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A Short Note On M-Symmetric Hyperelliptic Riemann Surfaces*

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ABSTRACT

We provide an argument, based on Schottky groups, of a result due to B. Maskit which states a necessary and sufficient condition for the double oriented cover of a planar compact Klein surface of algebraic genus at least two to be a hyperelliptic Riemann surface.

RESUMEN

Damos un argumento, basado en grupos de Schottky, de un resultado debido a B. Maskit el cual establece una condición necesária y suficiente para el cubrimiento duplo orientado de una superficie de Klein compacta planar de genero algebrico al menos dos ser una superficie de Riemann hipereliptica.

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1 Preliminaries

Let us consider a collection of $(g+1)$ pairwise disjoint round circles on the Riemann sphere, say C_1,..., C_{g+1}, bounding a common domain $\mathcal{D}$ of connectivity $(g+1)$. If we denote by τ_j the reflection on the circle C_j, then the group $G = \langle \tau_1, ..., \tau_{g+1} \rangle$ is an extended Kleinian group, isomorphic to the free product of $(g+1)$ copies of $\mathbb{Z}_2$. We say that G is a *planar extended Schottky group of rank* g. The region of discontinuity Ω of a planar extended Schottky group of rank g is connected (the complement of a Cantor set for $g \geq 2$) and $S = \Omega/G$ is a planar compact Klein surface of algebraic genus g, that is, holomorphically equivalent to the closure of $\mathcal{D}$. Quasiconformal deformation theory asserts that every planar compact Klein surface of algebraic genus g is obtained in this way.

Let $G = \langle \tau_1, ..., \tau_{g+1} \rangle$ a planar extended Schottky group of rank g. Let G^+ be its index two subgroup of orientation preserving transformations. It turns out that G^+ is a (classical) Schottky group of genus g, freely generated by the transformations $a_j = \tau_{g+1}\tau_j$, for $j = 1, ..., g$. The closed Riemann surface $S^+ = \Omega/G^+$ is the double oriented cover of the planar compact Klein surface $S = \Omega/G$. Any of the transformation in $G - G^+$ induces an anticonformal involution $\tau : S^+ \to S^+$ (that is, a real structure on S^+) so that $S = S^+/\langle \tau \rangle$. It follows that the number of ovals of τ (its connected components of fixed points) is equal to $(g+1)$, in particular, (S^+, τ) is a M-symmetric Riemann surface. Let us denote by $\pi : S^+ \to S$ the two-fold (branched) Klein cover induced by τ and by $P : \Omega \to S^+$ the Schottky covering of S^+ induced by the Schottky group G^+. In [4] B. Maskit proved the following result.

Theorem 1.1. *Let G be a planar extended Schottky group of rank $g \geq 2$, defined by tye circles C_1,..., C_{g+1}. Then the Riemann surface Ω/G^+ is hyperelliptic if and only if there is a circle which is orthogonal to all C_j, $j = 1, ..., g+1$.*

The aim of this note is to provide a different proof of Theorem 1.1 relaying more on the Schottky groups spirit.

We need to recall some extra definitions. Let Σ_1,... Σ_{g+1} be pairwise disjoint simple loops on the Riemann sphere, all of them bounding a common domain $\mathcal{D}$ of connectivity $g+1$. Assume that for each $j = 1, ..., g+1$, there is a Möbius transformation of order 2, say E_j, so that E_j permutes both topological discs discs bounded by Σ_j (in particular, both fixed points of E_j belong to Σ_j). The group $K = \langle E_1, ..., E_{g+1} \rangle$ is a Kleinian group, isomorphic to a free product of $g+1$ copies of $\mathbb{Z}_2$, called a *Whittaker group of rank* g [3]. If Ω is the region of discontinuity of K, then Ω is connected (the complement of a Cantor set for $g \geq 2$) and $S = \Omega/K$ is an orbifold of signature $(0, 2g+2; 2, ..., 2)$, that is, the Riemann sphere with exactly $2(g+1)$ conical points, all of them of conical order 2. Inside K there is exactly one index two torsion free subgroup, say $K^{(2)}$. It turns out that $K^{(2)}$ is a Schottky group of rank g, called a *hyperelliptic Schottky group*, which is freely generated by the transformations $E_{g+1}E_1$,..., $E_{g+1}E_g$. In this case, $S^{(2)} = \Omega/K^{(2)}$ turns out to be a hyperelliptic Riemann surface, the hyperellitic involution (unique for $g \geq 2$ [2]) is induced by any of the transformations in $K - K^{(2)}$. The projection of the fixed points of E_1,..., E_{g+1} to $S^{(2)}$ provides the $2(g+1)$ fixed points of the hyperelliptic involution.

2 The Necessary Part

Let us consider a planar extended Schottky group G of rank $g \geq 2$, say generated by the reflections τ_1,..., τ_{g+1} on a collection of $(g+1)$ pairwise disjoint round circles on the Riemann sphere, say C_1,..., C_{g+1}, bounding a common domain $\mathcal{D}$ of connectivity $(g+1)$. Let G^+ be the index two orientation preserving Schottky subgroup and let $S^+ = \Omega/G^+$, where Ω is the region of discontinuity of G (the same as for G^+). As before, we denote by $\tau : S^+ \to S^+$ the real structure induced on S^+ by the action of G. Let us denote by $\mathcal{O}_1 = P(C_1)$,...., $\mathcal{O}_{g+1} = P(C_{g+1})$ the ovals of τ.

Let us assume S^+ is a hyperelliptic Riemann surface and let $j : S^+ \to S^+$ be its hyperelliptic involution. As the hyperelliptic involution is unique [2], j and τ should commute, in particular, the collection of ovals of τ is invariant under j. The Schottky group G^+ is defined by the ovals $\mathcal{O}_1$,..., $\mathcal{O}_{g+1}$, that is, by the normalizer (in the fundamental group) of them. It follows that the hyperelliptic involution lifts to a conformal automorphism $\widehat{j} : \Omega \to \Omega$ under $P : \Omega \to S^+$, that is, $jP = P\widehat{j}$. We have that $\widehat{j}^2 \in G^+$. As j has fixed point, we may assume that $\widehat{j}$ also has fixed points, in particular, $\widehat{j}^2 = I$.

It is known that Ω is of class O_{AD}; that is, it admits no holomorphic function with finite Dirichlet norm (see [1, pg 241]). It follows from this (see [1, pg 200]) that every conformal map from Ω into the Riemann sphere is a Möbius transformation. In this way, $\widehat{j}$ is the restriction of a Möbius transformation of order two.

Lemma 2.1. *Each oval has exactly two fixed points of j and each fixed point belongs to some oval. Moreover, each oval is invariant under j.*

Proof. Let us denote by D_1 and D_2 the two connected components of $S^+ - \cup_{j=1}^{g+1}\mathcal{O}_j$. If one of the fixed points of j is not contained in $\cup_{j=1}^{g+1}\mathcal{O}_j$, then we should have that $j(D_1) = D_1$. But as D_1 is planar (isomorphic to the closure of $\mathcal{D}$) we will have that the restriction of j onto D_1 coincides with a Möbius transformation of order 2. It will follows then that j must have at most 4 fixed points on S, a contradiction. In particular, every fixed point of j is contained in some oval. Also, if the oval $\mathcal{O}_k$ contains a fixed point of j, then we should have that $j(\mathcal{O}_k) = \mathcal{O}_k$. In that case, we have that $\mathcal{O}_k$ should have exactly two fixed points of j. As j contains exactly $2(g+1)$ fixed points and we have exactly $(g+1)$ ovals, we have that: (i) each oval has exactly two fixed points of j and (ii) each oval is invariant unde! r j. ☐

By the previous lemma, for each $k \in \{1, ..., g+1\}$, $j(\mathcal{O}_k) = \mathcal{O}_k$. It follows that we may choose liftings $\widehat{j}_1$,..., $\widehat{j}_{g+1}$, of the hyperelliptic involution, each one of order 2 so that $\widehat{j}_k(C_k) = C_k$ and both fixed point of $\widehat{j}_k$ are contained in C_k. Let us consider the Whittaker group

$$\widehat{G} = \langle \widehat{j}_1, ..., \widehat{j}_{g+1} \rangle,$$

and its hyperelliptic Schottky group

$$\widehat{G}^{(2)} = \langle \widehat{j}_{g+1}\widehat{j}_1, ..., \widehat{j}_{g+1}\widehat{j}_g \rangle.$$

We have, by the construction, that $G^+ = \widehat{G}^{(2)}$.

Lemma 2.2. $\widehat{j}_{g+1}\widehat{j}_k = \tau_{g+1}\tau_k$, $k = 1, ..., g$.

Proof. Let us first observe that the circles C_k, $C'_k = \tau_{g+1}(C_k)$ and $\widehat{j}_{g+1}(C_k)$ are lifting of the oval $\mathcal{O}_k$. The circles C_1, C'_1,..., C_g, C'_g (respectively, the circles C_1, $\widehat{j}_{g+1}(C_1)$,..., C_g and $\widehat{j}_{g+1}(C_g)$) form a standard fundamental domain for G^+.

As $\widehat{j}_{g+1}(C_k)$ must belong to the disc bounded by the circle C_{g+1} which does not contains the circle C_k, we should have that $\widehat{j}_{g+1}(C_k)$ should be one of the discs C'_1,...., C'_g. But as mentioned, the only such disc which is a lifting of $\mathcal{O}_k$ is exactly C'_k. We have that $\widehat{j}_{g+1}(C_k) = C'_k$.

Now, we have that the loxodromic transformations $\widehat{j}_{g+1}\widehat{j}_k, \tau_{g+1}\tau_k \in G^+$ send C_k onto C'_k and each maps the exterior of C_k onto the interior of C'_k. It follows that the transformation $\eta = \tau_k\tau_{g+1}\widehat{j}_{g+1}\widehat{j}_k \in G^+$ keeps invariant the circle C_k and each of its bounded discs. As C_k is contained on the region of discontinuity of G^+, it follows that η cannot be loxodromic. As G^+ only contains loxodromic transformations besides the identity, we should have $\eta = I$.

□

Let us recall that, if C is a circle on the Riemann sphere and p and q are any two different points on it, then there is a unique orthogonal circle to it passing through these two given points. The previous fact together with the fact that a circle is uniquely determined by 3 points on it and the following lemma asserts the existence of a common orthogonal circle as desired to prove the necessary part of the theorem.

Lemma 2.3. *Let us consider two pairwise disjoint circles, say C_1 and C_2. Let σ_j be the reflection of C_j and t_j be an elliptic transformation of order 2 preserving C_j whose fixed points belong to C_j. Then $\sigma_2\sigma_1 = t_2t_1$ if and only if there is a circle C such that:*

(i) the fixed points of t_1 and t_2 belong to C and

(ii) C is orthogonal to both C_1 and C_2.

Proof. We may normalize by a suitable Möbius transformation in order to assume that C_1 is the unit circle and C_2 is the circle centered at the origin and a positive radius $r > 1$. In this case we have that $\sigma_2\sigma_1(z) = r^2 z$. The equality $t_2t_1 = \sigma_2\sigma_1$ then obligates to have that the fixed points of both t_1 and t_2 on a line through 0. □

3 The Sufficiency Part

Let us assume we have $(g+1)$ circles, say C_1,..., C_{g+1}, each one of them orthogonal to a common circle C_0. Let us denote by τ_k the reflection on the circle C_k, for $k = 0, 1, .., g+1$. Let us denote by $\eta_k = \tau_0\tau_k$, for $k = 1, ..., g+1$, which are elliptic transformations of order 2. Let G be the planar extended Schottky group generated by the reflections τ_1,..., τ_{g+1} and let $\widehat{G}$ be the Whittaker group generated by the involutions η_1,..., η_{g+1}. It easy to see that G^+ is the hyperelliptic subgroup of $\widehat{G}$. It follows then that the uniformized surface by G^+ is hyperelliptic, with hyperelliptic involution induced by η_0.

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