B]\}$ for $A, B \in P(X)$.
The following property of H is well-known.

Lemma 1.1. Let (X,d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

Definition 1.1. Let (X,d) be a metric space and $T : (X, d) \to P(X)$ a multifunction. We say that T is a weakly Picard multifunction [3],[4] if for each $x \in X$ and for every $y \in T(x)$, there exists a sequence $(x_n)_{n\in N}$ such that:
(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for each $n \in N^*$;
(iii) the sequence $(x_n)_{n\in N}$ is convergent and its limit is a fixed point of T.

Definition 1.2. Let (X,d) be a metric space and $T_1, T_2 : X \to P(X)$ two multifunctions. We say that (T_1, T_2) is a weakly Picard pair of multifunctions if for each $x \in X$ and for every $y \in T_1(x) \cup T_2(x)$, there exists a sequence $(x_n)_{n\in N}$ such that
(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{2n+1} \in T_i(x_{2n})$ and $x_{2n+2} \in T_j(x_{2n+1})$, for $n \in N$, where $i, j \in \{1, 2\}$, $i \neq j$;
(iii) the sequence $(x_n)_{n\in N}$ is convergent and its limit is a common fixed point of T_1 and T_2.

Problem 1.1 [4]. Let (X,d) be a metric space and $T_1, T_2 : (X, d) \to P(X)$ two multifunctions. Determine the metric conditions which implies (T_1, T_2) is a weakly Picard pair of multifunctions and T_1, T_2 are weakly Picard multifunctions.
Partial answers to Problem 1.1. was established by Sintămărian in [4]-[7].
In [1] and [2] is introduced the study of fixed point for mappings satisfying implicit relations.
The purpose of this paper is to prove two general fixed points theorems for multifunctions which satisfy a new type of implicit contractive relation which generalize some results from [6], [7].

2 Implicit Relations

Let $\mathcal{F}$ be the set of all continuous multifunctions $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:
(F_1): F is increasing in variable t_1 and nonincreasing in variables $t_3, ..., t_6$;
(F_2): there exists $k > 1$, $h \in [0, 1)$ and $g \geq 0$ such that for every $u \geq 0$, $v \geq 0$, $w \geq 0$, such that
(F_a): $u \leq kt$ and $F(t, v, v + w, u + w, u + v + w, w) \leq 0$, or
(F_b): $u \leq kt$ and $F(t, v, u + w, v + w, w, u + v + w) \leq 0$ implies $u \leq hv + gw$.

Example 2.1. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, when $0 < a + 2b + 2c < 1$.
(F_1): Obviously.
(F_2): $F(t, v, v + w, u + w, u + v + w, w) = t - av - b(u + v + 2w) - c(u + v + 2w) \leq 0$, where $1 < k < \frac{1}{a+2b+2c}$.
Then $u \leq kt \leq k[av+b(u+v+2w)+c(u+v+2w)]$. Hence $u \leq hv+gw$, where $0 < h = \frac{k(a+b+c)}{1-k(b+c)} < 0$ and $g = \frac{2k(b+c)}{1-k(b+c)} \geq 0$
Similarly, $F(t, v, u + w, w + v, w, u + v + w \leq 0$ implies $u \leq hv + gw$.

Remark 2.1. If $a + 4b + 4c < 1$ and $1 < k < \frac{1}{a+4b+4c}$ then $h + g < 1$.

Example 2.2. $F(t_1, ..., t_6) = t_1 - mmax\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$ where $0 < m < \frac{1}{2}$.
(F_1): Obviously.
(F_2): Let $u \geq 0, v \geq 0, w \geq 0$, $1 < k < \frac{1}{2m}$ and
$F(t, v, v + w, u + w, u + v + w, w) = t - mmax\{v, u + w, u + v, \frac{1}{2}(u + v + 2w)\} \leq 0$
which implies $t \leq m(u + v + w)$.
Then $u \leq kt \leq km(u + v + w)$. Hence, $u \leq hv + gw$ where $0 < h = \frac{km}{1-km} < 1$ and $g = \frac{km}{1-km} \geq 0$.
Similarly, $F(t, v, u + w, v + w, w, u + v + w) \leq 0$ implies $u \leq hv + gw$.

Remark 2.2. If $0 < m < \frac{1}{3}$ and $1 < k < \frac{1}{3m}$ then $h + g < 1$.

Example 2.3. $F(t_1, ..., t_6) = t_1^2 - mmax\{t_2^2, t_3t_4, t_5t_6\}$, where $0 \leq m < \frac{1}{4}$.
(F_1): Obviously.
(F_2): Let $u \geq 0, v \geq 0, w \geq 0$, $1 < k < \frac{1}{2\sqrt{m}}$ and
$F(t, v, v + w, u + w, u + v + w, w) = t^2 - mmax\{v^2, (v + w)(u + w), w(u + v + w)) \leq 0$
which implies $t^2 \leq m(u+v+w)^2$ and $t \leq \sqrt{m}(u+v+w)$. Then $u \leq kt \leq k\sqrt{m}(u+v+w)$. Hence, $u \leq hv + gw$, where $0 \leq h \leq \frac{k\sqrt{m}}{1=k\sqrt{m}} < 1$ and $g = \frac{k\sqrt{m}}{1=k\sqrt{m}} \geq 0$.
Similarly, $F(t, v, u + w, v + w, w, u + v + w) \leq 0$ implies $u \leq hv + gw$.

Remark 2.3. If $o \leq m < \frac{1}{9}$ and $1 < k < \frac{1}{3\sqrt{m}}$ then $h + g < 1$.

Example 2.4. $F(t_1, ..., t_6) = t_1^3 + t_2^2 + \frac{1}{1+t_5+t_6} - m(t_2^2 + t_3^2 + t_4^2)$, where $0 < m < \frac{1}{12}$.
(F_1): Obviously.
(F_2): Let $u \geq 0, v \geq 0, w \geq 0$ and $1 < k < \frac{1}{2\sqrt{m}}$ and
$F(t, v, v + w, u + w, u + v + w, w) = t^3 + t^2 + \frac{t}{1+u+v+w} - m(v^2 + (v + w)^2 + (u + w)^2) \leq 0$
which implies

$t^2 \le m(v^2+(u+v)^2+(u+w)^2) \le 3m(u+v+w)^2$ and $t \le \sqrt{3m}(u+v+w)$. If $u \le kt \le k\sqrt{3}m(u+v+w)$ then $u \le hv+gw$, where $0 \le h = \frac{k\sqrt{3}m}{1-k\sqrt{3}m} < 1$ and $g = \frac{k\sqrt{3}m}{1-k\sqrt{3}m} \ge 0$.
Similarly, $F(t,v,u+w,v+w,w,u+v+w) \le 0$ implies $u \le hv+gw$.

Remark 2.4. If $0 < m < \frac{1}{27}$ and $1 < k < \frac{1}{3\sqrt{3m}}$ then $h+g<1$.

3 Main Results

Theorem 3.1. Let $T_1, T_2 : (X,d) \to Pcl(X)$ be two multifunctions. If the inequality
(1) $\Phi(H(T_1(x),T_2(y)), d(x,y), D(x,T_1(x)), d(y,T_2(y)), D(x,T_2(y)), D(y,T_1(x)) \le 0$
holds for all $x,y \in X$, where $F \in \mathcal{F}$ and $F_{T_1} \ne \Phi$ or $F_{T_2} \ne \Phi$, then $F_{T_1} = F_{T_2}$.

Proof. Let $u \in F_{T_1}$, then $u \in T_1(u)$ and by (1) we have
$\Phi(H(T_1(u),T_2(u)), d(u,u), d(u,T_1(u)), D(u,T_2(u)), D(u,T_2(u)), D(u,T_1(u)) \le 0$
By $D(u,T_2(u)) \le H(T_1(u),T_2(u))$ it follows that
$\Phi(D(u,T_2(u)),0,0,D(u,T_2(u)),D(u,T_2(u)),0) \le 0$
Since $D(u,T_2(u)) \le kD(u,T_2(u))$ by (F_a) we have that $D(u,T_2(u))=0$. Since $T_2(u)$ is closed we obtain $u \in T_2(u)$ i.e. $u \in F_{T_2}$ and $F_{T_1} \subset F_{T_2}$. Similarly, by (F_b) we obtain $F_{T_2} \subset F_{T_1}$. Similarly, if $u \in F_{T_2}$, then $F_{T_1} = F_{T_2}$.

Theorem 3.2. Let (X,d) be a complete metric space and $T_1, T_2 : (X,d) \to Pcl(X)$ two multifunctions. If (1) holds for all $x,y \in X$, where $F \in \mathcal{F}$, then T_1 and T_2 have a common fixed point and $F_{T_1} = F_{T_2} \in Pcl(X)$.

Proof. Let $x_0 \in X$ and $x_1 \in T_1(x_0)$. Then there exists $x_2 \in T_2(x_1)$ so that

$$d(x_1,x_2) \le kH(T_1(x_0),T_2(x_1))$$

Suppose that $x_2, x_3, ..., x_{2n-1}, x_{2n}, ...$ such that $x_{2n-1} \in T_1 x_{2n-2}$, $x_{2n} \in T_2 x_{2n-1}, n \in N^*$ and
(2) $d(x_{2n-1},x_{2n}) \le kH(T_1(x_{2n-2}),T_2(x_{2n-1}))$,
(3) $d(x_{2n-2},x_{2n-1}) \le kH(T_1(x_{2n-2}),T_2(x_{2n-3}))$.
By (1) we have successively
$\Phi(H(T_1(x_{2n-2}),T_2(x_{2n-1})), d(x_{2n-2},x_{2n-1}), D(x_{2n-2},T_1(x_{2n-2})),$
$D(x_{2n-1},T_2(x_{2n-1})), D(x_{2n-2},T_2(x_{2n-1})), D(x_{2n-1},T_1(x_{2n-2})) \le 0$
$\Phi(H(T_1(x_{2n-2}),T_2(x_{2n-1})), d(x_{2n-2},x_{2n-1}), d(x_{2n-1},x_{2n-2}),$
$d(x_{2n-1},x_{2n}), d(x_{2n-2},x_{2n}),0) \le 0$
(4) $\Phi(H(T_1(x_{2n-2}),T_2(x_{2n-1})), d(x_{2n-2},x_{2n-1}), d(x_{2n-2},x_{2n-1}),$
$d(x_{2n-1},x_{2n}), d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n}),0) \le 0$
Since $\Phi \in \mathcal{F}$ then by (2),(4) and (F_a) we obtain
(5) $d(x_{2n-1},x_{2n}) \le hd(x_{2n-2},x_{2n-1})$
Similarly, by (3) and (F_b) we obtain
(6) $d(x_{2n-2},x_{2n-1}) \le hd(x_{2n-2},x_{2n-3})$

Then by a rutine calculation one can show that $(x_n)_{n\in N}$ is a Cauchy sequence and since (X,d) is complete we have $lim x_n = x$ for some $x \in X$.
Now, if $n \in N^*$, (1) implies
$\Phi(H(T_1(x), T_2(x_{2n-1})), d(x, x_{2n-1}), D(x, T_1(x)), D(x_{2n-1}, T_2(x_{2n-1})), D(x, T_2(x_{2n-1})),$
$D(x_{2n-1}, T_1 x) \leq 0$
As $D(x_{2n}, T_1(x)) \leq H(T_2(x_{2n-1}), T_1(x))$ we have
$\Phi(D(x_{2n}, T_1(x)), d(x, x_{2n-1}), D(x, T_1(x)), d(x_{2n-1}, x_{2n}), d(x, x_{2n}), D(x_{2n-1}, T_1(x)) \leq 0$
Letting n tend to infinity we obtain
$\Phi(D(x, T_1(x)), 0, D(x, T_1(x)), 0, 0, D(x, T_1(x)) \leq 0$
Since $D(x, T_1(x)) \leq kD(x, T_1(x))$ by (F_b) we obtain $D(x, T_1(x)) = 0$. Since $T_1(x)$ is closed, $x \in T_1(x)$. Hence $x \in F_{T_1}$. By Theorem 3.1 $F_{T_1} = F_{T_2}$.
Let us prove that $F_{T_1} = F_{T_2} \in Pcl(X)$. For this purpose that $y_n \in F_{T_1} = F_{T_2}$ for each $n \in N$ such that $y_n \to y^*$ as $n \to \infty$. For example $y_n \in T_1(y_n)$.
Then by Lemma 1.1 there exists $v_n \in T_2 y^*$ such that
(7) $d(y_n, v_n) \leq kH(T_1(y_n), T_2(y^*))$.
By (1) and (F_1) we have successively
$\Phi(H(T_1(y_n), T_2(y^*)), d(y_n, y^*), D(y_n, T_1(y_n)), D(y^*, T_2(y^*)),$
$D(y_n, T_2(y^*)), D(y^*, T_1(y_n)) \leq 0$
$\Phi(H(T_1(y_n), T_2(y^*)), d(y_n, y^*), 0, d(y^*, v_n), d(y_n, v_n), d(y^*, y_n)) \leq 0$
(8) $\Phi(H(T_1(y_n), T_2(y^*)), d(y_n, y^*), d(y_n, y^*) + d(y_n, y^*), d(y^*, y_n) + d(y_n, v_n), d(y_n, v_n) + d(y_n, y^*) + d(y_n, y^*), d(y_n, y^*)) \leq 0$
Since $\Phi \in \mathcal{F}$ by (7) and (8) it follows that

$$d(y_n, v_n) \leq hd(y_n, y^*) + gd(y^*, y_n)$$

Using the triangle inequality we obtain
$d(y^*, v_n) \leq d(y^*, y_n) + d(y_n, v_n) \leq (1 + h + g)d(y^*, y_n)$
Letting n tend to infinity we obtain that $lim v_n = y^*$. Since $v_n \in T_2(y^*)$, for each $n \in N^*$ and $T_2(y^*)$is closed, it follows that $y^* \in T_2(y^*)$, hence $y^* \in F_{T_2} = F_{T_1}$ and F_{T_1} is closed.
Therefore, $F_{T_1} = F_{T_2} \in Pcl(X)$.

Theorem 3.3. Let (X,d) be a complete metric space and $T_1, T_2 : (X, d) \to Pcl(X)$. If (1) holds for all $x, y \in X$, where $\Phi \in \mathcal{F}$, then $F_{T_1} = F_{T_2} \in Pcl(X)$ and (T_1, T_2) is a weakly Picard pair of multifunctions. If in adition we have that $h + g < 1$, then T_1 and T_2 are weakly Picard multifunctions.

Proof. The first part it follows from Theorem 3.2.
Let $x_0 \in X$ and $x_1 \in T_1(x_0)$. There exists $y_1 \in T_2(x_1)$ such that
(9) $d(x_1, y_1) \leq kH(T_1(x_0), T_2(x_1))$
By (1) and (F_1) we have successively
$\Phi(H(T_1(x_0), T_2(x_1)), d(x_0, x_1), D(x_0, T_1(x_0)), D(x_1, T_2(x_1)), D(x_0, T_2(x_1)), D(x_1, T_1(x_0)) \leq 0$
$\Phi(H(T_1(x_0), T_2(x_1)), d(x_0, x_1), d(x_0, x_1), d(x_1, y_1), d(x_0, y_1), 0) \leq 0$

(10) $\Phi(H(T_1(x_0),T_2(x_1)),d(x_0,x_1),d(x_0,x_1),d(x_1,y_1),d(x_0,x_1)+d(x_1,y_1),0)\leq 0$
Since $\Phi\in\mathcal{F}$ by (9) and (10) it follows that

$$d(x_1,y_1)\leq hd(x_0,x_1)$$

Also, there exists $x_2\in T_1(x_1)$ such that
(11) $d(x_2,y_1)\leq kH(T_1(x_1),T_2(x_1))$
By (1) we have successively
$\Phi(H(T_1(x_1),T_2(x_1)),0,D(x_1,T_1(x_1)),D(x_1,T_2(x_1)),D(x_1,T_2(x_1)),D(x_1,T_1(x_1))\leq 0$
$\Phi(H(T_1(x_1),T_2(x_1)),0,d(x_1,x_2),d(x_1,y_1),d(x_1,y_1),d(x_1,x_2))\leq 0$
(12) $\Phi(H(T_1(x_1),T_2(x_1)),0,d(x_1,x_2),d(x_1,x_2)+d(x_2,y_1),d(x_1,x_2)+d(x_2,y_1),d(x_1,x_2))\leq 0$
Since $\Phi\in\mathcal{F}$ by (11) and (12) it follows that

$$d(y_1,x_2)\leq gd(x_1,x_2)$$

Using the triangle inequality we have
$d(x_1,x_2)\leq d(x_1,y_1)+d(y_1,x_2)\leq hd(x_0,x_1)+gd(x_1,x_2)$
which implies that

$$d(x_1,x_2)\leq \tfrac{h}{1-g}d(x_0,x_1)$$

Now, there exists $y_2\in T_2(x_2)$ such that
(13) $d(x_2,y_2)\leq kH(T_1(x_1),T_2(x_2))$
By (1) we have successively
$\Phi(H(T_1(x_1),T_2(x_2)),d(x_1,x_2),D(x_1,T_1(x_1)),D(x_2,T_2(x_2)),D(x_1,T_2(x_2)),D(x_2,T_1(x_1))\leq 0$
$\Phi(H(T_1(x_1),T_2(x_2)),d(x_1,x_2),d(x_1,x_2),d(x_2,y_2),d(x_1,y_2),0)\leq 0$
(14) $\Phi(H(T_1(x_1),T_2(x_2)),d(x_1,x_2),d(x_1,x_2),d(x_2,y_2),d(x_1,x_2)+d(x_2,y_2),0)\leq 0$
Since $\Phi\in\mathcal{F}$ by (13) and (14) it follows that

$$d(x_2,y_2)\leq hd(x_1,x_2)$$

Also, there exists $x_3\in T_1(x_2)$ such that
(15) $d(x_3,y_2)\leq kH(T_1(x_2),T_2(x_2))$
By (1) we have successively
$\Phi(H(T_1(x_2),T_2(x_2)),0,D(x_2,T_1(x_2)),D(x_2,T_2(x_2)),D(x_2,T_2(x_2)),D(x_2,T_1(x_2))\leq 0$
$\Phi(H(T_1(x_2),T_2(x_2)),0,d(x_2,x_3),d(x_2,y_2),d(x_2,y_2),d(x_2,x_3))\leq 0$
(16) $\Phi(H(T_1(x_2),T_2(x_2)),0,d(x_2,x_3),d(x_2,x_3)+d(x_3y_2),d(x_2,x_3)+d(x_3,y_2),d(x_2,x_3))\leq 0$
Since $\Phi\in\mathcal{F}$ by (15) and (16) it follows that

$$d(x_3,y_2)\leq gd(x_2,x_3)$$

Using again the triangle inequality we obtain
$d(x_2,x_3)\leq d(x_2,y_2)+d(y_2,x_3)\leq hd(x_1,x_2)+gd(x_2,x_3)$
and so

$$d(x_2, x_3) \leq \frac{h}{1-g} d(x_1, x_2)$$

By induction we obtain that there exists a sequence $(x_n)_{n\in N}$ starting from x_0, x_1 with $x_{n+1} \in T_1(x_n)$ such that

$$d(x_n, x_{n+1}) \leq \frac{h}{1-g} d(x_{n-1}, x_n)$$

for each $n \in N^*$. Since $\frac{h}{1-g} < 1$ it follows that $(x_n)_{n\in N}$ is a convergent sequence, because (X,d) is a complete metric space. Let $x^* = lim x_n$.
By (1) we have
$\Phi(T_1(x_n), T_2(x^*)), d(x^*, x_n), D(x_n, T_1(x_n)), D(x^*, T_2(x^*)), D(x_n, T_2(x^*)), D(x^*, T_1(x_n))) \leq 0$
Since $D(x_{n+1}, T_2(x^*)) \leq H(T_1(x_n), T_2(x^*)$ we obtain
$\Phi(D(x_{2n+1}), T_2(x^*)), d(x^*, x_n), d(x_n, x_{n+1}), D(x^*, T_2(x^*)), D(x_n, T_2(x^*)), D(x^*, x_{n+1}))) \leq 0$
Letting n tend to infinity we obtain
$\Phi(D(x^*, T_2(x^*)), 0, 0, D(x^*, T_2(x^*)), D(x^*, T_2(x^*)), 0) \leq 0$
Since $D(x^*, T_2(x^*)) \leq kD(x^*, T_2(x^*))$ and $\Phi \in \mathcal{F}$ we obtain $D(x^*, T_2(x^*)) = 0$ and since $T_2(x^*)$ is closed we have that $x^* \in T_2(x^*)$ and $x^* \in F_{T_2} = F_{T_1}$.
Hence T_1 is a weakly Picard multifunction. The fact that T_2 is a weakly Picard multifunction is similar proved.

Remark 3.1. By Theorems 2 and 3 and Ex. 2.1 we obtain generalizations of the results from Theorem 2.1 [6] and Theorem 2.1 [7].
By Ex. 2.2 -2.4 we obtain new results.

Received: May, 2008. Revised: October, 2009.

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