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Dispersive Estimates for the Schrödinger Equation with Potentials of Critical Regularity

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ABSTRACT

We prove $L^1 \to L^\infty$ dispersive estimates with a logarithmic loss of derivatives for the Schrödinger group $e^{it(-\Delta+V)}$ for a class of real-valued potentials $V \in C^{(n-3)/2}(\mathbf{R}^n)$, $V(x) = O(\langle x\rangle^{-\delta})$, where $n = 4, 5$, $\delta > 3$ if $n = 4$ and $\delta > 5$ if $n = 5$.

RESUMEN

Probamos $L^1 \to L^\infty$ estimativas dispersivas con una perdida logaritmica de derivadas para el grupo de Schrödinger $e^{it(-\Delta+V)}$ para una clase de potenciales a valores reales $V \in C^{(n-3)/2}(\mathbf{R}^n)$, $V(x) = O(\langle x\rangle^{-\delta})$, donde $n = 4, 5$, $\delta > 3$ si $n = 4$ y $\delta > 5$ si $n = 5$.

Key words and phrases: *Schrodinger equation, dispersive estimates.*

Math. Subj. Class.: *35L15, 35B40, 47F05.*

1 Introduction and Statement of Results

The present paper is a continuation of our previous one [2] where $L^1 \to L^\infty$ dispersive estimates without loss of derivatives for the Schrödinger group $e^{it(-\Delta+V)}$ have been proved for potentials $V \in C^k(\mathbf{R}^n)$, $V(x) = O(\langle x\rangle^{-\delta})$, where $n = 4, 5$, $k > (n-3)/2$, $\delta > 3$ if $n = 4$ and $\delta > 5$ if $n = 5$. To be more precise, denote by G_0 and G the self-adjoint realizations of the operators $-\Delta$ and $-\Delta + V$ on $L^2(\mathbf{R}^n)$, $n \geq 4$, where $V \in L^\infty(\mathbf{R}^n)$ is a real-valued potential satisfying

$$|V(x)| \leq C\langle x\rangle^{-\delta}, \quad \forall x \in \mathbf{R}^n, \tag{1.1}$$

with constants $C > 0$, $\delta > (n+2)/2$. In fact, we are interested in finding the biggest possible class of real-valued potentials for which the perturbed Schrödinger group satisfies the following analogue of the well-known dispersive estimate for the free one:

$$\left\|e^{itG}P_{ac}\right\|_{L^1\to L^\infty} \leq C|t|^{-n/2}, \quad t \neq 0, \tag{1.2}$$

where P_{ac} denotes the spectral projection onto the absolutely continuous spectrum of G. There have been many works studying this problem. In general, the proof of (1.2) goes in studying separately and in a different maner three regions of frequencies - (1) low ones belonging to an interval $[0, \varepsilon]$, $0 < \varepsilon \ll 1$, (assuming additionally that zero is neither an eigenvalue nor a resonance), (2) intermediate ones in $[\varepsilon, \varepsilon^{-1}]$, and (3) high frequencies in $[\varepsilon^{-1}, +\infty)$. It became clear that in dimensions one, two and three no regularity of the potential is needed to prove (1.2). In higher dimensions the same conclusion remains true as far as the frequencies from the first two regions are concerned, but it is no longer true at high frequencies. In fact, from purely mathematical point of view the problem of proving (1.2) turns out to be interesting and difficult at high frequencies, only. That is why, in the present paper we will be only interested in the following high frequency analogue of (1.2):

$$\left\|e^{itG}\chi_a(G)\right\|_{L^1\to L^\infty} \leq C|t|^{-n/2}, \quad t \neq 0, \tag{1.3}$$

where $\chi_a \in C^\infty((-\infty, +\infty))$, $\chi_a(\lambda) = 0$ for $\lambda \leq a$, $\chi_a(\lambda) = 1$ for $\lambda \geq a+1$, $a \gg 1$. Note that when $n = 1$ the estimate (1.3) is proved by Goldberg and Shlag [4] for potentials $V \in L^1$, while in dimension $n = 2$ it is proved by Moulin [7] for potentials satisfying

$$\sup_{y\in\mathbf{R}^2} \int_{\mathbf{R}^2} \frac{|V(x)|}{|x-y|^{1/2}}dx < +\infty.$$

When $n = 3$ Rodnianski and Shlag [10] proved (1.3) for small potentials belonging to a subclass of Kato class with Kato norm satisfying

$$\sup_{y\in\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|V(x)|}{|x-y|}dx < 4\pi,$$

while for large potentials Goldberg [3] proved (1.3) for $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$, $0 < \epsilon \ll 1$. The situation, however, changes drastically when $n \geq 4$. Indeed, in this case Goldberg and Visan [5] showed the existence of potentials $V \in C_0^k(\mathbf{R}^n)$, $\forall k < (n-3)/2$, for which (1.3) fails to hold. On the other hand, Journé, Soffer and Sogge [6] proved (1.3) for potentials satisfying (1.1) with $\delta > n$ as well as the following regularity condition

$$\widehat{V} \in L^1. \tag{1.4}$$

Note that (1.3) has been recently proved by Moulin and Vodev [8] for potentials satisfying (1.1) with $\delta > n-1$ and (1.4). Without any regularity conditions on the potential Vodev [12] proved dispersive estimates with a loss of $(n-3)/2$ derivatives. More precisely, it was shown in [12] that under the condition (1.1) only, we have the estimates

$$\left\| e^{itG}\chi_a(G)f \right\|_{L^\infty} \le C|t|^{-n/2} \left\| \langle G\rangle^{(n-3)/4} f \right\|_{L^1}, \tag{1.5}$$

$$\left\| e^{itG}\chi_a(G)f \right\|_{L^\infty} \le C|t|^{-n/2} \left\| \langle x\rangle^{n/2+\epsilon} f \right\|_{L^2}, \tag{1.6}$$

for every $0 < \epsilon \ll 1$. So, the natural question which arises when $n \ge 4$ is that one of finding the smallest possible regularity of the potential in order to have (1.3). In other words, is it possible to replace the condition (1.4) by another one requiring less regularity on the potential? In view of the counterexample of [5] mentioned above, when $n \ge 4$ it is quite natural to make the following

Conjecture 1. *The dispersive estimate (1.3) holds true for all potentials $V \in C_0^{(n-3)/2}(\mathbf{R}^n)$.*

To our best knowledge, this is still an open problem. The following weaker statement, however, is more likely to be valid.

Conjecture 2. *The dispersive estimate (1.3) holds true for all potentials $V \in C_0^k(\mathbf{R}^n)$, where $k > (n-3)/2$.*

Indeed, when $n = 4$ or $n = 5$ Conjecture 2 follows from the recent results of [2]. However, it is still open when $n \ge 6$. In fact, in [2] more general potentials are treated not necessarily compactly supported. To describe this in more detials, introduce the spaces $\mathcal{C}_\delta^k(\mathbf{R}^n)$ and $\mathcal{V}_\delta^k(\mathbf{R}^n)$ of all functions $V \in C^k(\mathbf{R}^n)$ satisfying

$$\|V\|_{\mathcal{C}_\delta^k} := \sup_{x\in\mathbf{R}^n} \sum_{0\le|\alpha|\le k_0} \langle x\rangle^\delta \left|\partial_x^\alpha V(x)\right|$$

$$+\nu \sup_{x\in\mathbf{R}^n} \sum_{|\beta|=k_0} \langle x\rangle^\delta \sup_{x'\in\mathbf{R}^n:|x-x'|\le 1} \frac{\left|\partial_x^\beta V(x) - \partial_x^\beta V(x')\right|}{|x-x'|^\nu} < +\infty,$$

$$\|V\|_{\mathcal{V}_\delta^k} := \sup_{x\in\mathbf{R}^n} \sum_{0\le|\alpha|\le k_0} \langle x\rangle^{\delta+|\alpha|} \left|\partial_x^\alpha V(x)\right|$$

$$+\nu \sup_{x\in\mathbf{R}^n} \sum_{|\beta|=k_0} \langle x\rangle^{\delta+k_0+\nu} \sup_{x'\in\mathbf{R}^n:|x-x'|\leq 1} \frac{\left|\partial_x^\beta V(x) - \partial_x^\beta V(x')\right|}{|x-x'|^\nu} < +\infty,$$

where $k_0 \geq 0$ is an integer and $\nu = k - k_0$ satisfies $0 \leq \nu < 1$. In [2] we have proved the following

Theorem 1.1. *Let $n = 4$ or $n = 5$ and let $V \in \mathcal{C}_\delta^k(\mathbf{R}^n)$ with $k > (n-3)/2$, $\delta > 3$ if $n = 4$ and $\delta > 5$ if $n = 5$. Then, the dispersive estimate (1.3) holds true.*

It is not clear, however, if this still holds with $k = (n-3)/2$. Using the results of [2] we prove in the present paper the following

Theorem 1.2. *Let $n = 4$ or $n = 5$ and let $V \in \mathcal{C}_\delta^{(n-3)/2}(\mathbf{R}^n)$, $\delta > 3$ if $n = 4$ and $\delta > 5$ if $n = 5$. Then, we have the dispersive estimate*

$$\left\|e^{itG}\chi_a(G)f\right\|_{L^\infty} \leq C_\epsilon |t|^{-n/2} \left\|\left(\log\left(2+G^2\right)\right)^{2+\epsilon} f\right\|_{L^1}, \tag{1.7}$$

for every $0 < \epsilon \ll 1$. Moreover, for every $2 \leq p < +\infty$ we have the optimal dispersive estimate

$$\left\|e^{itG}\chi_a(G)\right\|_{L^{p'}\to L^p} \leq C|t|^{-n(1/2-1/p)}, \tag{1.8}$$

where $1/p + 1/p' = 1$.

Note that it is not clear if (1.7) and (1.8) hold true when $n \geq 6$. To prove the dispersive estimates above one needs to bound the quantity

$$A(t,h) = |t|^{n/2} \left\|e^{itG}\psi(h^2G)\right\|_{L^1\to L^\infty}$$

uniformly in both t and h, where $\psi \in C_0^\infty((0,+\infty))$ and $0 < h \ll 1$ is a semi-classical parameter. It was shown in [12] that, under the assumption (1.1) only, we have the bound

$$A(t,h) \leq Ch^{-(n-3)/2} \tag{1.9}$$

in all dimensions $n \geq 4$, where $C > 0$ is a constant independent of t and h. On the other hand, if we suppose (1.1) fulfilled with $\delta > n-1$ as well as (1.4), then we have the optimal bound (see [8])

$$A(t,h) \leq C. \tag{1.10}$$

Note that (1.10) still holds under the assumptions of Theorem 1.1 (see [2]). To prove the estimates (1.7) and (1.8) we show in the present paper that, under the assumptions of Theorem 1.2, we have the bound

$$A(t,h) \leq C\log\frac{1}{h}. \tag{1.11}$$

It is an open problem, however, to show that (1.11) still holds for potentials $V \in C_0^{(n-3)/2}(\mathbf{R}^n)$ when $n \geq 6$. Indeed, the strategy of proving (1.11) proposed in [1] leads to the study of a finite number ($\sim n/2$) of operators, $T_j(t,h)$, $t > 0$, $j = 0, 1, ...$, with explicit kernels defined as follows

$$T_j(t,h) = i\int_0^t e^{i(t-\tau)G_0}\psi_1(h^2G_0)VT_{j-1}(\tau,h)d\tau, \quad T_0(t,h) = e^{itG_0}\psi(h^2G_0),$$

where $\psi_1 \in C_0^\infty((0,+\infty))$, $\psi_1 = 1$ on $\operatorname{supp}\psi$. Roughly speaking, one needs to show that if $V \in C_0^{(n-3)/2}(\mathbf{R}^n)$, then each T_j satisfies the bound

$$\|T_j(t,h)\|_{L^1\to L^\infty} \le C_j t^{-n/2}\log\frac{1}{h}, \quad j\ge 1. \tag{1.12}$$

In the present paper we prove (1.12) with $j = 1$ in all dimensions $n \ge 4$ (actually for a larger class of potentials - see Section 3). However, this is hard to show for $j \ge 2$. In fact, under the assumption (1.1) only, we have the bounds (see [1])

$$\|T_j(t,h)\|_{L^1\to L^\infty} \le C_j t^{-n/2}h^{j-n/2}, \quad j\ge 1. \tag{1.13}$$

On the other hand, without any regularity assumption on V, the kernel of T_j behaves like $C_j t^{-n/2}h^{-j(n-3)/2}$. These observations show that if one wants to prove (1.12) (for $j \ge 2$) it suffices to do it for $j < n/2$ only, and secondly one should better avoid using the kernels of T_j for this purpose, unless one menages to show that some regularity on the potential improves the behaviour in h of the kernels (which is far from being clear when $j \ge 2$). Note that (1.11) follows from (1.12) with $j = 1$ and the following theorem proved in [2].

Theorem 1.3. *If $n = 4$ we suppose $V \in \mathcal{C}_\delta^\varepsilon(\mathbf{R}^4)$ with $\varepsilon > 0$, $\delta > 3$, while if $n = 5$ we suppose $V \in \mathcal{C}_\delta^1(\mathbf{R}^5)$ with $\delta > 5$. Then, there exist constants $C, \varepsilon_0 > 0$ so that we have the estimate*

$$\left\| e^{itG}\psi(h^2G) - \sum_{j=0}^{1} T_j(t,h)\right\|_{L^1\to L^\infty} \le Ch^{\varepsilon_0}t^{-n/2}. \tag{1.14}$$

Note again that it is a difficult open problem to show that (1.14) holds true for potentials $V \in C_0^{(n-3)/2}(\mathbf{R}^n)$ when $n \ge 6$. However, (1.14) holds in all dimensions $n \ge 4$ for potentials satisfying (1.1) with $\delta > n-1$ as well as (1.4) (see Appendix B of [8]). It is shown in [12], [1] that, under the assumption (1.1) only, we have the bound (1.14) with $h^{-(n-4)/2}$ in place of h^{ε_0} in the right-hand side.

2 Proof of Theorem 1.2

In this section we will show that Theorem 1.2 follows from the estimate (1.11). To this end we will use the identity

$$\chi_a(\sigma) = \int_0^1 \psi(\theta\sigma)\frac{d\theta}{\theta},$$

where $\psi(\sigma) = \sigma\chi_a'(\sigma) \in C_0^\infty((0,+\infty))$. So we can write

$$\chi_a(\sigma)\left(\log\left(2+\sigma^2\right)\right)^{-2-\epsilon} = \int_0^1 \psi_\theta(\theta\sigma)\frac{d\theta}{\theta\left(\log\left(4\theta^{-2}\right)\right)^{2+\epsilon}},$$

where

$$\psi_\theta(\sigma)=\psi(\sigma)\left(1-\frac{\log(\theta^2/2+\sigma^2/4)}{\log(\theta^2/4)}\right)^{-2-\epsilon}$$

belongs to $C_0^\infty((0,+\infty))$ uniformly in θ and having a support independent of θ. Therefore, we have

$$e^{itG}\chi_a(G)\left(\log\left(2+G^2\right)\right)^{-2-\epsilon}=\int_0^1 e^{itG}\psi_\theta(\theta G)\frac{d\theta}{\theta\left(\log\left(4\theta^{-2}\right)\right)^{2+\epsilon}}.$$

Hence, by (1.11) we get

$$\left\|e^{itG}\chi_a(G)\left(\log\left(2+G^2\right)\right)^{-2-\epsilon}\right\|_{L^1\to L^\infty}\le\int_0^1\left\|e^{itG}\psi_\theta(\theta G)\right\|_{L^1\to L^\infty}\frac{d\theta}{\theta\left(\log\left(4\theta^{-2}\right)\right)^{2+\epsilon}}$$

$$\le C|t|^{-n/2}\int_0^1\frac{d\theta}{\theta\left(\log\left(2\theta^{-1}\right)\right)^{1+\epsilon}}\le C|t|^{-n/2}\int_0^1\frac{-d\log\left(2\theta^{-1}\right)}{\left(\log\left(2\theta^{-1}\right)\right)^{1+\epsilon}}\le C|t|^{-n/2},$$

which proves (1.7). To prove (1.8) we will use the following estimate proved in [12] (see Theorem 3.1)

$$\left\|e^{itG}\psi(\theta G)-e^{itG_0}\psi(\theta G_0)\right\|_{L^2\to L^2}\le Ch. \tag{2.1}$$

On the other hand, by (1.11) we have

$$\left\|e^{itG}\psi(\theta G)-e^{itG_0}\psi(\theta G_0)\right\|_{L^1\to L^\infty}\le C_\epsilon h^{-\epsilon}|t|^{-n/2}, \tag{2.2}$$

for every $0<\epsilon\ll 1$. An interpolation between (2.1) and (2.2) leads to the estimate

$$\left\|e^{itG}\psi(\theta G)-e^{itG_0}\psi(\theta G_0)\right\|_{L^{p'}\to L^p}\le C_\epsilon h^{1-(1+\epsilon)(1-2/p)}|t|^{-n(1/2-1/p)}, \tag{2.3}$$

for every $2\le p\le+\infty$, where $1/p+1/p'=1$. Let $2\le p<+\infty$. By (2.3) we get

$$\left\|e^{itG}\chi_a(G)-e^{itG_0}\chi_a(G_0)\right\|_{L^{p'}\to L^p}\le\int_0^1\left\|e^{itG}\psi(\theta G)-e^{itG_0}\psi(\theta G_0)\right\|_{L^{p'}\to L^p}\frac{d\theta}{\theta}$$

$$\le C|t|^{-n(1/2-1/p)}\int_0^1\theta^{-1+1/p-\epsilon(1/2-1/p)}d\theta$$

$$\le C|t|^{-n(1/2-1/p)}\int_0^1\theta^{-1+1/(2p)}d\theta\le C|t|^{-n(1/2-1/p)}, \tag{2.4}$$

provided $\epsilon>0$ is taken small enough. Clearly, (1.8) follows from (2.4).

3 Study of the Operator T_1 in all Dimensions $n\ge 4$

Let $\gamma=t/2$ if $0<t\le 2$, $\gamma=1$ if $t\ge 2$, and decompose the operator T_1 as follows

$$T_1=\left(\int_0^\gamma+\int_{t-\gamma}^t\right)...+\int_\gamma^{t-\gamma}...:=T_1^{(1)}+T_1^{(2)}.$$

In this section we will prove the following

Proposition 3.1. *Let $n \ge 4$ and let $V \in \mathcal{V}_\delta^{(n-3)/2-k}(\mathbf{R}^n)$, where $0 \le k < (n-3)/2$ and $\delta > 2+k$. Then, we have the estimate*

$$\left\| T_1^{(1)}(t,h) \right\|_{L^1 \to L^\infty} \le C t^{-n/2} h^{-k} \log \frac{1}{h}. \tag{3.1}$$

Moreover, if $V \in \mathcal{V}_\delta^m(\mathbf{R}^n)$ with an integer $0 \le m < (n-3)/2$, and if $k \ge 0$ is such that $n-1-2m-\delta < k < (n-3)/2 - m$, then we have

$$\left\| T_1^{(2)}(t,h) \right\|_{L^1 \to L^\infty} \le C t^{-n/2} h^{-k}, \quad t \ge 2. \tag{3.2}$$

Remark. It is proved in [12] that if V satisfies (1.1) with $\delta > (n+1)/2$, then we have

$$\| T_1(t,h) \|_{L^1 \to L^\infty} \le C t^{-n/2} h^{-(n-3)/2}.$$

Note also that (3.2) with $m = k = 0$ is proved in [8] (see Appendix B), and this is enough for the proof of Theorem 1.2.

Proof. We will make use of the fact that the kernel of the operator $e^{itG_0}\psi(h^2 G_0)$ is of the form $K_h(|x-y|, t)$, where

$$K_h(\sigma, t) = \frac{\sigma^{-2\nu}}{(2\pi)^{\nu+1}} \int_0^\infty e^{it\lambda^2} \psi(h^2\lambda^2) \mathcal{J}_\nu(\sigma\lambda) \lambda d\lambda = h^{-n} K_1(\sigma/h, t/h^2), \tag{3.3}$$

where $\mathcal{J}_\nu(z) = z^\nu J_\nu(z)$, $J_\nu(z) = \left(H_\nu^+(z) + H_\nu^-(z)\right)/2$ is the Bessel function of order $\nu = (n-2)/2$. Thus, the kernel of T_1 is of the form

$$\mathcal{T}(x,y,t,h) = \int_0^t \int_{\mathbf{R}^n} \widetilde{K}_h(|x-\xi|, t-\tau) K_h(|y-\xi|, \tau) V(\xi) d\xi d\tau,$$

where $\widetilde{K}_h$ denotes the kernel of the operator $e^{itG_0}\psi_1(h^2 G_0)$. It is shown in [12] (see Proposition 2.1) that the function K_h satisfies the bound

$$|K_h(\sigma,t)| \le C (h\sigma)^{s-(n-1)/2} t^{-s-1/2}, \quad \forall\, t, \sigma > 0,\ 0 < h \le 1,\ 0 \le s \le (n-1)/2. \tag{3.4}$$

Set

$$a_h(\sigma,t) = t^{n/2} e^{i\sigma^2/4t} K_h(\sigma,t) = a_1(\sigma/h, t/h^2). \tag{3.5}$$

Clearly, (3.4) can be rewritten as

$$|a_h(\sigma,t)| \le C \left(\frac{t}{h\sigma} \right)^s, \quad \forall\, t, \sigma > 0,\ 0 < h \le 1,\ 0 \le s \le (n-1)/2. \tag{3.6}$$

Denote $a_h' = da_h/dt$. We need the following

Lemma 3.2. *For every $t, \sigma > 0$, $0 < h \le 1$, $0 \le s \le (n-1)/2$ and every integer $k \ge 0$ such that $k + s \le n/2$, we have the bound*

$$\left| \partial_\sigma^k a_h(\sigma,t) \right| \le C \left(\frac{1}{\sigma} \right)^k \left(\frac{t}{h\sigma} \right)^s. \tag{3.7}$$

Moreover, if $k+s\le (n-2)/2$, we have

$$\left|\partial_\sigma^k a_h'(\sigma,t)\right| \le Ct^{-1}\left(\frac{1}{\sigma}\right)^k\left(\frac{t}{h\sigma}\right)^s. \tag{3.8}$$

Proof. In view of (3.5), it suffices to prove (3.7) and (3.8) with $h=1$. Consider first the case $0<\sigma\le 1$. We will use that $\mathcal{J}_\nu(z)=z^{2\nu}g_\nu(z)$ with a function $g_\nu(z)$ analytic at $z=0$. Hence, given any integer $k\ge 0$ we have

$$\partial_\sigma^k K_1(\sigma,t)=\int_0^\infty e^{it\lambda^2}\psi_k(\lambda)g_\nu^{(k)}(\sigma\lambda)d\lambda,$$

where $\psi_k\in C_0^\infty((0,+\infty))$ and $g_\nu^{(k)}(z)=d^kg_\nu(z)/dz^k$. Let $t\ge 1$. Then, in the same way as in the proof of Proposition 2.1 of [12] (see (2.10)) we have

$$\left|\partial_\sigma^k K_1(\sigma,t)\right|\le C_{k,m}t^{-m-1/2}, \tag{3.9}$$

for every integer $m\ge 0$, and hence for all real $m\ge 0$. Using (3.9) we get

$$\left|\partial_\sigma^k a_1(\sigma,t)\right|\le Ct^{n/2}\sum_{j=0}^k\left|\partial_\sigma^j\left(e^{i\sigma^2/4t}\right)\right|\left|\partial_\sigma^{k-j}K_1(\sigma,t)\right|$$

$$\le Ct^{(n-1)/2-m}\sum_{j=0}^k\left|\partial_\sigma^j\left(e^{i\sigma^2/4t}\right)\right|\le Ct^{(n-1)/2-m},$$

which proves (3.7) (with $h=1$) in this case. Let $0<t\le 1$. Then we have

$$\left|\partial_\sigma^k K_1(\sigma,t)\right|\le C_k. \tag{3.10}$$

Using (3.10) we get

$$\left|\partial_\sigma^k a_1(\sigma,t)\right|\le Ct^{n/2}\sum_{j=0}^k\left|\partial_\sigma^j\left(e^{i\sigma^2/4t}\right)\right|\left|\partial_\sigma^{k-j}K_1(\sigma,t)\right|$$

$$\le Ct^{n/2}\sum_{j=0}^k\left|\partial_\sigma^j\left(e^{i\sigma^2/4t}\right)\right|\le Ct^{n/2-k}\le Ct^s.$$

Consider now the case $\sigma\ge 1$. We will use that $\mathcal{J}_\nu(z)=e^{iz}b_\nu^+(z)+e^{-iz}b_\nu^-(z)$ with functions $b_\nu^\pm$ satisfying

$$\left|\partial_z^j b_\nu^\pm(z)\right|\le C_jz^{(n-3)/2-j},\quad \forall z\ge z_0, \tag{3.11}$$

for every integer $j\ge 0$ and every $z_0>$ with a constant $C_j>0$ depending on j and z_0. We can write $K_1=K_1^++K_1^-$ with $K_1^\pm$ defined by replacing in the definition of K_1 the function $\mathcal{J}_\nu(z)$ by $e^{\pm iz}b_\nu^\pm(z)$. Then the functions

$$a_1^\pm(\sigma,t)=t^{n/2}e^{i\sigma^2/4t}K_1^\pm(\sigma,t)$$

can be written in the form

$$a_1^{\pm}(\sigma,t)=t^{n/2}\int_0^\infty e^{it(\lambda\pm\sigma/2t)^2}\widetilde{b}_\nu^{\pm}(\sigma\lambda)\varphi(\lambda)d\lambda,$$

where $\varphi(\lambda)=(2\pi)^{-\nu-1}\lambda^{1+2\nu}\psi(\lambda^2)$, $\widetilde{b}_\nu^{\pm}(z)=z^{-2\nu}b_\nu^{\pm}(z)$. Hence

$$\partial_\sigma^k a_1^{\pm}(\sigma,t)=t^{n/2}\int_0^\infty\sum_{j=0}^k c_j\partial_\sigma^j\left(e^{it(\lambda\pm\sigma/2t)^2}\right)\partial_\sigma^{k-j}\widetilde{b}_\nu^{\pm}(\sigma\lambda)\varphi(\lambda)d\lambda.$$

Using the identity

$$\partial_\sigma^j\left(e^{it(\lambda\pm\sigma/2t)^2}\right)=(\mp 2t)^{-j}\partial_\lambda^j\left(e^{it(\lambda\pm\sigma/2t)^2}\right)$$

and integrating by parts, we get

$$\partial_\sigma^k a_1^{\pm}(\sigma,t)=\sum_{j=0}^k t^{n/2-j}e^{i\sigma^2/4t}\int_0^\infty e^{it\lambda^2\pm i\sigma\lambda}\widetilde{\varphi}(\lambda)B_{\nu,j}^{\pm}(\lambda,\sigma)d\lambda,$$

where $\widetilde{\varphi}\in C_0^\infty((0,+\infty))$, $\widetilde{\varphi}=1$ on $\operatorname{supp}\varphi$, and

$$B_{\nu,j}^{\pm}(\lambda,\sigma)=c_j\,(\pm 2)^{-j}\,\partial_\lambda^j\left(\partial_\sigma^{k-j}\widetilde{b}_\nu^{\pm}(\sigma\lambda)\varphi(\lambda)\right).$$

It is easy to deduce from (3.11) that the functions $B_{\nu,j}^{\pm}$ satisfy the bounds

$$\left|\partial_\lambda^\ell B_{\nu,j}^{\pm}(\lambda,\sigma)\right|\le C_{\ell,j}\sigma^{-(n-1)/2-k+j}, \tag{3.12}$$

for all integers $\ell,j\ge 0$. Using (3.12), in the same way as in the proof of Proposition 2.1 of [12] (see (2.13)), we get

$$\left|\int_0^\infty e^{it\lambda^2\pm i\sigma\lambda}\widetilde{\varphi}(\lambda)B_{\nu,j}^{\pm}(\lambda,\sigma)d\lambda\right|\le C_{m,j}t^{-m-1/2}\sigma^{-(n-1)/2-k+j+m}, \tag{3.13}$$

for all integers m, and hence for all real m. By (3.13) with $m=(n-1)/2-s-j$ we obtain

$$\left|\partial_\sigma^k a_1(\sigma,t)\right|\le C\sigma^{-k}\left(\frac{t}{\sigma}\right)^s,$$

which is the desired result in this case.

To prove (3.8) with $h=1$, observe that

$$a_1'(\sigma,t)=t^{n/2}e^{i\sigma^2/4t}\left(K_1'(\sigma,t)+\frac{n}{2t}K_1(\sigma,t)-\frac{i\sigma^2}{4t^2}K_1(\sigma,t)\right). \tag{3.14}$$

On the other hand, integrating by parts twice with respect to the variable λ^2 and using that the function $g_\nu(z)=z^{-2\nu}\mathcal{J}_\nu(z)=z^{-\nu}J_\nu(z)$ satisfies the equation

$$g_\nu''(z)+(n-1)z^{-1}g_\nu'(z)+g_\nu(z)=0,$$

we get

$$K_1'(\sigma,t)+\frac{n}{2t}K_1(\sigma,t)-\frac{i\sigma^2}{4t^2}K_1(\sigma,t)=t^{-1}\left(K_1^{(0)}(\sigma,t)+K_1^{(1)}(\sigma,t)\right), \tag{3.15}$$

where

$$K_1^{(j)}(\sigma,t)=\left(\frac{\sigma}{t}\right)^j\int_0^\infty e^{it\lambda^2}\psi^{(j)}(\lambda)g_\nu^{(j)}(\sigma\lambda)d\lambda,\quad j=0,1,$$

where $\psi^{(j)}\in C_0^\infty((0,+\infty))$, $g_\nu^{(0)}(z)=g_\nu(z)$, $g_\nu^{(1)}(z)=dg_\nu(z)/dz$. By (3.14) and (3.15),

$$a_1'(\sigma,t)=t^{-1}\left(a_1^{(0)}(\sigma,t)+a_1^{(1)}(\sigma,t)\right), \tag{3.16}$$

where

$$a_1^{(j)}(\sigma,t)=t^{n/2}e^{i\sigma^2/4t}K_1^{(j)}(\sigma,t),\quad j=0,1.$$

Now, in the same way as above one can see that the functions $a_1^{(j)}$ satisfy (3.7) with $h=1$, provided $k+s\le(n-2)/2$. $\square$

The kernel of the operator T_1 is of the form

$$\mathcal{T}(x,y,t,h)=\int_0^t\int_{\mathbf{R}^n}e^{-i\varphi}(t-\tau)^{-n/2}\tau^{-n/2}\widetilde{a}_h(|x-\xi|,t-\tau)a_h(|y-\xi|,\tau)V(\xi)d\xi d\tau,$$

where

$$\varphi=\frac{|x-\xi|^2}{4(t-\tau)}+\frac{|y-\xi|^2}{4\tau},$$

and $\widetilde{a}_h$ is defined by replacing in the definition of a_h the function K_h by $\widetilde{K}_h$. Observe that by Lemma 3.2 we have the bounds

$$\left|\partial_\xi^\alpha a_h(|x-\xi|,t)\right|\le C(t/h)^{k+\epsilon}|x-\xi|^{-|\alpha|-k-\epsilon}, \tag{3.17}$$

$$\left|\partial_\xi^\alpha a_h'(|x-\xi|,t)\right|\le Ct^{-1}(t/h)^{k+\epsilon}|x-\xi|^{-|\alpha|-k-\epsilon}, \tag{3.18}$$

for every $0\le\epsilon\ll1$, $0\le k<(n-3)/2$, and all multi-indices α such that $|\alpha|\le(n-2)/2-k-\epsilon$. Define the functions $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ by replacing $\int_0^t$ in the definition of the function $\mathcal{T}$ by $\int_0^\gamma$ and $\int_\gamma^{t/2}$, respectively. Let $\phi\in C_0^\infty(\mathbf{R})$, $\phi(\lambda)=1$ for $|\lambda|\le1/2$, $\phi(\lambda)=0$ for $|\lambda|\ge1$, and write

$$1=\sum_{q=0}^\infty\phi_q(\lambda),$$

where $\phi_0=\phi$, $\phi_q(\lambda)=\widetilde{\phi}(2^{-q}\lambda)$, $q\ge1$, with a function $\widetilde{\phi}\in C_0^\infty(\mathbf{R})$, $\widetilde{\phi}(\lambda)=0$ for $|\lambda|\le1/2$ and $|\lambda|\ge1$. We can write

$$\mathcal{F}^{(1)}=\sum_{p=1}^\infty\sum_{q=0}^\infty\mathcal{F}_{p,q}^{(1)},\quad \mathcal{F}^{(2)}=\sum_{q=0}^\infty\mathcal{F}_q^{(2)},$$

where

$$\mathcal{F}_{p,q}^{(1)}=\int_0^\gamma\int_{\mathbf{R}^n}e^{-i\varphi}(t-\tau)^{-n/2}\tau^{-n/2}\phi_p\left(\frac{\gamma}{\tau}\right)\phi_q(|\xi|)\widetilde{a}_h(|x-\xi|,t-\tau)a_h(|y-\xi|,\tau)V(\xi)d\xi d\tau$$

$$= t^{1-n}\int_{t/\gamma}^{\infty}\int_{\mathbf{R}^n} e^{-i\varphi}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}\phi_p\left(\gamma\mu/t\right)\widetilde{a}_h(|x-\xi|,t(\mu-1)/\mu)a_h(|y-\xi|,t/\mu)V_q(\xi)d\xi d\mu,$$

$$\mathcal{F}_q^{(2)} = \int_1^{t/2}\int_{\mathbf{R}^n} e^{-i\varphi}(t-\tau)^{-n/2}\tau^{-n/2}\phi_q(|\xi|)\widetilde{a}_h(|x-\xi|,t-\tau)a_h(|y-\xi|,\tau)V(\xi)d\xi d\tau$$

$$= t^{1-n}\int_2^{t}\int_{\mathbf{R}^n} e^{-i\varphi}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}\widetilde{a}_h(|x-\xi|,t(\mu-1)/\mu)a_h(|y-\xi|,t/\mu)V_q(\xi)d\xi d\mu,$$

where we have made a change of variables $\mu = t/\tau$ and set $V_q(\xi) = \phi_q(|\xi|)V(\xi)$. Clearly, it suffices to prove the following

Proposition 3.3. *Under the assumptions of Proposition 3.1, there exist constants $C, \varepsilon' > 0$ such that we have the bounds*

$$\left|\mathcal{F}_{p,q}^{(1)}\right| \le C2^{-\varepsilon p-\varepsilon' q}t^{-n/2}h^{-k-\varepsilon}, \tag{3.19}$$

for every $0 < \varepsilon \ll 1$, and

$$\left|\mathcal{F}_q^{(2)}\right| \le C2^{-\varepsilon' q}t^{-n/2}h^{-k}, \quad t \ge 2. \tag{3.20}$$

Indeed, by (3.19) we have

$$\left|\mathcal{F}^{(1)}\right| \le Ct^{-n/2}h^{-k}(\varepsilon h^{\varepsilon})^{-1} = C't^{-n/2}h^{-k}\log\frac{1}{h},$$

if we take ε so that $h^{-\varepsilon} = 2$, while (3.20) yields

$$\left|\mathcal{F}^{(2)}\right| \le Ct^{-n/2}h^{-k}, \quad t \ge 2.$$

Proof. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be a real-valued function such that $\int\rho(x)dx = 1$, and set $\rho_\theta(x) = \theta^{-n}\rho(x/\theta)$, $0 < \theta \le 1$, $V_{q,\theta} = \rho_\theta * V_q$. Let $k_0 \ge 0$ be an integer such that $(n-3)/2 - k = k_0 + \nu$ with $0 \le \nu < 1$. Since $V \in \mathcal{V}_\delta^{k_0+\nu}(\mathbf{R}^n)$, we have

$$\left|\partial_\xi^\alpha V_q(\xi)\right| \le C2^{-q(\delta+|\alpha|)}, \quad 0 \le |\alpha| \le k_0, \tag{3.21}$$

$$\left|\partial_\xi^\alpha V_q(\xi) - \partial_\xi^\alpha V_q(\xi')\right| \le C2^{-q(\delta+k_0+\nu)}|\xi-\xi'|^\nu, \quad |\xi-\xi'| \le 1, \quad |\alpha| = k_0. \tag{3.22}$$

It is easy to see that these bounds imply

$$\left|\partial_\xi^\alpha V_{q,\theta}(\xi)\right| \le C2^{-q(\delta+|\alpha|)}, \quad 0 \le |\alpha| \le k_0, \tag{3.23}$$

$$\left|\partial_\xi^\alpha V_{q,\theta}(\xi)\right| \le C\theta^{-1+\nu}2^{-q(\delta+k_0+\nu)}, \quad |\alpha| = k_0+1, \tag{3.24}$$

$$\left|\partial_\xi^\alpha V_q(\xi) - \partial_\xi^\alpha V_{q,\theta}(\xi)\right| \le C\theta 2^{-q(\delta+|\alpha|+1)}, \quad 0 \le |\alpha| \le k_0-1, \tag{3.25}$$

$$\left|\partial_\xi^\alpha V_q(\xi) - \partial_\xi^\alpha V_{q,\theta}(\xi)\right| \le C\theta^\nu 2^{-q(\delta+|\alpha|+\nu)}, \quad |\alpha| = k_0. \tag{3.26}$$

Integrating by parts with respect to the variable ξ as in Section 4 of [12] (see the proof of (4.15)) we obtain the following

Lemma 3.4. *Let $0 \le m < (n-1)/2$ be an integer and let $W(\mu,\cdot) \in C_0^m(\mathbf{R}^n)$. Then we have the estimate*

$$\left| t^{1-n} \int_{t/\gamma}^{\infty} \int_{\mathbf{R}^n} e^{-i\varphi} \left(\frac{\mu}{\mu-1} \right)^{n/2} \mu^{n/2-2} \phi_p\left(\gamma\mu/t\right) W(\mu,\xi) d\xi d\mu \right|$$

$$\le C t^{-n/2} \left(2^p/\gamma\right)^{(n-3)/2-m} \sum_{0\le|\alpha|\le m} \int_{\mathbf{R}^n} |\xi - y|^{-2m+|\alpha|-1} \left|\partial_\xi^\alpha W(\infty,\xi)\right| d\xi$$

$$+ C t^{-n/2} \left(2^p/\gamma\right)^{(n-3)/2-m} \sum_{0\le|\alpha|\le m} \int_{\mathbf{R}^n} |y-\xi|^{-1} \left|\xi - y - \gamma t^{-1}(x-y)\right|^{-2m+|\alpha|} \left|\partial_\xi^\alpha W(t/\gamma,\xi)\right| d\xi$$

$$+ C t^{-n/2-1} \left(2^p/\gamma\right)^{(n-3)/2-m-1} \sum_{0\le|\alpha|\le m} \int_{2^{p-1}t/\gamma}^{2^p t/\gamma} \int_{\mathbf{R}^n} \Big(|y-\xi|^{-1} \left|\xi - y - \mu^{-1}(x-y)\right|^{-2m+|\alpha|}$$

$$+ \left|\xi - y - \mu^{-1}(x-y)\right|^{-2m+|\alpha|-1} \Big) \left|\partial_\xi^\alpha W(\mu,\xi)\right| d\xi d\mu$$

$$+ C t^{-n/2} \left(2^p/\gamma\right)^{(n-3)/2-m} \sum_{0\le|\alpha|\le m} \int_{2^{p-1}t/\gamma}^{2^p t/\gamma} \int_{\mathbf{R}^n} |y-\xi|^{-1} \left|\xi - y - \mu^{-1}(x-y)\right|^{-2m+|\alpha|}$$

$$\times \left|\partial_\mu \partial_\xi^\alpha W(\mu,\xi)\right| d\xi d\mu. \tag{3.27}$$

We would like to apply this lemma with a function W of the form

$$W(\mu,\xi) = \widetilde{a}_h(|x-\xi|, t(\mu-1)/\mu) a_h(|y-\xi|, t/\mu) Q(\xi)$$

where $Q \in C_0^m(\mathbf{R}^n)$ is independent of the variable μ. In view of (3.17) and (3.18) we have

$$\left|\partial_\xi^\alpha W(\mu,\xi)\right| \le C \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|} \left|\partial_\xi^{\alpha_1} \widetilde{a}_h(|x-\xi|, t(\mu-1)/\mu)\right| \left|\partial_\xi^{\alpha_2} a_h(|y-\xi|, t/\mu)\right| \left|\partial_\xi^{\alpha_3} Q(\xi)\right|$$

$$\le C h^{-k-\varepsilon} (t/\mu)^{k+\varepsilon} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|} |x-\xi|^{-|\alpha_1|} |y-\xi|^{-|\alpha_2|-k-\varepsilon} \left|\partial_\xi^{\alpha_3} Q(\xi)\right|, \tag{3.28}$$

$$\left|\partial_\mu \partial_\xi^\alpha W(\mu,\xi)\right| \le t\mu^{-2} \left|\partial_\xi^\alpha \left(\widetilde{a}_h'(|x-\xi|, t(\mu-1)/\mu) a_h(|y-\xi|, t/\mu) Q(\xi)\right)\right|$$

$$+ t\mu^{-2} \left|\partial_\xi^\alpha \left(\widetilde{a}_h(|x-\xi|, t(\mu-1)/\mu) a_h'(|y-\xi|, t/\mu) Q(\xi)\right)\right|$$

$$\le C t\mu^{-2} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|} \left|\partial_\xi^{\alpha_1} \widetilde{a}_h'(|x-\xi|, t(\mu-1)/\mu)\right| \left|\partial_\xi^{\alpha_2} a_h(|y-\xi|, t/\mu)\right| \left|\partial_\xi^{\alpha_3} Q(\xi)\right|$$

$$+ C t\mu^{-2} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|} \left|\partial_\xi^{\alpha_1} \widetilde{a}_h(|x-\xi|, t(\mu-1)/\mu)\right| \left|\partial_\xi^{\alpha_2} a_h'(|y-\xi|, t/\mu)\right| \left|\partial_\xi^{\alpha_3} Q(\xi)\right|$$

$$\le C \mu^{-1} h^{-k-\varepsilon} (t/\mu)^{k+\varepsilon} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|} |x-\xi|^{-|\alpha_1|} |y-\xi|^{-|\alpha_2|-k-\varepsilon} \left|\partial_\xi^{\alpha_3} Q(\xi)\right|. \tag{3.29}$$

By (3.27), (3.28) and (3.29), one can easily get the estimate

$$\left| t^{1-n} \int_{t/\gamma}^{\infty} \int_{\mathbf{R}^n} e^{-i\varphi} \left(\frac{\mu}{\mu-1} \right)^{n/2} \mu^{n/2-2} \phi_p\left(\gamma\mu/t\right) W(\mu,\xi) d\xi d\mu \right|$$

$$\leq Ct^{-n/2}h^{-k-\varepsilon}\left(\gamma 2^{-p}\right)^{m+k+\varepsilon-(n-3)/2}\mathcal{M}(m,Q), \tag{3.30}$$

where

$$\mathcal{M}(m,Q)=\sum_{0\leq|\alpha|\leq m}\sup_{\mathbf{R}^n}\langle\xi\rangle^{n+2\varepsilon'-2m-1-k+|\alpha|}\left|\partial_\xi^\alpha Q(\xi)\right|,$$

for every $0<\varepsilon'\ll 1$ and $0<\varepsilon\leq\varepsilon'$. Consider first the case $k_0<(n-3)/2$. Then, we are going to apply (3.30) with $m=k_0$, $Q=V_q-V_{q,\theta}$, and $m=k_0+1$, $Q=V_{q,\theta}$, respectively. Choose ε' such that $\delta\geq k+2+4\varepsilon'$. In view of (3.25) and (3.26), we have

$$\mathcal{M}(k_0,V_q-V_{q,\theta})\leq C\theta^\nu\left(2^q\right)^{\nu-\varepsilon'}, \tag{3.31}$$

while by (3.23) and (3.24), we have

$$\mathcal{M}(k_0+1,V_{q,\theta})\leq C\theta^{\nu-1}\left(2^q\right)^{\nu-1-\varepsilon'}. \tag{3.32}$$

Combining (3.30), (3.31) and (3.32) we conclude

$$\left|\mathcal{F}_{p,q}^{(1)}\right|\leq Ct^{-n/2}h^{-k-\varepsilon}\left(\gamma 2^{-p}\right)^{\varepsilon}2^{-\varepsilon' q}\left(\left(\theta 2^{p+q}/\gamma\right)^{\nu}+\left(\theta 2^{p+q}/\gamma\right)^{\nu-1}\right). \tag{3.33}$$

Taking $\theta=\gamma 2^{-p-q}$ we deduce (3.19) from (3.33). Let now $k_0=(n-3)/2$. This implies that n is odd and $\nu=0$. Then we apply (3.30) with $m=k_0$ and $Q=V_q$. In view of (3.21) we have

$$\mathcal{M}(k_0,V_q)\leq C2^{-\varepsilon' q}, \tag{3.34}$$

with some constant $0<\varepsilon'\ll 1$, so in this case (3.19) follows from (3.30) and (3.34).

Proceeding as in the proof of (3.30) we obtain in the same way the estimate

$$\left|t^{1-n}\int_2^t\int_{\mathbf{R}^n}e^{-i\varphi}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}W(\mu,\xi)d\xi d\mu\right|$$

$$\leq Ch^{-k}\left(t^{-n/2}+t^{m+k-n+3/2}\right)\mathcal{M}(m,Q)$$

$$+Ch^{-k}t^{m+k-n+3/2}\mathcal{M}(m,Q)\int_2^t\mu^{(n-3)/2-m-k-1}d\mu\leq C'h^{-k}t^{-n/2}\mathcal{M}(m,Q), \tag{3.35}$$

where we have used that $m+k<(n-3)/2$. On the other hand, since $V\in\mathcal{V}_\delta^m(\mathbf{R}^n)$ with $\delta>n-1-2m-k$, it is easy to check that we have the bound

$$\mathcal{M}(m,V_q)\leq C2^{-\varepsilon' q}, \tag{3.36}$$

with some constant $0<\varepsilon'\ll 1$. Now, combining (3.35) and (3.36) leads to (3.20). □

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