CUBO A Mathematical Journal
Vol.11, Nº04, (87–107). September 2009

Pseudo-differential operators with smooth symbols on modulation spaces

Joachim Toft
Department of Mathematics and Systems Engineering,
Växjö University, Sweden
email: `joachim.toft@vxu.se`

ABSTRACT

Let $M^{p,q}_{(\omega_0)}$ be the modulation space with parameters p,q and weight function ω_0. If $\partial^\alpha a/\omega \in L^\infty$ for all α, then we prove that the pseudo-differential operator $a_t(x,D)$ is continuous from $M^{p,q}_{(\omega_0\omega)}$ to $M^{p,q}_{(\omega_0)}$. More generally, if $\mathscr{B}$ is a translation invariant BF-space, then we prove that $a_t(x,D)$ is continuous from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M_{(\omega_0)}(\mathscr{B})$. We use these results to establish identifications between such spaces with different weights.

RESUMEN

Sea $M^{p,q}_{(\omega_0)}$ el espacio de modulación con parámetros p,q y función de peso ω_0. Si $\partial^\alpha a/\omega \in L^\infty$ para todo α, entonces probamos que el operador pseudo-diferencial $a_t(x,D)$ es continuo de $M^{p,q}_{(\omega_0\omega)}$ a $M^{p,q}_{(\omega_0)}$. En general, si $\mathscr{B}$ es una translación invariante en el espacio-BF, entonces probamos que $a_t(x,D)$ es continuo de $M_{(\omega_0\omega)}(\mathscr{B})$ en $M_{(\omega_0)}(\mathscr{B})$. Usamos estos resultados para establecer las identificaciones entre dichos espacios con diferentes pesos.

Key words and phrases: *Pseudo-differential operators, Modulation spaces, Coorbit spaces, BF-spaces, Sobolev spaces, Besov spaces.*

Math. Subj. Class.: *35S05, 47B37, 47G30, 42B35.*

1 Introduction

In this paper we establish continuity properties for certain pseudo-differential operators with smooth symbols when acting on general class of modulation spaces. These modulation spaces involve the usual modulation spaces, as well as certain type of weighted spaces related to Wiener amalgam spaces. Furthermore, we establish bijectivity properties for multiplication operators and Fourier multipliers, and use these properties to establish identification properties between modulation spaces with different weights.

In particular we cover Theorem 2.1 in [30], where Tachizawa considers pseudo-differential operators with symbols in $S_{(\omega)}(\mathbf{R}^{2d})$, the set of all smooth functions a on $\mathbf{R}^{2d}$ such that $(\partial^\alpha a)/\omega \in L^\infty(\mathbf{R}^{2d})$. Here ω is an appropriate weight function on $\mathbf{R}^{2d}$, which takes the form of

$$\omega(x,\xi) = \langle x\rangle^t \langle \xi\rangle^s \tag{1}$$

in [30], where $s,t \in \mathbf{R}$ and $\langle x\rangle = (1+|x|^2)^{1/2}$. (We use the usual notation for function and distribution spaces, see e. g. [22].) In this context, Tachizawa extends Calderon-Vaillancourt's theorem, and proves that if ω_0 is appropriate, and $p,q \in [1,\infty]$, then the corresponding pseudo-differential operators are continuous from the modulation space $M^{p,q}_{(\omega_0\omega)}$ to $M^{p,q}_{(\omega_0)}$. (Cf. Section 2 for the definition of modulation spaces and pseudo-differential operators.) Tachizawa's result were thereafter generalized in Theorem 3.2 in the report [38], where the conditions on the weight ω are relaxed in the sense that it is only assumed that ω should be v-moderate for some polynomial v.

A similar and interesting result comparing to [30, 38], concerns [32, Theorem 5.3], where Teofanov discuss similar properties in context of ultra-modulation spaces. In this approach, the condition on v here above is relaxed in the sense that v is permitted to grow subexponentially, instead of polynomially. This in turn implies that symbols to the pseudo-differential operators might grow subexponentially. However, the classes of pseudo-differential operators in [32] do not contain those in [30] or [38], because the symbols in [32] have to fulfill certain conditions of Gelfand-Shilov type, which is not the case in [30, 38].

Other related results are Theorem 3 in [25], and Theorem 3, Corollary 2 and Remark 3 in [26], where Pilipović and Teofanov consider mapping properties for pseudo-differential operators with symbols in ultra-modulation space and which fulfill certain ellipticity conditions.

In Section 3 we generalize [38, Theorem 3.2], and prove continuity for such pseudo-differential operators on a broad class of modulation spaces, which contains the modulation spaces in [38], and their Fourier transforms. These modulation spaces are in turn special cases of so called coorbit spaces (see [11, 12] for the definition of coorbit spaces, and [9] for an updated definition of modulation spaces). (See Theorem 3.2 and Theorem 3.2′.) Furthermore we establish bijectivity properties for pseudo-differential operators, if they, in addition, are appropriate multiplication operators or Fourier multipliers. (See Corollary 3.6.) Thereafter we give links on how these results can be used to establish identification properties between modulation spaces with different weights. (See Remark 3.7 and Theorems 3.9.) Here we also present some immediate consequences in modulation space theory and for spaces related to Wiener amalgam spaces. (See Corollary 3.6′, Theorem 3.9′ and Theorem 3.9″.)

The (classical) modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$, as introduced by Feichtinger in [6], consist of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L^{p,q}$ norm. It follows that the parameters p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}$. The theory of modulation spaces was developed further and generalized in [11–13, 16], where Feichtinger and Gröchenig established the theory of coorbit spaces. In particular, the modulation space $M^{p,q}_{(\omega)}$, where ω denotes a weight function on phase (or time-frequency shift) space, appears as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{(\omega)}$.

A major idea behind the design of these spaces was to find useful Banach spaces, which are defined in a way similar to Besov spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov spaces, with a *uniform* decomposition. From the construction of these spaces, it turns out that modulation spaces and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [1, 29, 35, 37]. (See also [15, 23] for other embeddings.)

During the last 15 years many results have been proved which confirm the usefulness of the modulation spaces in time-frequency analysis, where they occur naturally. For example, in [13, 17, 21], it is shown that all modulation spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of (i) the study of continuity of pseudo-differential operators with smooth symbols acting on modulation spaces, and (ii) the use of modulation spaces as symbol classes. Tachizawa pioneered topic (i) in [30]. (See at the above.)

In [28], Sjöstrand founded topic (ii) and introduced the modulation space $M^{\infty,1}$, which contains non-smooth functions, as a symbol class. He proved that the symbol class $M^{\infty,1}$ corresponds to an algebra of operators which are bounded on L^2.

Gröchenig and Heil thereafter proved in [17, 18] that each operator with symbol in $M^{\infty,1}$ is continuous on all modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$. This extends Sjöstrand's result since $M^{2,2} = L^2$. Some generalizations to operators with symbols in general unweighted modulation spaces were obtained in [19, 35], and in [36, 38, 39] some further extensions involving weighted modulation spaces are presented. Modulation spaces in pseudodifferential calculus is currently an active field of research (see e. g. [18–20, 25, 31, 32]).

2 Preliminaries

In this section we discuss basic properties for modulation spaces and other related spaces. The proofs are in many cases omitted since they can be found in [4–6, 11–13, 17, 33–36].

We start by recalling some properties of the involved weight functions. The positive function $\omega \in L^\infty_{loc}(\mathbf{R}^d)$ is called *v-moderate* for some appropriate function $v \in L^\infty_{loc}(\mathbf{R}^d)$, if there is a constant $C > 0$ such that

$$\omega(x_1 + x_2) \le C\omega(x_1)v(x_2), \quad x_1, x_2 \in \mathbf{R}^d. \tag{2}$$

If v can be chosen as polynomial, then ω is called polynomially moderate. The function v is called *submultiplicative*, if (2) holds for $\omega = v$.

As in [36] we let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderate functions on $\mathbf{R}^d$. We also let $\mathscr{P}_0(\mathbf{R}^d)$ be the set of all smooth $\omega \in \mathscr{P}(\mathbf{R}^d)$ such that $(\partial^\alpha \omega)/\omega$ is bounded for every α.

Note that if $\omega \in \mathscr{P}(\mathbf{R}^d)$, then $\omega(x) + \omega(x)^{-1} \le P(x)$, for some polynomial P on $\mathbf{R}^d$.

In most of the applications, it is no restriction to assume that the weight functions belong to $\mathscr{P}_0$, which is a consequence of the following lemma. (See also [36].)

Lemma 2.1. *Assume that $\omega \in \mathscr{P}(\mathbf{R}^d)$. Then there is a function $\omega_0 \in \mathscr{P}(\mathbf{R}^d)$ and a constant $C > 0$ such that $C^{-1}\omega \le \omega_0 \le C\omega$.*

Proof. The assertion follows by letting $\omega_0 = \omega * \varphi$ for some $0 \le \varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$. □

The duality between a topological vector space and its dual is denoted by $\langle \,\cdot\,, \,\cdot\, \rangle$. For admissible a and b in $\mathscr{S}'(\mathbf{R}^d)$, we set $(a, b) = \langle a, \overline{b} \rangle$, and it is obvious that $(\,\cdot\,, \,\cdot\,)$ on L^2 is the usual scalar product.

Next let V_1 and V_2 be vector spaces such that $V_1 \oplus V_2 = \mathbf{R}^d$ and $V_2 = V_1^\perp$, and assume that $v_0 \in \mathscr{S}'(V_1)$ and $v \in \mathscr{S}'(\mathbf{R}^d)$ are such that $v(x_1, x_2) = (v_0 \otimes 1)(x_1, x_2)$, when $x_j \in V_j$ for $j = 1, 2$. Then $v(x_1, x_2)$ is identified with $v_0(x_1)$, and we set $v(x_1, x_2) = v(x_1)$.

In order to discuss modulation spaces, we recall the definition of short-time Fourier transform. Assume that $\chi \in \mathscr{S}'(\mathbf{R}^d) \setminus 0$ and let $\tau_x\chi(y) = \chi(y - x)$ when $x, y \in \mathbf{R}^d$. Then the *short-time Fourier transform* $V_\chi f$ of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to the *window function* χ is the distribution in $\mathscr{S}'(\mathbf{R}^{2d})$, defined by the formula

$$(V_\chi f)(x, \xi) = \mathscr{F}(f \cdot \overline{\tau_x \chi})(\xi).$$

Here $\mathscr{F}$ denotes the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$, which takes the form

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int f(y) e^{-i\langle y, \xi \rangle}\, dy$$

when $f \in \mathscr{S}(\mathbf{R}^d)$. We note that $V_\chi f$ is well-defined (as an element in $\mathscr{S}'$), since it is the partial Fourier transform of the tempered distribution $(x, y) \mapsto f(y)\overline{\chi(y - x)}$ with respect to the y-variable.

(Cf. [14].) If $f,\chi\in \mathscr S(\mathbf R^d)$, then $V_\chi f$ is given by the formula

$$(V_\chi f)(x,\xi) = (2\pi)^{-d/2}\int f(y)\overline{\chi(y-x)}e^{-i\langle y,\xi\rangle}\, dy.$$

Assume that $\chi\in \mathscr S(\mathbf R^d)\setminus 0$, $p,q\in [1,\infty]$ and $\omega \in \mathscr P(\mathbf R^{2d})$ are fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf R^d)$ consists of all $f\in \mathscr S'(\mathbf R^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int\Big(\int |V_\chi f(x,\xi)\omega(x,\xi)|^p\, dx\Big)^{q/p}d\xi\Big)^{1/q} <\infty \tag{3}$$

(with the obvious modifications when $p=\infty$ and/or $q=\infty$). Furthermore, the space $W^{p,q}_{(\omega)}(\mathbf R^d)$ consists of all $f\in \mathscr S'(\mathbf R^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} \equiv \Big(\int\Big(\int |V_\chi f(x,\xi)\omega(x,\xi)|^q\, d\xi\Big)^{p/q}dx\Big)^{1/p} <\infty . \tag{4}$$

Note that the latter space is related to certain types of Wiener amalgam spaces. (Cf. Definition 4 in [13].)

We recall that $W^{p,q}_{(\omega)} = \mathscr F M^{q,p}_{(\omega_0)}$ when $\omega_0(x,\xi)=\omega(-\xi,x)\in \mathscr P(\mathbf R^{2d})$. In fact, let $\check\chi(x)=\chi(-x)$ as usual. Then Parseval's formula and a change of the order of integration shows that

$$|\mathscr F^{-1}(\widehat f\,\tau_\xi\widehat\chi)(x)| = |\mathscr F(f\,\tau_x\check\chi)(\xi)|. \tag{5}$$

Hence, by applying the $L^{q,p}_{(\omega)}$ norm, the assertion follows.

The convention of indicating weight functions with parenthesis is used also in other situations. For example, if $\omega\in \mathscr P(\mathbf R^d)$, then $L^p_{(\omega)}(\mathbf R^d)$ is the set of all measurable functions f on $\mathbf R^d$ such that $f\omega\in L^p(\mathbf R^d)$, i.e. such that $\|f\|_{L^p_{(\omega)}}\equiv \|f\omega\|_{L^p}$ is finite.

The following proposition is a consequence of well-known facts in [6, 17]. Here and in what follows, we let p' denotes the conjugate exponent of p, i.e. $1/p+1/p'=1$.

Proposition 2.2. *Assume that $p,q,p_j,q_j\in [1,\infty]$ for $j=1,2$, $\omega,\omega_1,\omega_2,v\in \mathscr P(\mathbf R^{2d})$ are such that ω is v-moderate, $\chi\in M^1_{(v)}(\mathbf R^d)\setminus 0$, and let $f\in \mathscr S'(\mathbf R^d)$. Then the following is true:*

1. *$f\in M^{p,q}_{(\omega)}(\mathbf R^d)$ if and only if (3) holds, i.e. $M^{p,q}_{(\omega)}(\mathbf R^d)$ is independent of the choice of χ. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (3), and different choices of χ give rise to equivalent norms;*

2. *$f\in W^{p,q}_{(\omega)}(\mathbf R^d)$ if and only if (4) holds, i.e. $W^{p,q}_{(\omega)}(\mathbf R^d)$ is independent of the choice of χ. Moreover, $W^{p,q}_{(\omega)}$ is a Banach space under the norm in (4), and different choices of χ give rise to equivalent norms;*

3. *if $p_1\le p_2$, $q_1\le q_2$ and $\omega_2\le C\omega_1$ for some constant C, then*

$$\mathscr S(\mathbf R^d)\subseteq M^{p_1,q_1}_{(\omega_1)}(\mathbf R^d)\subseteq M^{p_2,q_2}_{(\omega_2)}(\mathbf R^d)\subseteq \mathscr S'(\mathbf R^d),$$

$$\mathscr S(\mathbf R^d)\subseteq W^{p_1,q_1}_{(\omega_1)}(\mathbf R^d)\subseteq W^{p_2,q_2}_{(\omega_2)}(\mathbf R^d)\subseteq \mathscr S'(\mathbf R^d).$$

Proposition 2.2 permits us to be rather vague about to the choice of $\chi \in M^1_{(v)} \setminus 0$ in (3) and (4). For example, if $C > 0$ is a constant and Ω is a subset of $\mathscr{S}'$, then $\|a\|_{M^{p,q}_{(\omega)}} \le C$ for every $a \in \Omega$, means that the inequality holds for some choice of $\chi \in M^1_{(v)} \setminus 0$ and every $a \in \Omega$. Evidently, for any other choice of $\chi \in M^1_{(v)} \setminus 0$, a similar inequality is true although C may have to be replaced by a larger constant, if necessary.

Next we discuss weight functions which are common in the applications. For any $s, t \in \mathbf{R}$, set

$$\sigma_t(x) = \langle x\rangle^t, \quad \sigma_{s,t}(x,\xi)\langle \xi\rangle^s\langle x\rangle^t, \tag{6}$$

when $x, \xi \in \mathbf{R}^d$. Then it follows that $\sigma_t \in \mathscr{P}_0(\mathbf{R}^d)$ and $\sigma_{s,t} \in \mathscr{P}_0(\mathbf{R}^{2d})$ for every $s, t \in \mathbf{R}$, and σ_t is $\sigma_{|t|}$-moderate and $\sigma_{s,t}$ is $\sigma_{|s|,|t|}$-moderate. Obviously, $\sigma_s(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2}$, and $\sigma_{s,t} = \sigma_t \otimes \sigma_s$. Moreover, if $\omega \in \mathscr{P}(\mathbf{R}^d)$, then ω is σ_t-moderate provided t is chosen large enough.

For conveniency we use the notations L^p_s, $M^{p,q}_s$ and $M^{p,q}_{s,t}$ instead of $L^p_{(\sigma_s)}$, $M^{p,q}_{(\sigma_s)}$ and $M^{p,q}_{(\sigma_{s,t})}$ respectively.

Remark 2.3. Assume that $p, q, q_1, q_2 \in [1,\infty]$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then the following properties for modulation spaces hold:

1. if $q_1 \le \min(p, p')$ and $q_2 \ge \max(p, p')$ and $\omega(x,\xi) = \omega(x)$, then
$$M^{p,q_1}_{(\omega)} \subseteq L^p_{(\omega)} \subseteq M^{p,q_2}_{(\omega)}, \quad W^{p,q_1}_{(\omega)} \subseteq L^p_{(\omega)} \subseteq W^{p,q_2}_{(\omega)};$$

2. $S^0_0 = \bigcap_{s\in\mathbf{R}} M^{\infty,1}_{s,0}$;

3. if $p \ge q$, then $M^{p,q}_{(\omega)} \subseteq W^{p,q}_{(\omega)}$. If instead $q \ge p$, then $W^{p,q}_{(\omega)} \subseteq M^{p,q}_{(\omega)}$;

4. $M^{1,\infty}(\mathbf{R}^d)$ and $W^{1,\infty}(\mathbf{R}^d)$ are convolution algebras such that if $\mathfrak{M}(\mathbf{R}^d)$ is the set of all measures on $\mathbf{R}^d$ with bounded mass, then $\mathfrak{M} \subseteq W^{1,\infty} \subseteq M^{1,\infty}$;

5. if Ω is a subset of $\mathscr{P}(\mathbf{R}^{2d})$ such that for any polynomial P on $\mathbf{R}^{2d}$, there is an element $\omega \in \Omega$ such that P/ω is bounded, then
$$\mathscr{S}(\mathbf{R}^d) = \bigcap_{\omega\in\Omega} M^{p,q}_{(\omega)}(\mathbf{R}^d), \quad \mathscr{S}'(\mathbf{R}^d) = \bigcup_{\omega\in\Omega} M^{p,q}_{(1/\omega)}(\mathbf{R}^d);$$

6. if $s, t \in \mathbf{R}$ are such that $t \ge 0$, then
$$M^2_{s,0} = H^2_s, \quad M^2_{0,s} = L^2_s, \quad \text{and} \quad M^2_t = L^2_t \cap H^2_t.$$

(See e. g. [4–6, 10–13, 17, 35, 36].)

We refer to [6, 11–13, 17, 27] for more facts about modulation spaces and $W^{p,q}_{(\omega)}$-spaces.

As anounced in the introduction we consider in Section 3 mapping properties for pseudo-differential operators when acting on certain types of coorbit spaces, which are defined by imposing certain types of translation invariant solid BF-space norms on the short-time Fourier transforms. (Cf. [6, 8, 11, 12].) This familly of coorbit spaces contains the modulation and Wiener amalgam spaces. In the following we recall the definition of these spaces.

Definition 2.4. Assume that $\mathscr{B}$ is a Banach space of complex-valued measurable functions on $\mathbf{R}^d$ and $v \in \mathscr{P}(\mathbf{R}^d)$. Then $\mathscr{B}$ is called a *translation invariant BF-space on* $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:

1. $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings);
2. if $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $\tau_x f \in \mathscr{B}$, and
$$\|\tau_x f\|_{\mathscr{B}} \le C v(x) \|f\|_{\mathscr{B}}; \tag{7}$$
3. if $f, g \in L^1_{loc}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$, then $f \in \mathscr{B}$ and
$$\|f\|_{\mathscr{B}} \le C \|g\|_{\mathscr{B}}.$$

Here the condition (3) in Definition 2.4 means that a translation invariant BF-space is a solid BF-space in the sense of (A.3) in [8]. It follows from this condition that if $f \in \mathscr{B}$ and $h \in L^\infty$, then $f \cdot h \in \mathscr{B}$, and
$$\|f \cdot h\|_{\mathscr{B}} \le C \|f\|_{\mathscr{B}} \|h\|_{L^\infty}. \tag{8}$$

Example 2.5. Assume that $p, q \in [1, \infty]$, and let $L^{p,q}_1(\mathbf{R}^{2d})$ be the set of all $f \in L^1_{loc}(\mathbf{R}^{2d})$ such that
$$\|f\|_{L^{p,q}_1} \equiv \Big(\int \Big(\int |f(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}$$
if finite. Also let $L^{p,q}_2(\mathbf{R}^{2d})$ be the set of all $f \in L^1_{loc}(\mathbf{R}^{2d})$ such that
$$\|f\|_{L^{p,q}_2} \equiv \Big(\int \Big(\int |f(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p}$$
is finite. Then it follows that $L^{p,q}_1$ and $L^{p,q}_2$ are translation invariant BF-spaces with respect to $v = 1$.

More generally, assume that $\omega, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate, and let $L^{p,q}_{j,(\omega)}(\mathbf{R}^{2d})$, for $j = 1, 2$, be the set of all $f \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|f\|_{L^{p,q}_{j,(\omega)}} \equiv \|f\,\omega\|_{L^{p,q}_j}$ is finite. Then $L^{p,q}_{j,(\omega)}$ is a translation invariant BF-space with respect to v.

Remark 2.6. The conclusion in the latter part of Example 2.5 is also a consequence of the first part in that example and the following observation. Assume that $\omega_0, v, v_0 \in \mathscr{P}(\mathbf{R}^d)$ are such that ω is v-moderate, and assume that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$ with respect to v_0. Also let $\mathscr{B}_0$ be the Banach space which consists of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}_0} \equiv \|f\,\omega\|_{\mathscr{B}}$ is finite. Then $\mathscr{B}_0$ is a translation invariant BF-space with respect to $v_0 v$.

For translation invariant BF-spaces we make the following observation.

Proposition 2.7. *Assume that $v \in \mathscr{P}(\mathbf{R}^d)$, and that $\mathscr{B}$ is a translation invariant BF-space with respect to v. Then the convolution mapping $(\varphi, f) \mapsto \varphi * f$ from $C_0^\infty(\mathbf{R}^d) \times \mathscr{B}$ to $\mathscr{S}'$ extends uniquely to a continuous mapping from $L^1_{(v)}(\mathbf{R}^d) \times \mathscr{B}$ to $\mathscr{B}$, and*

$$\|\varphi * f\|_{\mathscr{B}} \le C\|\varphi\|_{L^1_{(v)}}\|f\|_{\mathscr{B}},$$

for some constant C which is independent of $\varphi \in L^1_{(v)}$ and $f \in \mathscr{B}$.

Proposition 2.9 is a consequence of the results in [6, 8]. In order to be more self-contained we give here a short motivation.

Proof. First assume that $\varphi \in C_0^\infty$ and that $f \in \mathscr{B}$. Then Minkowski's inequality and (8) give

$$\begin{aligned}\|\varphi * f\|_{\mathscr{B}} &= \Big\| \int f(\,\cdot\, - y)\varphi(y)\, dy \Big\|_{\mathscr{B}} \\ &\le \int \|f(\,\cdot\, - y)\varphi(y)\|_{\mathscr{B}}\, dy = \int \|f(\,\cdot\, - y)\|_{\mathscr{B}} |\varphi(y)|\, dy \\ &\le C \int \|f\|_{\mathscr{B}}\, v(y)|\varphi(y)|\, dy = C\|f\|_{\mathscr{B}}\|\varphi\|_{L^1_{(v)}},\end{aligned}$$

which proves the result in this case. For general $\varphi \in L^1_{(v)}$, the result follows from the fact that C_0 is dense in $L^1_{(v)}$. □

Next we consider the general type of modulation spaces which we are especially interested in.

Definition 2.8. Assume that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and that $\chi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$. Then the modulation space $M_{(\omega)} = M_{(\omega)}(\mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M_{(\omega)}} = \|f\|_{M_{(\omega)}(\mathscr{B})} \equiv \|V_\chi f\, \omega\|_{\mathscr{B}}$$

is finite.

Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ is fix, and consider the familly of distribution spaces which consists of all spaces of the form $M_{(\omega)}(\mathscr{B})$ such that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$. Then it follows by Remark 2.6 that this familly is invariant under ω. Consequently we do not increase the set of possible spaces in Definition 2.8 by permitting ω that are not identically 1.

From this observation it seems to be superfluous to include the weight ω in Definition 2.8. However, it will be convenient for us to permit such ω dependency when investigating mapping properties for pseudo-differential operators in Section 3, when acting on spaces of the form $M_{(\omega)}(\mathscr{B})$.

Obviously, we have

$$M^{p,q}_{(\omega)}(\mathbf{R}^d) = M_{(\omega)}(\mathscr{B}_1) \quad \text{and} \quad W^{p,q}_{(\omega)}(\mathbf{R}^d) = M_{(\omega)}(\mathscr{B}_2)$$

when $\mathscr{B}_1 = L_1^{p,q}(\mathbf{R}^{2d})$ and $\mathscr{B}_2 = L_2^{p,q}(\mathbf{R}^{2d})$ (cf. Example 2.5). It follows that many properties which are valid for the modulation spaces also hold for the spaces of the form $M_{(\omega)}(\mathscr{B})$. For example we have the following proposition, which shows that the definition of $M_{(\omega)}(\mathscr{B})$ is independent of the choice of χ. We omit the proof since it can be found in e. g. [8,11,12]. It also follows by similar arguments as in the proof of Proposition 11.3.2 in [17].

Proposition 2.9. *Assume that $\mathscr{B}$ is a translation invariant BF-space with respect to $v_0 \in \mathscr{P}(\mathbf{R}^{2d})$ for $j = 1, 2$. Also assume that $\omega, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate, $M_{(\omega)}(\mathscr{B})$ is the same as in Definition 2.8, and let $\chi \in M^1_{(v_0 v)}(\mathbf{R}^d) \setminus 0$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. Then $f \in M_{(\omega)}(\mathscr{B})$ if and only if $V_\chi f\, \omega \in \mathscr{B}$, and different choices of χ gives rise to equivalent norms in $M_{(\omega)}(\mathscr{B})$.*

Next we recall some facts in Chapter XVIII in [22] concerning pseudo-differential operators. Assume that $t \in \mathbf{R}$ is fixed and that $a \in \mathscr{S}(\mathbf{R}^{2d})$. Then the pseudo-differential operator $a_t(x, D)$ is the continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$\begin{aligned}(a_t(x,D)f)(x) &= (\mathrm{Op}_t(a)f)(x) \\ &= (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi.\end{aligned} \tag{9}$$

The definition of $a_t(x, D)$ extends to any $a \in \mathscr{S}'(\mathbf{R}^{2d})$, and then $a_t(x, D)$ is continuous from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$. Moreover, for every fixed $t \in \mathbf{R}$, it follows that there is a one to one correspondance between such operators, and pseudo-differential operators of the form $a_t(x, D)$. (See e. g. [22].) If $t = 1/2$, then $a_t(x, D)$ is equal to the Weyl operator $a^w(x, D)$ for a. If instead $t = 0$, then the standard (Kohn-Nirenberg) representation $a(x, D)$ is obtained.

Consequently, for every $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $s, t \in \mathbf{R}$, there is a unique $b \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $a_s(x, D) = b_t(x, D)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$a_s(x, D) = b_t(x, D) \quad \Longleftrightarrow \quad b(x, \xi) = e^{i(t-s)\langle D_x, D_\xi\rangle} a(x, \xi). \tag{10}$$

(Cf. [22].)

In the next section we discuss continuity for pseudo-differential operators with symbols in $S_{(\omega)}(\mathbf{R}^{2d})$, the set of all smooth functions a on $\mathbf{R}^{2d}$ such that $\partial^\alpha a/\omega \in L^\infty(\mathbf{R}^{2d})$. Here $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. If $\omega = 1$, then we use the notation $S_0^0(\mathbf{R}^{2d})$ instead of $S_{(\omega)}(\mathbf{R}^{2d})$.

3 Continuity for pseudo-differential operators with symbols in $S_{(\omega)}$

In this section we discuss continuity for operators in $\mathrm{Op}(S_{(\omega_0)})$ when acting on modulation spaces. In the first part we prove in Theorem 3.2 below that if $\omega, \omega_0 \in \mathscr{P}$, $t \in \mathbf{R}$ and $a \in S_{(\omega)}$, then $a_t(x, D)$ is continuous from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M_{(\omega_0)}(\mathscr{B})$. In particular, Theorem 2.1 in [30] as well as Theorem 2.2 in [36] are covered.

In the second part we present some applications and prove that certain properties which are valid for Sobolev spaces carry over to modulation spaces.

We start by giving some remarks on $S_{(\omega)}(\mathbf{R}^{2d})$ when $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. By straight-forward computations it follows that $S_{(\omega)}(\mathbf{R}^{2d})$ agrees with $S(\omega, g)$ when $g_{(x,\xi)}(y,\eta) = |y|^2 + |\eta|^2$ is the standard euclidean metric on $\mathbf{R}^{2d}$. (See Section 18.4–18.6 in [22].) Since the metric g is constant it follows that it is trivially slowly varying and σ-temperate, where σ denotes the standard symplectic form on $\mathbf{R}^{2d}$. Moreover, from the fact that ω is σ_t-moderate when t is large enough, it follows by straight-forward computations that ω is σ,g-temperate. The following lemma is therefore a consequence of Theorem 18.5.10 in [22].

Lemma 3.1. *Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $s, t \in \mathbf{R}$, and that $a, b \in \mathscr{S}'(\mathbf{R}^{2d})$ are such that $a_s(x, D) = b_t(x, D)$. Then*

$$a \in S_{(\omega)}(\mathbf{R}^{2d}) \qquad \Longleftrightarrow \qquad b \in S_{(\omega)}(\mathbf{R}^{2d}).$$

We have now the following result.

Theorem 3.2. *Assume that $t \in \mathbf{R}$, $\omega, \omega_0 \in \mathscr{P}(\mathbf{R}^{2d})$, $a \in S_{(\omega)}(\mathbf{R}^{2d})$, $t \in \mathbf{R}$, and that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$. Then $a_t(x, D)$ is continuous from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M_{(\omega_0)}(\mathscr{B})$.*

We need some preparations for the proof, and start by recalling Minkowski's inequality in a somewhat general form. Assume that $d\mu$ is a positive measure, and that $f \in L^1(d\mu; \mathscr{B})$ for some Banach space $\mathscr{B}$. Then Minkowski's inequality asserts that

$$\Big\| \int f(x)\, d\mu(x) \Big\|_{\mathscr{B}} \le \int \|f(x)\|_{\mathscr{B}}\, d\mu(x).$$

We also need some lemmas.

Lemma 3.3. *Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $a \in S_{(\omega)}(\mathbf{R}^{2d})$, $f \in \mathscr{S}(\mathbf{R}^d)$, $\chi \in \mathscr{S}(\mathbf{R}^d)$, $\chi_2 = \sigma_s\chi$ and $0 \le s \in \mathbf{R}$. If*

$$\Phi(x, \xi, z, \zeta) = \frac{a(x + z, \xi + \zeta)}{\omega(x, \xi)\langle z\rangle^s \langle \zeta\rangle^s} \tag{11}$$

and

$$H(x, \xi, y) = \iint \overline{\Phi(x, \xi, z, \zeta)}\chi_2(z)\langle\zeta\rangle^s e^{i\langle y-x-z,\zeta\rangle}\, dzd\zeta,$$

then

$$V_\chi(a(\cdot, D)f)(x, \xi) = (2\pi)^{-d}(f, e^{i\langle\,\cdot\,,\xi\rangle} H(x, \xi,\, \cdot\,))\omega(x, \xi). \tag{12}$$

Proof. For simplicity we assume that a is real-valued. By straight-forward computations we get

$$\begin{aligned} V_\chi(a(\cdot, D)f)(x, \xi) &= (a(\cdot, D)f, \tau_x\chi\, e^{i\langle\cdot,\xi\rangle}) \\ &= (f, a(\cdot, D)^*(\tau_x\chi\, e^{i\langle\cdot,\xi\rangle})) \\ &= (2\pi)^{-d}(f, e^{i\langle\,\cdot\,,\xi\rangle}\widetilde{H}(x, \xi,\, \cdot\,))\omega(x, \xi), \end{aligned} \tag{13}$$

where

$$\widetilde{H}(x,\xi,y) = (2\pi)^d e^{-i\langle y,\xi\rangle}(a(\cdot,D)^*(\tau_x\chi\, e^{i\langle\cdot,\xi\rangle}))(y)/\omega(x,\xi)$$
$$= \iint \frac{a(z,\zeta)}{\omega(x,\xi)}\chi(z-x)e^{i\langle y-z,\zeta-\xi\rangle}\,dzd\zeta$$
$$= \iint \Phi(x,\xi,z-x,\zeta-\xi)\chi_2(z-x)\langle\zeta-\xi\rangle^s e^{i\langle y-z,\zeta-\xi\rangle}\,dzd\zeta.$$

If $z-x$ and $\zeta-\xi$ are taken as new variables of integrations, it follows that the right-hand side is equal to $H(x,y,\xi)$. This proves the assertion. □

Lemma 3.4. *Let $s\ge 0$ be an even integer, Φ and H be the same as in Lemma 3.3, and set*

$$\Phi_\beta(x,\xi,z,\zeta) = \partial_z^\beta\Phi(x,\xi,z,\zeta),\qquad \chi_{2,\gamma} = \partial^\gamma\chi_2. \tag{14}$$

Also let $\Psi_\beta(x,\xi,y,\,\cdot\,)$ be the inverse partial Fourier transform of $\overline{\Phi_\beta(x,\xi,y,\eta)}$ with respect to the η variable, and let

$$H_{\beta,\gamma}(x,\xi,y) = \int \Psi_\beta(x,\xi,y-z-x,z)\chi_{2,\gamma}(y-z-x)\,dz. \tag{15}$$

Then there are constants $C_{\beta,\gamma}$ which depend on β, γ, s and d only such that

$$H(x,\xi,y) = \sum_{|\beta+\gamma|\le s} C_{\beta,\gamma}H_{\beta,\gamma}(x,\xi,y).$$

Proof. By integrating by parts we get

$$H(x,\xi,y) = \iint \overline{\Phi(x,\xi,z,\zeta)}\chi_2(z)\langle\zeta\rangle^{s/2}e^{i\langle y-x-z,\zeta\rangle}\,dzd\zeta$$
$$= \iint \overline{\Phi(x,\xi,z,\zeta)}\chi_2(z)(1-\Delta_z)^{s/2}(e^{i\langle y-x-z,\zeta\rangle})\,dzd\zeta$$
$$= \sum_{|\beta+\gamma|\le N} C_{\beta,\gamma}\widetilde{H}_{\beta,\gamma}(x,\xi,y),$$

where

$$\widetilde{H}_{\beta,\gamma}(x,\xi,y) = (2\pi)^{-d/2}\iint \overline{\Phi_\beta(x,\xi,z,\zeta)}\chi_{2,\gamma}(z)e^{i\langle y-x-z,\zeta\rangle}\,dzd\zeta.$$

If we take $y-x-z$ and ζ as new variables of integrations, and perform the integration with respect to the ζ variable, it follows that $\widetilde{H}_{\beta,\gamma} = H_{\beta,\gamma}$, which gives the result. □

For the next lemma we observe that if $f\in\mathscr{S}'(\mathbf{R}^d)$ is fixed, then there are positive constants s_0, N and C_0 such that

$$|V_{\chi_0}f(x,\xi)| \le C_0\langle x,\xi\rangle^N,\quad \text{where}\quad \chi_0=\sigma_{-s}\quad\text{and}\quad s\ge s_0. \tag{16}$$

Lemma 3.5. *Assume that* $\omega \in \mathscr{P}(\mathbf{R}^{2d})$*,* $a \in S_{(\omega)}(\mathbf{R}^{2d})$*,* $\varphi \in \mathscr{S}(\mathbf{R}^d)$*, and* $f \in \mathscr{S}'(\mathbf{R}^d)$*. If* $s \ge 0$ *is large enough and* $\chi_0 = \sigma_{-s}$*, then there is a constant* C *such that*

$$|V_\varphi(a_t(\,\cdot\,,D)f)(x,\xi)| \le C(F(x,\,\cdot\,) * \chi_0)(\xi), \tag{17}$$

where

$$F(x,\xi) = |V_{\chi_0} f(x,\xi)\omega(x,\xi)|. \tag{18}$$

Proof. It is no restriction to assume that a is real-valued, and by Lemmas 2.1 and 3.1 it follows that we may assume that $t = 0$ and that $\omega \in \mathscr{P}_0$. Furthermore, by Lemma 3.3, Lemma 3.4 and (13), the result follows if we prove the following:

1. the right-hand side of (12) is well-defined for the fixed $f \in \mathscr{S}'(\mathbf{R}^d)$ when s is chosen large enough, and that (12) holds also in this case;

2. for each multi-indices β and γ, there is a constant C such that

$$I_{\beta,\gamma}(x,\xi) \equiv |(f, e^{i\langle\,\cdot\,,\xi\rangle} H_{\beta,\gamma}(x,\xi,\,\cdot\,))\omega(x,\xi)| \le C(F(x,\,\cdot\,) * \sigma_{-s})(\xi). \tag{9}$$

Let C_0, s_0 and N be chosen such that (16) is fulfilled, let N_1 be an even and large integer, and let Φ_β be as in (14). The assertion (1) follows if we prove that for each multi-indices α and β, there is a constant $C_{\alpha,\beta} = C_{N_1,\alpha,\beta}$ such that

$$|(\partial^\alpha \Phi_\beta)(x,\xi,z,\zeta)| \le C_{\alpha,\beta}\langle z\rangle^{-N_1}\langle\zeta\rangle^{-N_1}. \tag{19}$$

In order to prove (19) we choose $M \ge 0$ and $s \ge M + N_1$ such that $\omega \in \mathscr{P}_0$ is σ_M-moderate, and assume first that $\alpha = \beta = 0$. Then (11) and the facts that $a \in S_{(\omega)}$ give

$$|\Phi(x,\xi,z,\zeta)| = \frac{|a(x+z,\xi+\zeta)|}{\omega(x,\xi)\langle z\rangle^s\langle\zeta\rangle^s} \le C_1\frac{|a(x+z,\xi+\zeta)|\langle z,\zeta\rangle^M}{\omega(x+z,\xi+\zeta)\langle z\rangle^s\langle\zeta\rangle^s} \le C_2\langle z\rangle^{-N_1}\langle\zeta\rangle^{-N_1}.$$

For general α and β, (19) follows from these computations in combination with Leibnitz rule, using the facts that $(\partial^\gamma a)/\omega \in L^\infty$ and $(\partial^\gamma \omega)/\omega \in L^\infty$ for each multi-index γ. This gives (1).

Assume that $N_2 \ge 0$ is arbitrary. Then it follows by choosing N_1 in (19) large enough, that for some constant C it holds

$$|\partial^\alpha \Psi_\beta(x,\xi,y-z,z)| \le C\langle y\rangle^{-N_2}\langle z\rangle^{-N_2} \tag{20}$$

for every multi-index α such that $|\alpha| \le N_2$.

If N_3 is a fixed integer, then it follows from (15) and (20) that

$$H_{\beta,\gamma}(x,\xi,y) = \sigma_{-N_3}(y-x)\overline{\varphi_{\beta,\gamma}(x,\xi,y-x)}, \tag{21}$$

where $\varphi_{\beta,\gamma}$ satisfies

$$|\partial^\alpha \varphi_{\beta,\gamma}(x,\xi,y)| \le C\langle y\rangle^{-N_3}, \quad |\alpha| \le N_3,$$

for some constant C, provided N_2 was chosen large enough. Hence, for any fixed $s \ge 0$, it follows by choosing N_3 large enough that

$$|\mathscr{F}(\varphi_{\beta,\gamma}(x,\xi,\,\cdot\,))(\eta)| \le C\langle\eta\rangle^{-s}, \tag{22}$$

for some constant C.

By choosing $s > d$, it follows from (21), (22) and straight-forward computations that

$$\begin{aligned} I_{\beta,\gamma}(x,\xi) &= |(f, e^{i\langle\cdot,\xi\rangle}\chi_0(\cdot - x)\overline{\varphi_{\beta,\gamma}(x,\xi,\cdot - x)})| \\ &= |\mathscr{F}((f\,\tau_x\chi_0)\varphi_{\beta,\gamma}(x,\xi,\,\cdot\, - x))(\xi)| \\ \le (2\pi)^{-d/2} &\int |\mathscr{F}(f\,\tau_x\chi_0)(\xi-\eta)||\mathscr{F}(\varphi_{\beta,\gamma}(x,\xi,\,\cdot\, - x))(\eta)|\,d\eta, \\ &\le C\int |V_{\chi_0}f(x,\xi-\eta)|\chi_0(\eta)\,d\eta, \end{aligned} \tag{23}$$

where

$$\begin{aligned} C &= (2\pi)^{-d/2}\int \sup_{x,\xi} |\big(\mathscr{F}(\varphi_{\beta,\gamma}(x,\xi,\,\cdot\, - x))(\eta)|\big)\,d\eta \\ &= (2\pi)^{-d/2}\int \sup_{x,\xi} |\big(\mathscr{F}(\varphi_{\beta,\gamma}(x,\xi,\,\cdot\,))(\eta)|\big)\,d\eta \\ &\le C_1\int \langle\eta\rangle^{-s}\,d\eta < \infty. \end{aligned}$$

This gives (17), and the proof is complete. □

Proof of Theorem 3.2. We use the same notations as in Lemma 3.5, and set

$$G = |V_\chi(a_t(\,\cdot\,,D)f)|.$$

Since $\omega_0 \in \mathscr{P}$, it follows that $\omega_0(x,\xi) \le C\omega_0(x,\xi-\eta)\langle\eta\rangle^{s_0}$, for some constants C and s_0. By Lemma 3.5 we get

$$\begin{aligned} G(x,\xi)\omega_0(x,\xi) &\le C_1\int F(x,\xi-\eta)\langle\eta\rangle^{-s}\omega_0(x,\xi)\,d\eta \\ &\le C_2\int F(x,\xi-\eta)\omega_0(x,\xi-\eta)\langle\eta\rangle^{s_0-s}\,d\eta, \\ &= C_2\int F_{\eta,\omega_0}(x,\xi)\langle\eta\rangle^{s_0-s}\,d\eta, \end{aligned}$$

for some constants C_1 and C_2, where

$$F_{\eta,\omega_0}(x,\xi) = F(x,\xi-\eta)\omega_0(x,\xi-\eta).$$

Now choose $s_1, s_2 \in \mathbf{R}$ in such way that $s_1 = s - s_0$ and $\mathscr{B}$ is a translation invariant BF-space with respect to σ_{s_2}, and let $\omega_1 = \omega_0\omega$. Then it follows for some constant C and Minkowki's inequality that

$$\|a_t(x,D)f\|_{M_{(\omega_0)}(\mathscr{B})} = \|G\omega_0\|_{\mathscr{B}} \le C_1 \int \|F_{\eta,\omega_0}\|_{\mathscr{B}} \langle \eta \rangle^{-s_1}\, d\eta$$

$$\le C_2 \int \|F\,\omega_0\|_{\mathscr{B}} \langle \eta \rangle^{s_2 - s_1}\, d\eta = C_3 \|f\|_{M_{(\omega_0\omega)}(\mathscr{B})},$$

where

$$C_3 = C_2 \|\sigma_{s_2 - s_1}\|_{L^1}.$$

Since s can be chosen arbitrary large, it follows that s_1 can be chosen larger than $s_2 + d$, which implies that $C_3 < \infty$. This gives the result. □

Next we show that [36, Theorem 2.2] is essentially a consequence of Theorem 3.2.

Corollary 3.6. *Assume that $t \in \mathbf{R}$, $\omega \in \mathscr{P}_0(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}(\mathbf{R}^{2d})$ are such that $\omega(x,\xi) = \omega(x)$ or $\omega(x,\xi) = \omega(\xi)$, and that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$. Then $\omega_t(x,D)$ is a homeomorphism from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M_{(\omega_0)}(\mathscr{B})$.*

Proof. Since it follows from the assumptions that $\omega \in S_{(\omega)}$, Theorem 3.2 shows that $\omega_t(x,D)$ is continuous from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M^{p,q}_{(\omega)}(\mathscr{B})$. On the other hand, since $\omega(x,\xi) = \omega(x)$ or $\omega(x,\xi) = \omega(\xi)$, it follows that the inverse of $\omega_t(x,D)$ on $\mathscr{S}'(\mathbf{R}^d)$ is equal to $(1/\omega)_t(x,D)$. Hence Theorem 3.2 together with the obvious fact that $1/\omega \in \mathscr{P}_0$ give

$$\|f\|_{M_{(\omega_0\omega)}(\mathscr{B})} = \|(1/\omega)_t(x,D)(\omega_t(x,D)f)\|_{M_{(\omega_0\omega)}(\mathscr{B})}$$

$$\le C\|\omega_t(x,D)f\|_{M_{(\omega_0)}(\mathscr{B})}$$

for some constant C. This proves that $\omega_t(x,D)$ is a bijective map from $M_{(\omega_0\omega)}(\mathscr{B})$ to $M_{(\omega_0)}(\mathscr{B})$, and the result follows. □

Remark 3.7. We remark that an immediate consequence of Corollary 3.6 is that if $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega(x,\xi) = \omega_1(x)\omega_2(\xi)$ where $\omega_j \in \mathscr{P}_0(\mathbf{R}^d)$ for $j = 1,2$, and $\omega_0 \in \mathscr{P}(\mathbf{R}^{2d})$, then

$$M_{(\omega_0\omega)}(\mathscr{B}) = \{\, f \in \mathscr{S}'(\mathbf{R}^d)\,;\, \omega_1(x)\omega_2(D)f \in M_{(\omega_0)}(\mathscr{B})\,\}$$

$$= \{\, f \in \mathscr{S}'(\mathbf{R}^d)\,;\, \omega_2(D)(\omega_1 f) \in M_{(\omega_0)}(\mathscr{B})\,\}.$$

In particular, if $s,t \in \mathbf{R}$, $\mathscr{B} = L^{p,q}_1$ or $\mathscr{B} = L^{p,q}_2$, and $\omega(x,\xi) = \sigma_{s,t}(x,\xi) = \langle x\rangle^t \langle \xi \rangle^s$, then

$$M^{p,q}_{(\sigma_{s,t}\omega_0)}(\mathbf{R}^d) = \{\, f \in \mathscr{S}'(\mathbf{R}^d)\,;\, \langle x\rangle^t \langle D\rangle^s f \in M^{p,q}_{(\omega_0)}(\mathbf{R}^d)\,\}$$

$$= \{\, f \in \mathscr{S}'(\mathbf{R}^d)\,;\, \langle D\rangle^s (\langle\,\cdot\,\rangle^t f) \in M^{p,q}_{(\omega_0)}(\mathbf{R}^d)\,\}$$

and

$$
\begin{aligned}
W^{p,q}_{(\sigma_{s,t}\omega_0)}(\mathbf{R}^d) &= \{\, f \in \mathscr{S}'(\mathbf{R}^d)\, ;\, \langle x\rangle^t \langle D\rangle^s f \in W^{p,q}_{(\omega_0)}(\mathbf{R}^d)\,\}\\
&= \{\, f \in \mathscr{S}'(\mathbf{R}^d)\, ;\, \langle D\rangle^s (\langle\,\cdot\,\rangle^t f) \in W^{p,q}_{(\omega_0)}(\mathbf{R}^d)\,\}.
\end{aligned}
$$

Remark 3.8. For certain ω it is possible to use Remark 2.12 in [36] to prove that the continuity assertions in Theorem 3.2 also holds when the symbols for the pseudo-differential operators belong to $M^{\infty,1}_{(\omega)}(\mathbf{R}^{2d})$.

Note that $\sigma_{s,t}(x,D)$ here above, appears frequently in harmonic analysis and in the pseudo-differential calculus. For example, if $p \in [1,\infty]$, then recall that $f \in \mathscr{S}'(\mathbf{R}^d)$ belongs to the Sobolev space $H^p_s(\mathbf{R}^d)$ if and only if $\|f\|_{H^p_s} \equiv \|\sigma_s(D)f\|_{L^p}$ is finite. It is well-known that if $s = N$ is a positive integer and $1 < p < \infty$, then H^p_s agrees with

$$
\{\, f \in L^p\, ;\, \partial^\alpha f \in L^p \text{ when } |\alpha| \le N\,\}.
$$

(See [2].)

In the following theorems we prove that similar properties in a somewhat extended form also hold for general spaces of the form $M_{(\omega)}(\mathscr{B})$.

Theorem 3.9. *Assume that $N_1, N_2 \ge 0$ are integers, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}$ is a translation invariant BF-spaces on $\mathbf{R}^{2d}$, and assume that $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following conditions are equivalent:*

1. *$f \in M_{(\sigma_{N_1,N_2}\omega)}(\mathscr{B})$;*
2. *$x^\beta \partial^\alpha f \in M_{(\omega)}(\mathscr{B})$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
3. *$\partial^\alpha (x^\beta f) \in M_{(\omega)}(\mathscr{B})$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
4. *$f, x_j^{N_2} f,\ \partial_k^{N_1} f,\ x_j^{N_2}\partial_k^{N_1} f \in M_{(\omega)}(\mathscr{B})$ for all $1 \le j,k \le d$;*
5. *$f, x_j^{N_2} f,\ \partial_k^{N_1} f,\ \partial_k^{N_1}(x_j^{N_2} f) \in M_{(\omega)}(\mathscr{B})$ for all $1 \le j,k \le d$.*

Proof. We only prove the equivalences

$$
(1) \quad \Longleftrightarrow \quad (2) \quad \Longleftrightarrow \quad (4).
$$

The equivalences

$$
(1) \quad \Longleftrightarrow \quad (3) \quad \Longleftrightarrow \quad (5)
$$

follow by similar arguments and are left for the reader.

Let M_0 be the set of all $f \in M_{(\omega)}(\mathscr{B})$ such that $x^\beta \partial^\alpha f \in M_{(\omega)}(\mathscr{B})$ when $|\alpha| \le N_1$ and $|\beta| \le N_2$, and let $\widetilde{M}_0$ be the set of all $f \in M_{(\omega)}(\mathscr{B})$ such that

$$x_j^{N_2} f, \partial_k^{N_1} f, x_j^{N_2}\partial_k^{N_1} f \in M_{(\omega)}(\mathscr{B})$$

for $j,k = 1,\dots,d$. We shall prove that $M_0 = \widetilde{M}_0 = M_{(\sigma_{N_1,N_2}\omega)}(\mathscr{B})$. Obviously, $M_0 \subseteq \widetilde{M}_0$. Since the symbol ξ^α of the operator D^α belongs to $S_{(\sigma_{N_1,N_2})}$ when $|\alpha| \le N$, it follows from Theorem 3.2 that the embedding $M_{(\sigma_{N_1,N_2}\omega)}(\mathscr{B}) \subseteq M_0$ holds. The result therefore follows if we prove that $\widetilde{M}_0 \subseteq M_{(\sigma_{N_1,N_2}\omega)}(\mathscr{B})$.

In order to prove this, assume first that $N_1 = N$, $N_2 = 0$, $f \in \widetilde{M}_0$, and choose open sets

$$\Omega_0 = \{\, \xi \in \mathbf{R}^d\, ;\, |\xi| < 2\,\}, \quad \text{and} \quad \Omega_j = \{\, \xi \in \mathbf{R}^d\, ;\, 1 < |\xi| < d|\xi_j|\,\}.$$

Then $\bigcup_{j=0}^d \Omega_j = \mathbf{R}^d$, and there are non-negative functions $\varphi_0,\dots,\varphi_d$ in S_0^0 such that $\operatorname{supp}\varphi_j \subseteq \Omega_j$ and $\sum_{j=0}^d \varphi_j = 1$. In particular, $f = \sum_{j=0}^d f_j$ when $f_j = \varphi_j(D)f$. The result follows if we prove that $f_j \in M_{(\sigma_{N,0}\omega)}(\mathscr{B})$ for every j.

Now set $\psi_0(\xi) = \sigma_N(\xi)\varphi_0(\xi)$ and $\psi_j(\xi) = \xi_j^{-N}\sigma_N(\xi)\varphi_j(\xi)$ when $j = 1,\dots,d$. Then $\psi_j \in S_0^0$ for every j. Hence Theorem 3.2 gives

$$\begin{aligned}\|f_j\|_{M_{(\sigma_{N,0}\omega)}(\mathscr{B})} &\le C_1\|\sigma_N(D)f_j\|_{M_{(\omega)}(\mathscr{B})}\\ &= C_1\|\psi_j(D)\partial_j^N f\|_{M_{(\omega)}(\mathscr{B})} \le C_2\|\partial_j^N f\|_{M_{(\omega)}(\mathscr{B})} < \infty\end{aligned}$$

and

$$\begin{aligned}\|f_0\|_{M_{(\sigma_{N,0}\omega)}(\mathscr{B})} &\le C_1\|\sigma_N(D)f_0\|_{M_{(\omega)}(\mathscr{B})}\\ &= C_1\|\psi_0(D)f\|_{M_{(\omega)}(\mathscr{B})} \le C_2\|f\|_{M_{(\omega)}(\mathscr{B})} < \infty\end{aligned}$$

for some constants C_1 and C_2. This proves that

$$\|f\|_{M_{(\sigma_{N,0}\omega)}(\mathscr{B})} \le C\Big(\|f\|_{M_{(\omega)}(\mathscr{B})} + \sum_{j=1}^d \|\partial_j^N f\|_{M_{(\omega)}(\mathscr{B})}\Big), \tag{24}$$

and the result follows in this case.

If we instead split up f into $\sum \varphi_j f$, then similar arguments show that

$$\|f\|_{M_{(\sigma_{0,N}\omega)}(\mathscr{B})} \le C\Big(\|f\|_{M_{(\omega)}(\mathscr{B})} + \sum_{k=1}^d \|x_k^N f\|_{M_{(\omega)}(\mathscr{B})}\Big), \tag{25}$$

and the result follows in the case $N_1 = 0$ and $N_2 = N$ from this estimate.

The general case now follows if we combine (24) with (25). The proof is complete. □

We finish the section by stating the previous results in the special cases of modulation spaces and corresponding Wiener amalgam related spaces. In fact, by letting $\mathscr{B} = L_1^{p,q}$ or $\mathscr{B} = L_2^{p,q}$, the following results are immediate consequences of the previous ones.

Theorem 3.2′. *Assume that $\omega, \omega_0 \in \mathscr{P}(\mathbf{R}^{2d})$, $a \in S_{(\omega)}(\mathbf{R}^{2d})$, $t \in \mathbf{R}$, and that $p,q \in [1,\infty]$. Then $a_t(x,D)$ is continuous from $M^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$, and from $W^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)$ to $W^{p,q}_{(\omega_0)}(\mathbf{R}^d)$.*

We note that if $t = 0$ and $\omega_0 = \sigma_{s_1,s_2}$ where $s_1, s_2 \in \mathbf{R}$, then Theorem 3.2′ agrees with Theorem 1.1 in [30].

Corollary 3.6′. *Assume that $\omega \in \mathscr{P}_0(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}(\mathbf{R}^{2d})$ are such that $\omega(x,\xi) = \omega(x)$ or $\omega(x,\xi) = \omega(\xi)$, and that $p,q \in [1,\infty]$. Then $\omega_t(x,D)$ is a homeomorphism from $M^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$, and from $W^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)$ to $W^{p,q}_{(\omega_0)}(\mathbf{R}^d)$.*

Theorem 3.9′. *Assume that $N_1, N_2 \ge 0$ are integers, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $p,q \in [1,\infty]$, and that $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following conditions are equivalent:*

1. *$f \in M^{p,q}_{(\sigma_{N_1,N_2}\omega)}(\mathbf{R}^d)$;*
2. *$x^\beta \partial^\alpha f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
3. *$\partial^\alpha (x^\beta f) \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
4. *$f, x_j^{N_2} f$, $\partial_k^{N_1} f$, $x_j^{N_2} \partial_k^{N_1} f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all $1 \le j,k \le d$;*
5. *$f, x_j^{N_2} f$, $\partial_k^{N_1} f$, $\partial_k^{N_1}(x_j^{N_2} f) \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all $1 \le j,k \le d$.*

Theorem 3.9″. *Assume that $N_1, N_2 \ge 0$ are integers, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $p,q \in [1,\infty]$, and that $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following conditions are equivalent:*

1. *$f \in W^{p,q}_{(\sigma_{N_1,N_2}\omega)}(\mathbf{R}^d)$;*
2. *$x^\beta \partial^\alpha f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
3. *$\partial^\alpha (x^\beta f) \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all multi-indices α and β such that $|\alpha| \le N_1$ and $|\beta| \le N_2$;*
4. *$f, x_j^{N_2} f$, $\partial_k^{N_1} f$, $x_j^{N_2} \partial_k^{N_1} f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all $1 \le j,k \le d$;*
5. *$f, x_j^{N_2} f$, $\partial_k^{N_1} f$, $\partial_k^{N_1}(x_j^{N_2} f) \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ for all $1 \le j,k \le d$.*

The following result was presented in [39, Remark 1.3]. Since the facts here do not seems to be well-known, we give some explicit motivations.

Corollary 3.10. *Assume that $p,q \in [1,\infty]$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ is such that $\omega(x,\xi) = \omega(x)$. Then the following is true:*

1. $M^{p,q}_{(\omega)}(\mathbf{R}^d) \hookrightarrow C(\mathbf{R}^d)$ *if and only if* $q = 1$;
2. $W^{p,q}_{(\omega)}(\mathbf{R}^d) \hookrightarrow C(\mathbf{R}^d)$ *if and only if* $q = 1$.

Proof. By Corollary 3.6′ it follows that we may assume that $\omega = 1$. If $f \in W^{\infty,1}$, then it follows that $\mathscr{F}(f\varphi) \in L^1$ for every $\varphi \in \mathscr{S}$, which implies that $f\varphi$ is a continuous function. Since $\varphi \in \mathscr{S}$ is arbitrary chosen, it follows that f is continuous. This gives

$$M^{p,1} \subseteq W^{p,1} \subseteq W^{\infty,1} \subseteq C, \tag{26}$$

which proves one part of the assertion.

Next assume that $q > 1$, and let f be the characteristic function of the cube $[0,1]^d$. Then $f \notin C$, and it follows by straight-forward computations that $f \in W^{1,q} \subseteq M^{1,q}$. Since $M^{p,q}$ and $W^{p,q}$ increases with the parameters p and q, it follows that

$$M^{p,q} \nsubseteq C, \quad \text{and} \quad W^{p,q} \nsubseteq C, \quad \text{when} \quad q > 1. \tag{27}$$

Hence (26) and (27) give the result. □

Remark 3.11. By using techniques of ultra-distributions, Pilipović and Teofanov prove in [24–26, 31, 32] parallel results comparing to Theorem 3.2′. Here they consider generalized modulation spaces, where less growth restrictions are assumed on the weight function ω. It is for example not necessary that ω should be bounded by polynomials.

Received: May 2008. Revised: September 2008.

References

[1] BAOXIANG, W. AND CHUNYAN, H., *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations*, J. Differential Equations, **239** (2007), 213–250.

[2] BERGH, J. AND LÖFSTRÖM, J., *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin Heidelberg NewYork, 1976.

[3] FEICHTINGER, H. G., *Un espace de Banach de distributions temperees sur les groupes localement compacts abeliens (French)*, C. R. Acad. Sci. Paris SŐr. A-B 290 **17** (1980), A791–A794.

[4] Feichtinger, H. G., *Banach spaces of distributions of Wiener's type and interpolation,* in: Ed. P. Butzer, B. Sz. Nagy and E. Görlich (Eds), Proc. Conf. Oberwolfach, Functional Analysis and Approximation, August 1980, Int. Ser. Num. Math. **69** Birkhäuser Verlag, Basel, Boston, Stuttgart, 1981, pp. 153–165.

[5] Feichtinger, H. G., *Banach convolution algebras of Wiener's type,* in: Proc. Functions, Series, Operators in Budapest, Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co., Amsterdam Oxford NewYork, 1980.

[6] Feichtinger, H. G., *Modulation spaces on locally compact abelian groups. Technical report*, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Hagpur Ahmedabad Bangalore Hyderbad Lucknow, 2003, pp.99–140.

[7] Feichtinger, H. G., *Atomic characterizations of modulation spaces through Gabor-type representations, in: Proc. Conf. on Constructive Function Theory,* Rocky Mountain J. Math. **19** (1989), 113–126.

[8] Feichtinger, H. G., *Wiener amalgams over Euclidean spaces and some of their applications,* in: Function spaces (Edwardsville, IL, 1990), Lect. Notes in pure and appl. math., **136**, Marcel Dekker, New York, 1992, pp. 123–137.

[9] Feichtinger, H. G., *Modulation spaces: Looking back and ahead*, Sampl. Theory Signal Image Process. **5** (2006), 109–140.

[10] Feichtinger, H. G. and Gröbner, P., *Banach Spaces of Distributions Defined by Decomposition Methods, I*, Math. Nachr. **123** (1985), 97–120.

[11] Feichtinger, H. G. and Gröchenig, K. H., *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal. **86** (1989), 307–340.

[12] Feichtinger, H. G. and Gröchenig, K. H., *Banach spaces related to integrable group representations and their atomic decompositions, II*, Monatsh. Math. **108** (1989), 129–148.

[13] Feichtinger, H. G. and Gröchenig, K. H., *Gabor frames and time-frequency analysis of distributions*, J. Functional Anal. (2) **146** (1997), 464–495.

[14] Folland, G. B., *Harmonic analysis in phase space*, Princeton U. P., Princeton, 1989.

[15] Gröbner, P., *Banachräume Glatter Funktionen und Zerlegungsmethoden, Thesis*, *University of Vienna*, Vienna, 1992.

[16] Gröchenig, K. H., *Describing functions: atomic decompositions versus frames*, Monatsh. Math.**112** (1991), 1–42.

[17] Gröchenig, K. H., *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[18] Gröchenig, K. H. and Heil, C., *Modulation spaces and pseudo-differential operators*, Integral Equations Operator Theory (4) **34** (1999), 439–457.

[19] Gröchenig, K. H. and Heil, C., *Modulation spaces as symbol classes for pseudodifferential operators* in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Hagpur Ahmedabad Bangalore Hyderbad Lucknow, 2003, pp. 151–170.

[20] Gröchenig, K. H. and Heil, C., *Counterexamples for boundedness of pseudodifferential operators*, Osaka J. Math. **41** (2004), 681–691.

[21] Gröchenig, K. and Leinert, M., *Wiener's lemma for twisted convolution and Gabor frames*, J. Amer. Math. Soc. (1) **17** (2004), 1–18.

[22] Hörmander, L., *The Analysis of Linear Partial Differential Operators*, vol I, III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[23] Okoudjou, K., *Embeddings of some classical Banach spaces into modulation spaces*, Proc. Amer. Math. Soc., **132** (2004), 1639–1647.

[24] Pilipović, S. and Teofanov, N., *Wilson Bases and Ultramodulation Spaces*, Math. Nachr. **242** (2002), 179–196.

[25] Pilipović, S. and Teofanov, N., *On a symbol class of Elliptic Pseudodifferential Operators*, Bull. Acad. Serbe Sci. Arts **27** (2002), 57–68.

[26] Pilipović, S. and Teofanov, N., *Pseudodifferential operators on ultra-modulation spaces*, J. Funct. Anal.**208** (2004), 194–228.

[27] Ruzhansky, M. Sugimoto, M. Tomita, N. and Toft, J., *Changes of variables in modulation and Wiener amalgam spaces*, Preprint, 2008, Available at arXiv:0803.3485v1.

[28] Sjöstrand, J., *An algebra of pseudodifferential operators*, Math. Res. L. **1** (1994), 185–192.

[29] Sugimoto, M. and Tomita, N., *The dilation property of modulation spaces and their inclusion relation with Besov Spaces*, J. Funct. Anal. (1), **248** (2007), 79–106.

[30] Tachizawa, K., *The boundedness of pseudo-differential operators on modulation spaces*, Math. Nachr. **168** (1994), 263–277.

[31] Teofanov, N., *Ultramodulation spaces and pseudodifferential operators*, Endowment Andrejević, Beograd, 2003.

[32] Teofanov, N., *Modulation spaces, Gelfand-Shilov spaces and pseudodifferential operators*, Sampl. Theory Signal Image Process, **5** (2006), 225–242.

[33] Toft, J., *Continuity and Positivity Problems in Pseudo-Differential Calculus, Thesis*, Department of Mathematics, University of Lund, Lund, 1996.

[34] Toft, J., *Subalgebras to a Wiener type Algebra of Pseudo-Differential operators*, Ann. Inst. Fourier (5) **51** (2001), 1347–1383.

[35] Toft, J., *Continuity properties for modulation spaces with applications to pseudo-differential calculus, I*, J. Funct. Anal. (2), **207** (2004), 399–429.

[36] Toft, J., *Continuity properties for modulation spaces with applications to pseudo-differential calculus, II*, Ann. Global Anal. Geom., **26** (2004), 73–106.

[37] Toft, J., *Convolution and embeddings for weighted modulation spaces in: P. Boggiatto, R. Ashino, M. W. Wong (eds) Advances in Pseudo-Differential Operators,* Operator Theory: Advances and Applications **155**, Birkhäuser Verlag, Basel 2004, pp. 165–186.

[38] Toft, J., *Continuity and Schatten-von Neumann Properties for Pseudo-Differential Operators and Toeplitz operators on Modulation Spaces*, The Erwin Schrödinger International Institute for Mathematical Physics, Preprint ESI **1732** (2005).

[39] Toft, J., *Continuity and Schatten properties for pseudo-differential operators on modulation spaces in: J. Toft, M. W. Wong, H. Zhu (Eds) Modern Trends in Pseudo-Differential Operators,* Operator Theory: Advances and Applications **172**, Birkhäuser Verlag, Basel, 2007, pp. 173–206.