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Strong Vector Equilibrium Problems in Topological Vector Spaces Via KKM Maps

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ABSTRACT

In this paper, we establish some existence results for strong vector equilibrium problems (for short, SVEP) in topological vector spaces. The solvability of the SVEP is presented using the

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Fan-KKM lemma. These results give a positive answer to an open problem proposed by Chen and Hou and generalize many important results in the recent literature.

RESUMEN

En este artículo, establecemos algunos resultados de existencia para problemas de equilibrio strong vector en espacios vectoriales topológicos (abreviadamente, SVEP). La salubilidad del SVEP es presentada usando el lema de Fan-KKM. Estos resultados dan una respuesta positiva a problemas abiertos propuestos por Chen y Hon y generalizan varios resultados importantes en la literatura reciente.

Key words and phrases: *Strong vector equilibrium, Upper sign continuity, Pseudomonotone bifunction, Quasimonotone bifunction.*

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1 Introduction

It is well known that the vector equilibrium problem provides a unified model for several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems (see [1,8]). Let X and Y be two real Hausdorff topological vector spaces, and K a nonempty subset of X. Let $F : K \times K \to Y$ be a given bifunction and $W : K \to 2^Y$ be a set valued mapping. Let $C \subseteq Y$. C is said a convex cone subset of Y if, $\lambda C \subseteq C$ for each $\lambda \geq 0$ and $C + C \subseteq C$. The convex cone C is called pointed if, $C \cap (-C) = \{0\}$. In this paper, we consider the following problem :

Strong vector equilibrium problem (for short, SVEP) which consists in finding $\overline{x} \in K$ such that

$$F(\overline{x}, y) \in W(\overline{x}), \qquad \forall y \in K.$$

If X and Y be two Banach spaces and $F(x, y) = \langle T(x), y - x\rangle$, where $T : K \to L(X, Y)$ and $L(X, Y)$ denotes the set of all continuous linear mappings from X into Y, then SVEP reduces to the strong vector variational inequality SVVI which was considered by Fang and Huang [7]. In [7] Fang and Huang obtained some existence results for SVVI which gave a positive answer to the open problem proposed by Chen and Hou [3] in a real Banach space. The main purpose of this work is to establish some existence results for SVEP in a real Hausdorff topological vector spaces. Our results extend and improve the corresponding results of Fang and Huang [6], Iusem and Sosa [9] and many others.

The paper is organized as follows. In the rest of this section we recall some notation,definitions, and the Fan-KKM lemma which are used in the next section. In section 2, we first present the solvability of SVEP without monotonicity by using the Fan-KKM lemma and then obtain an existence result for SVEP with generalized monotonicity.

Throughout the paper, unless otherwise specified, let X and Y be two real Hausdorff topological vector spaces. For given bifunction $F : K \times K \to Y$ and $f \in Y^* = L(Y, \mathbb{R})$ consider $f \circ F : K \times K \to \mathbb{R}$ as $\langle f \circ F, (x, y)\rangle = f(\langle F, (x, y)\rangle)$. We denote the duality pairing between X^* and X, by $\langle ., .\rangle$, and the open line segment joining between $x, y \in K$ by $]x, y[$. Let A be a nonempty subset of a topological space X. We denote by 2^A the family of all subsets of the set A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. In a topological vector space X, let *int*, *cl*, and *co* denote the interior, closure and convex hull respectively.
Let K be a nonempty convex subset of X and let K_0 be a subset of K. A multivalued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if

$$coA \subseteq \bigcup_{x \in A} \Gamma(x), \quad \forall \ A \ \in \mathcal{F}(K_0).$$

Let X and Y be two topological spaces. The set-valued mapping $T : X \to 2^Y$ is called upper semi-continuous (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$ there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$. The map T is said to be closed if the set $Gr(T) = \{(x, y) : x \in X, \ y \in T(x)\}$ is a closed set in $X \times Y$.

Proposition 1.1. *If T is closed and $\overline{T(X)} = \overline{\bigcup_{x \in K} T(x)}$ is compact, then T is u.s.c.*

We need the following theorems which are special cases of [5, Theorem 3.1] and [12, Theorem 2.1], respectively, in the sequel.

Theorem 1.2. *([5]). Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset B of K such that $D = \bigcap_{x \in B} F(x)$ is compact. Then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

2 The Main Results

The following theorem provides sufficient conditions in order to guarantee nonemptyness and compactness of the solution set of SVEP.

Theorem 2.1. *Let X and Y be real Hausdorff topological vector spaces, K a nonempty closed subset of X, $F : K \times K \to Y$ and $W : K \to 2^Y$ be two set valued mappings. Assume the following hypotheses hold*

(a) for all finite subsets A of K and for all $x \in coA$ there exists $y \in A$ such that $F(x, y) \in W(x)$;

(b) *for each $y \in K$, the set $\{x \in K : F(x,y) \in W(x)\}$ is closed*

(c) *there exist a nonempty compact subset B of K and a nonempty convex compact subset D of K such that, for each $x \in K \backslash B$ there exists $y \in D$ such that $F(x,y) \notin W(x)$.*

then, the solution set of SVEP is nonempty and compact.

Proof. We define $\Gamma : K \to 2^K$ as follows

$$\Gamma(y) = \{x \in K : F(x,y) \in W(x)\}.$$

By (a) , Γ is a KKM mapping. Applying (b) and (c), we deduce that $\bigcap_{y \in D} \Gamma(y)$ is a closed subset of B. Now, Γ satisfies all of the assumptions of Theorem 1.2 and hence $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$. This means that SVEP has a solution. By (b), the solution set of SVEP is closed and by (c) it is subset of the compact set B. Thus the solution set of SVEP is compact and the proof is complete. □

Remark 2.2. (i) If K is convex, $F(x,x) \in W(x)$, $C(x) = Y \backslash W(x)$ is a convex cone, for all $x \in K$, and the mapping F is concave in y with respect to $C(x)$, that is,

$$F(x, ty_1 + (1-t)y_2) - (tF(x,y_1) + (1-t)F(x,y_2)) \in C(x),$$

for each $x, y_1, y_2 \in K$ and $t \in]0,1[$ then condition (a) of Theorem 2.1 holds. To see this let $A = \{y_1, ..., y_n\}$ be a finite subset of K and $x = \Sigma_{i=1}^n \lambda_i y_i \in co(A)$ where $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. On the contrary, we suppose that (a) does not hold. Then for each $1 \leq i \leq n$, $F(x, y_i) \notin W(x)$. Thus

$$F(x, y_i) \in C(x), \quad \text{for each} \quad 1 \leq i \leq n. \tag{2.1}$$

Since F is concave in y with respect to $C(x)$ we get

$$F(x, \Sigma_{i=1}^n \lambda_i y_i) - \sum_{i=1}^n \lambda_i F(x, y_i) \in C(x). \tag{2.2}$$

By (2.1), note $C(x)$ is convex, we have

$$\sum_{i=1}^n \lambda_i F(x, y_i) \in C(x). \tag{2.3}$$

Now since $C(x)$ is a convex cone, by (2.2) and (2.3) we obtain

$$F(x,x) = F(x, \Sigma_{i=1}^n \lambda_i y_i) \in C(x),$$

which contradicts our assumption. For example if we let $X = Y = \mathbb{R}, W(x) = (-\infty, 0]$, and define $F(x,y) = x - y$ then F satisfies condition (a).

(ii) It is clear that condition (a) is different from diagonally quasiconvex, in the single valued case, defined in [4, page 114].

(iii) If the mapping F is continuous in x, for all fixed $y \in K$ and the graph of the mapping W is closed then $D = \{x \in K : F(x,y) \in W(x)\}$ is closed and so condition (b) holds. To see this Let $x_n \in D$ and $x_n \to x \in K$. Then for each $n \in \mathbb{N}$, $F(x_n, y) \in W(x_n)$. Since $F(.,y)$ is continuous and the graph of W is closed then $F(x,y) \in W(x)$ which shows that $x \in D$. Finally if K is compact then condition (c) trivially holds.

In the following, as an application of Theorem 2.1, we give a topological vector space version of Theorem 2.1 of Fang and Huang [7] for a family of moving closed pointed convex cones $\{C(x) : x \in K\}$.

Corollary 2.3. *Let K be a nonempty closed convex subset of X and $T : K \to L(X,Y)$ be a mapping such that*

(a′) for each $y \in K$, the set $\{x \in K : \langle Tx, y - x\rangle \ \in -C(x)\backslash\{0\}\}$ is open in K,

(b′) there exist a nonempty compact subset B of K and a nonempty convex compact subset D of K such that, for each $x \in K\backslash B$ there exists $y \in D$ such that $\langle Tx, y - x\rangle \in -C(x)\backslash\{0\}$.

Then the set $\{x \in K : \langle Tx, y - x\rangle \notin -C(x)\backslash\{0\}, \ \ \forall \ \ y \in K\}$ is a nonempty and compact subset of K.

Proof. Let $F(x,y) = \langle T(x), y - x\rangle$ and $W(x) = Y\backslash(-C(x)\backslash\{0\})$. We claim that F and W satisfy all of the assumptions of Theorem 2.1. Indeed, by (a'), the set $\{x \in K : F(x,y) \in W(x)\}$ is closed in K (so closed in X), for all $y \in K$. Now if for a finite subset $A = \{y_1, y_2, ..., y_n\}$ of K there exists $x \in coA$ such that $F(x,y) \notin W(x)$, for all $y \in A$. Then

$$F(x, y_i) \in -C(x)\backslash\{0\}, \quad \text{for each} 1 \leq i \leq n.$$

Since $C(x)$ is closed pointed convex cone then $-C(x)\backslash\{0\}$ is a convex cone and so from the definition of F we get (here $x = \Sigma_{i=1}^{n}\lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n}\lambda_i = 1$)

$$0 = F(x,x) = F\left(x, \sum_{i=1}^{n}\lambda_i y_i\right) = \sum_{i=1}^{n}\lambda_i\langle Tx, y_i - x\rangle \in -C(x)\backslash\{0\},$$

which is a contradiction.This proves (a) of Theorem 2.1. Finally (b') guarantees condition (c) of Theorem 2.1 and so the proof is complete. ☐

In what follows we give another application of Theorem 2.1 in order to provide sufficient conditions that a function defined on a nonempty convex subset of a normed space has a fixed point.

Corollary 2.4. *Let K be a nonempty, closed convex subset of a real normed vector space X and let $f : K \to K$ be a continuous mapping. If there is a nonempty compact subset B of K and a compact convex subset D of K such that for every $x \in K$ there exists $y \in D$ satisfying $\| x - f(x) \| > \| y - f(x) \|$, then there exists $\overline{x} \in K$ such that $\overline{x} = f(\overline{x})$. Furthermore, the set of all such elements is a compact subset of B.*

Proof. Define $F : K \times K \to \mathbb{R}$ as

$$F(x,y) = - \parallel y - f(x) \parallel + \parallel x - f(x) \parallel, \qquad \forall x, y \in K,$$

and let $W(x) = [-\infty, 0) \subset Y = \mathbb{R}$, for all $x \in K$. The function F is concave in the second variable with respect to the set $(Y \backslash W(x)) = [0, \infty)$ which is a convex subset of Y. Hence by Remark 2.2 (i), F satisfies condition (a) of Theorem 2.1. By the continuity of F in the first variable and the closedness of graph of W (see Remark 2.2 (iii)) we deduce that F satisfies condition (b) of Theorem 2.1. Finally F satisfies condition (c) of Theorem 2.1 by assumption. Thus there exists $\overline{x} \in K$ such that

$$F(\overline{x}, y) = - \parallel y - f(\overline{x}) \parallel + \parallel \overline{x} - f(\overline{x}) \parallel \leq 0, \text{ for all } y \in K.$$

Now, if, in the previous relation, we take $y = f(\overline{x})$ then $\parallel \overline{x} - f(\overline{x}) \parallel \leq 0$ which implies the result requested. □

Remark 2.5. Corollary 2.4 generalizes Theorem 3.2 in [12] for infinite dimensional normed spaces.

Now using a scalarization method, we establish an existence theorem for SVEP. In particular, we prove the following theorem which is an extension of Theorem 2.2 in [7].

Theorem 2.6. *Let K be a nonempty, closed convex subset of a real Hausdorff topological vector space X, W be a nonempty subset of X. Let $F : K \times K \to Y$ be a bifunction . Suppose that there exists $x_0 \in K$ and $f \in \{y^* \in Y^* : \langle y^*, c\rangle < 0, \ \ \forall \ \ c \in Y \backslash W\}$ such that*

(a) $F(x,x) = 0$, for all $x \in K$;

(b) for each $x, y, z \in K$, if $foF(x,y) \leq 0$ and $foF(x,z) < 0$, then $foF(x,u) < 0, \ \ \forall u \in]y,z[$;

(c) for each $y \in K$, the set $\{x \in K : foF(x,y) \geq 0\}$ is closed in K;

(d) there exist a nonempty compact subset B of K such that,

$$\forall \ \ x \in K \backslash B, \qquad foF(x, x_0) < 0.$$

Then there exists $\overline{x} \in K$ such that $F(\overline{x}, y) \in W$, for all $y \in K$.

Proof. Define $G : K \to 2^K$ by $G(y) = \{x \in K : foF(x,y) \geq 0\}$. By (c), $G(y)$ is closed for each $y \in K$. Now we show that G is a KKM map. Let $A = \{y_1, y_2, ..., y_n\}$ be a finite subset of K and $x = \Sigma_{i=1}^n \lambda_i y_i \in co(A)$ where $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. We prove that $x \in \cup_{i=1}^n G(y_i)$. On the contrary, suppose for each $i = 1, ..., n$, $x \notin G(y_i)$. Consequently

$$f \circ F(x, y_i) < 0, \qquad \text{for each} \quad i = 1, ..., n. \tag{2.4}$$

Notice that using induction, we can deduce from (b) that for each $x, y_1, ..., y_n \in K$, if for each $i = 1, ..., n$, $f \circ F(x, y_i) < 0$ then $f \circ F(x, u) < 0$ for each $u \in co(y_1, y_2, ..., y_n)$. Thus by (2.4) we get

$$0 = f \circ F(x, x) < 0,$$

which is a contradiction. Let $D = \{x_0\}$, then $\cap_{y \in D} G(y) = G(x_0) \subseteq B$ by (d). Thus Theorem 1.2. implies $\bigcap_{y \in K} G(y) \neq \emptyset$. Let $\overline{x} \in \bigcap_{y \in K} G(y)$, then

$$foF(\overline{x}, y) \geq 0, \quad \text{for all} \quad y \in K. \tag{2.5}$$

Now, if $F(\overline{x}, y) \notin W$, for some $y \in K$, then from $f \in \{y^* \in Y^* : \langle y^*, c\rangle < 0, \quad \forall \ \ c \in Y \backslash W\}$, we get $foF(\overline{x}, y) < 0$, which is a contradiction by (2.5). □

As an application of Theorem 2.6 we now present two corollaries. The first is a topological version of Theorem 2.2 in [7] with mild assumptions and the second improves Theorem 3.12 in [10].

Corollary 2.7. *Let K be a nonempty closed convex subset of a topological vector space X, C be a nonempty closed pointed convex cone of X and let $T : K \to L(X, Y)$ be a nonlinear mapping. Suppose that there exist $x_0 \in K$, $f \in \{y^* \in Y^* : \langle y^*, c\rangle > 0, \quad \forall \ \ c \in C\backslash\{0\}\}$, and a nonempty compact subset B of K such that*

$$\forall \ \ x \in K\backslash B, \qquad foT(x)(x - x_0) < 0.$$

Then there exists $\overline{x} \in K$ such that $\langle T\overline{x}, y - \overline{x}\rangle \notin -C\backslash\{0\}$, for all $y \in K$.

Proof. Define $F : K \times K \to Y$ by $F(x, y) = \langle Tx, y - x\rangle$ and $W = Y\backslash -C$. Now the result follows of Theorem 2.6. □

Corollary 2.8. *Suppose that $(X, \|.\|)$ is a real reflexive Banach space and K is a nonempty closed convex subset of X. Let $F : K \times K \to \mathbb{R}$ be a bifunction such that:*

(a) $F(x, x) = 0$, for each $x \in K$,

(b) for each $x, y, z \in K$, if $F(x, y) \geq 0$ and $F(x, z) > 0$, then $F(x, u) > 0$, $\forall u \in]y, z[$;

(c) for each $y \in K$, the set $\{x \in K : F(x, y) \geq 0\}$ is closed;

(d) there exists $r_0 > 0$ such that for each $x \in K\backslash K_{r_0}$ there exists $y \in K_{r_0}$ with $F(x, y) \leq 0$, where $K_{r_0} = \{x \in K : \|x\| \leq r_0\}$.

Then, the solution set of the equilibrium problem,i.e., $\{x \in K : F(x, y) \leq 0, \quad \forall y \in K\}$, is nonempty and compact.

Proof. Set $Y = \mathbb{R}$, $W(x) = [0, \infty)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ for each $x \in K$. By (b), note $f \circ F = F$, so we get condition (b) of Theorem 2.6. Pick $B = D = K_{r_0}$, since K_{r_0} is weakly compact convex, then condition (d) of the Theorem 2.6 trivially holds and so Theorem 2.6 implies that there is $\bar{x} \in K$ such that $F(\bar{x}, y) \geq 0$, for all $y \in K$. Moreover, by (b) and (d), the set $\{x \in K : F(x, y) \geq 0, \quad \forall x \in K\}$, is a closed subset of K_{r_0}, respectively. This completes the proof. □

Definition 2.9. Let $F : K \times K \to Y$, be a given bifunction and $\mathcal{C} = \{C(x) : x \in K\}$ is a family of closed convex cone proper subsets of Y. We say F is $\mathcal{C}-$psudomonotone if the following implication holds

$$F(x, y) \notin -C(x) \backslash \{0\} \Rightarrow F(y, x) \in -C(y).$$

A nonlinear mapping $T : K \to L(X, Y)$ is C-psudomonotone if the bifunction $F(x, y) = \langle Tx, y - x\rangle$ is C-psudomonotone.

Remark 2.10. For the single-valued bifunction F, our definition of pseudomonotonicity reduces to that in [11]. Moreover, if we let $F(x, y) = \langle T(x), y - x\rangle$, where $T : K \to L(X, Y)$ is a nonlinear mapping, and $C(x) = C$ for all $x \in K$, C is a convex cone of Y, then the previous definition reduces to Definition 2.1 in [7].

Definition 2.11. Let $F : K \times K \to Y$, and $\mathcal{C} = \{C(x) : x \in K\}$ is a family of convex cone proper subsets of Y. We say that F is $\mathcal{C}-$upper sign continuous if the following implication holds for every $x, y \in K$,

$$F(u, y) \notin -C(u) \backslash \{0\}, \quad \forall u \in]x, y[\Rightarrow F(x, y) \notin -C(x) \backslash \{0\}.$$

A nonlinear mapping $T : K \to L(X, Y)$ is $\mathcal{C}$-upper sign continuous if the bifunction $F(x, y) = \langle Tx, y - x\rangle$ is $\mathcal{C}$-upper sign continuous.

Remark 2.12. If $Y = \mathbb{R}$ and $C(x) = [0, \infty)$, for all $x \in \mathbb{R}$, then our definition of upper sign continuity, reduces to the definition of upper sign continuity introduced by Bianchi and Pini in [2]. Also, if the graph of the set-valued map $W : K \to 2^Y$ defined by $W(x) = Y \backslash -C(x) \backslash \{0\}$ is closed and the function $t \to F(x_t, y)$ is continuous at $t = 0$, where $x_t = (1 - t)x + ty$, then F is $\mathcal{C}-$ upper sign continuous. This shows that if we set $F(x, y) = \langle T(x), y - x\rangle$, where T is nonlinear mapping, and $C(x) = C$ for all $x \in K$, C is a closed convex cone of Y, then hemicontinuity of T (Definition 2.2 in [6]) implies $\mathcal{C}-$upper sign continuity of F.

Lemma 2.13. *Let K be a nonempty closed convex subset of X, and $\mathcal{C} = \{C(x) : x \in K\}$ is a family of convex cone proper subsets of Y. Let $F : K \times K \to Y$ be a $\mathcal{C}-$pseudomonotone and $\mathcal{C}-$upper sign continuous function. Assume that the following assumptions hold,*

- *(a)* $F(x, x) = 0$, *for each* $x \in K$,
- *(b) for each* $x, y, z \in K$, *if* $F(x, y) \in -C(x) \backslash \{0\}$ *and* $F(x, z) \in -C(x)$, *then* $F(x, u) \in -C(x) \backslash \{0\}$, *for all* $u \in]y, z[$,

then for every $x_0 \in K$,

$$F(x_0, y) \notin -C(x_0)\backslash\{0\}, \qquad \forall y \in K$$

if and only if

$$F(y, x_0) \in -C(y), \qquad \forall y \in K.$$

Proof. Let $x_0 \in K$ be such that

$$F(x_0, y) \notin -C(x_0)\backslash\{0\}, \quad \forall y \in K.$$

Now the $\mathcal{C}-$Pseudomonotonicity of F, implies that

$$F(y, x_0) \in -C(y), \qquad \forall y \in K.$$

Conversely, suppose that

$$F(y, x_0) \in -C(y), \qquad \forall y \in K.$$

We first show that for each $y \in K$,

$$u \in]x_0, y[\Rightarrow F(u, y) \notin -C(u)\backslash\{0\}. \tag{2.6}$$

On the contrary, we suppose that there exist $y \in K$ and $u \in]x_0, y[$ such that

$$F(u, y) \in -C(u)\backslash\{0\}. \tag{2.7}$$

By our assumption,

$$F(u, x_0) \in -C(u). \tag{2.8}$$

From (2.7), (2.8) and (b) we get

$$0 = F(u, u) \in -C(u)\backslash\{0\},$$

which is a contradiction. Thus, (2.6) holds. Since F is $\mathcal{C}-$upper sign continuous, (2.6) implies that

$$F(x_0, y) \notin -C(x_0)\backslash\{0\}.$$

□

The following result improves Proposition 2.5 in [10].

Corollary 2.14. *Let* K *be a nonempty convex subset of* X *and* $F : K \times K \to \mathbb{R}$ *be a pseudomonotone bifunction satisfying the following conditions:*

(a) for each $x \in K, \quad F(x,x) = 0,$

(b) for each $x, y, z \in K$*, if* $F(x,y) < 0$ *and* $F(x,z) \leq 0$*, then* $F(x,u) < 0$*, for all* $u \in]y,z[$,

(c) for every x *and* y *in* K *the following implication holds:*

$$F(u,y) \geq 0, \ \forall \ u \in]x,y[\Rightarrow F(x,y) \geq 0.$$

Let $x_0 \in K$*, and then*

$$F(x_0,y) \geq 0, \quad \forall y \in K \Leftrightarrow F(y,x_0) \leq 0, \quad \forall y \in K.$$

Proof. In the previous Lemma, let $Y = \mathbb{R}$ and $C(x) = [0,\infty)$, for every $x \in K$. Obviously, F is $\mathcal{C}-$pseudomonotone by (b) and $\mathcal{C}-$upper sign continuous by (c). Now, the result follows from Lemma 2.13. □

Theorem 2.15. *Let* K *be a nonempty convex subset of* X *and* $\mathcal{C} = \{C(x) : x \in K\}$ *is a family of convex cone proper subsets of* Y*. Let* $F : K \times K \to Y$ *be a* $\mathcal{C}-$*pseudomonotone and* $\mathcal{C}-$*upper sign continuous bifunction such that:*

(a) for each $x \in K, \quad F(x,x) = 0,$

(b) for each $y \in K$*, the set* $\{x \in K : F(y,x) \in -C(y)\}$ *is closed in* K;

(c) for each $x, y, z \in K$*, if* $F(x,y) \in -C(x)\backslash\{0\}$ *and* $F(x,z) \in -C(x)$*, then* $F(x,u) \in -C(x)\backslash\{0\}$*, for all* $u \in]y,z[$,

(d) there exist a nonempty compact subset B *of* K *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K\backslash B$ *there exists* $y \in D$ *such that* $F(y,x) \notin -C(y)$.

Then, the solution set of SVEP with respect to the family $W(x) = Y\backslash(-C(x)\backslash\{0\})$*, is nonempty and compact.*

Proof. We define $\Gamma, \hat{\Gamma} : K \to 2^K$ by

$$\hat{\Gamma}(y) = \{x \in K : F(x,y) \notin -C(x)\backslash\{0\}\},$$

$$\Gamma(y) = \{x \in K : F(y,x) \in -C(y)\}.$$

By (a), $\Gamma(y)$ and $\hat{\Gamma}(y)$ are nonempty for each $y \in K$. By the $\mathcal{C}$-pseudomonotoncity of F we get,

$$\hat{\Gamma}(y) \subset \Gamma(y), \quad \forall y \in K.$$

Now, we show that $\hat{\Gamma}$ is a KKM mapping. Indeed, assume that $\hat{\Gamma}$ is not a KKM mapping, then there exist $y_1, y_2, ..., y_n$ in K and $z \in co\{y_1, y_2, ..., y_n\}$ such that $z \notin \bigcup_{i=1}^{n} \hat{\Gamma}(y_i)$. Hence, we have

$$F(z, y_i) \in -C(z)\backslash\{0\}, \quad \forall \;\; i = 1, 2, ..., n.$$

Now, it follows from (c) that $F(z, z) \in -C(z)\backslash\{0\}$, which contradicts (a). Thus $\hat{\Gamma}$ is a KKM mapping. Now since for each $y \in K$, $\hat{\Gamma}(y) \subseteq \Gamma(y)$ we deduce (use the same reasoning as above with in this case if $z \notin \cup_{i=1}^{n}\Gamma(y_i)$ then $z \notin \Gamma(y_i)$ so $z \notin \hat{\Gamma}(y_i)$ for each $1 \leq i \leq n$) that Γ is a KKM mapping. The other conditions of Theorem 1.2 are fulfilled by (b) and (d) and hence,

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset. \tag{2.9}$$

Also, Lemma 2.13 implies that

$$\bigcap_{x \in K} \Gamma(x) = \bigcap_{x \in K} \hat{\Gamma}(x). \tag{2.10}$$

From (2.9) and (2.10), SVEP has a solution. Since the solution set of SVEP is $\bigcap_{x \in K} \hat{\Gamma}(x) = \bigcap_{x \in K} \Gamma(x) \neq \emptyset$, then is is closed and a subset of the compact set B. This completes the proof.

□

The next corollary generalize Theorem 2.3 in [7].

Corollary 2.16. *Let K be a nonempty pointed closed convex subset of X and $\mathcal{C} = \{C(x) : x \in K\}$ is a family of proper pointed closed convex cone subsets of Y. Let $T : K \to L(X, Y)$ be a $\mathcal{C}$-pseudomonotone and $\mathcal{C}$-upper sign continuous bifunction such that there exist a nonempty compact subset B of K and a nonempty convex compact subset D of K such that for each $x \in K\backslash B$ there exists $y \in D$ such that $\langle Ty, y - x\rangle \notin C(y)$. Then there exists an $\overline{x} \in K$ such that*

$$\langle T\overline{x}, y - \overline{x}\rangle \notin -C(\overline{x}\backslash\{0\}), \qquad \forall y \in K.$$

Proof. Let $F(x, y) = \langle T(x), y - x\rangle$. We show that F satisfies the conditions of Theorem 2.15. Note (a) obviously holds. For (b) notice for each $y \in K$, the set $\{x \in K : F(y, x) \in -C(y)\} = \{x \in K : \langle T(y), x - y\rangle \in -C(y)\}$ is closed in K, since $C(y)$ is a closed set and $T(y) \in L(X, Y)$ for each $y \in K$. To show (c) let $x, y, z \in K$ and $t \in]0, 1[$. If $F(x, y) = \langle T(x), y - x\rangle \in -C(x)\backslash\{0\}$ and $F(x, z) = \langle T(x), z - x\rangle \in -C(x)$, then

$$\langle T(x), ty + (1-t)z - x\rangle = \langle T(x), t(y - x) + (1-t)(z - x)\rangle =$$

$$t\langle T(x), y - x\rangle + (1-t)\langle T(x), z - x\rangle \in$$

$$-tC(x)\backslash\{0\} + (1-t)(-C(x)) \subseteq -C(x)\backslash\{0\}, \qquad \text{for all} \quad t \in]0, 1[$$

since $-C(x)\backslash\{0\}$ is convex, note $C(x)$ is a pointed convex cone. To show (d) note for all $x \in K\backslash B$ there exists $y \in D$ such that $\langle T(y), y - x\rangle \notin C(y)$ which implies $F(y, x) \notin -C(y)$. Now apply Theorem 2.15 so there exists an $\bar{x} \in K$ such that $F(\bar{x}, y) = \langle T\bar{x}, y - \bar{x}\rangle \notin -C(\bar{x})$ for all $y \in K$. □

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