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Some General Theorems on Uniform Boundedness for Functional Differential Equations

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ABSTRACT

Consider the functional differential equation with bounded delay

$$X' = F(t, X_t), \;\; X \in \mathbf{R^n}.$$

We discuss uniform boundedness and uniform ultimate boundedness by Liapunov's second method with conditions such as:

(i) $W_1(|X(t)|) \leq V(t, X_t) \leq W_2(|X(t)| + \int_{t-h}^{t} D(u, X_u)du)$;

(ii) $V(t, \phi) \leq W_3(\|\phi\|)$;

(iii) $V'_{(1)}(t, X_t) \leq -\gamma_1(t)W_4(m(X_t)) - \gamma_2(t)W_5(\int_{t-h}^{t} D(u, X_u)du) + M$;

where $m(\phi) = \min_{-h \leq s \leq 0} |\phi(s)|$.

The theorem discussed in this paper generalizes some results on uniform boundedness and uniform ultimate boundedness for functional differential equations with bounded delay. Some examples are also discussed in this paper.

RESUMEN

Considere la ecuación diferencial con retardo acotado

$$X' = F(t, X_t), \; X \in \mathbf{R^n}.$$

Discutimos acotamiento uniforme y acotamiento uniforme definitivo mediante el segundo metodo de Liapunov con las condiciones:

(i) $W_1(|X(t)|) \leq V(t, X_t) \leq W_2(|X(t)| + \int_{t-h}^{t} D(u, X_u)du)$;

(ii) $V(t, \phi) \leq W_3(\|\phi\|)$;

(iii) $V'_{(1)}(t, X_t) \leq -\gamma_1(t)W_4(m(X_t)) - \gamma_2(t)W_5(\int_{t-h}^{t} D(u, X_u)du) + M$;

donde $m(\phi) = \min_{-h \leq s \leq 0} |\phi(s)|$.

El teorema discutido en este artículo generaliza algunos resultados de acotamiento uniforme y acotamiento uniforme definitivo para ecuaciones diferenciales funcionales con retardo acotado. Algunos ejemplos son presentados.

Key words and phrases: *Uniform boundedness, stability, Liapunov's second method, functional differential equations.*

Math. Subj. Class.: *34D20, 34D40, 34K20.*

1 Introduction

We consider the system

$$X'(t) = F(t, X_t), \; X \in \mathbf{R^n}, \tag{1}$$

where $X_t(\theta) = X(t + \theta)$ for $-h \leq \theta \leq 0$ and h is a positive constant. The following notation and terminology will be used.

Denote by $\mathbf{C}$ the space of continuous functions $\phi : [-h, 0] \to \mathbf{R^n}$. For $\phi \in \mathbf{C}$ we will use the norm $\|\phi\| := \max |\phi(s)|$, where $|\cdot|$ is any convenient norm in $\mathbf{R^n}$. Given $H > 0$, $\mathbf{C_H}$ denotes the set of $\phi \in \mathbf{C}$ with $\|\phi\| < H$. $X'(t)$ denotes the right-hand derivative at t if it exists and is finite. It is supposed that $F : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R^n}$, that F is continuous, and that F takes bounded sets into bounded sets. Here, $\mathbf{R}_+ = [0, \infty)$. Then it is known [2, 6, 7, 15] that for each $t_0 \in \mathbf{R}_+$ and each $\phi \in \mathbf{C}$ there is at least one solution $X(t_0, \phi)$ of (1) satisfying $X_{t_0}(t_0, \phi) = \phi$ defined on an interval $[t_0, t_0 + \alpha)$ for some $\alpha > 0$ and if there is an $H_1 < H$ with $|X(t, t_0, \phi)| \leq H_1$, then $\alpha = \infty$.

By means of Liapunov's second method, throughout this paper we work with wedges, denoted by $W_i : \mathbf{R}_+ \to \mathbf{R}_+$, which are continuous and strictly increasing. We also work with continuous functionals $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ (called Liapunov functionals) with $V(t, 0) \equiv 0$, whose derivative V' with respect to (1) is defined by

$$V'_{(1)}(t, \phi) = \lim_{\delta \to 0^+} sup[V(t + \delta, X_{t+\delta}(t, \phi)) - V(t, \phi)]/\delta.$$

Definition 1.1. Solutions of (1) are *uniformly bounded* (U.B.) if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[t_0 \geq 0, \|\phi\| \leq B_1, t \geq t_0]$ imply that $|X(t, t_0, \phi)| < B_2$. Solutions of (1) are *uniformly ultimately bounded* (U.U.B.) for bound B if for each $B_3 > 0$ there exists $T > 0$ such that $[t_0 \geq 0, \|\phi\| \leq B_3, t \geq t_0 + T]$ imply $|X(t, t_0, \phi)| < B$.

Because we are also going to state some stability results, it is necessary to tell the difference of conditions between stability and boundedness. When we discuss stability, we always assume, in addition to the above general assumptions:

(i) $F : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R^n}$, and $F(t, 0) \equiv 0$ so that $X \equiv 0$ is a solution of (1), and is called the zero solution.

(ii) $V : \mathbf{R}_+ \times \mathbf{C_H} \to \mathbf{R}_+$, and $V(t, 0) \equiv 0$.

(iii) $W_i(0) = 0$ for each wedge $W_i(r)$.

It is a common idea that stability theory can be generalized in a manner parallel to boundedness theory. But the fact is that the development of boundedness theory is much slower than that of stability theory. For instance, for the system of ordinary differential equations

$$X' = f(t, X), \ X \in \mathbf{R^n}, \tag{2}$$

where $f : \mathbf{R}_+ \times \mathbf{D} \to \mathbf{R^n}$ continuous, and $\mathbf{D} \subset \mathbf{R^n}$ an open set with $0 \in D$, two classical results may be stated as the following:

Theorem 1.1. *Let $V : \mathbf{R}_+ \times \mathbf{D} \to \mathbf{R}_+$ be continuous and suppose*

(i) $W_1(|X|) \leq V(t, X) \leq W_2(|X|)$,

and

(ii) $V'_{(2)}(t, X(t)) \leq -W_3(|X(t)|)$.

Then $X \equiv 0$ of (2) is uniformly asymptotically stable.

Theorem 1.2. *Let $V : \mathbf{R}_+ \times \mathbf{D} \to \mathbf{R}_+$ be continuous and suppose*

(i) $W_1(|X|) \leq V(t, X) \leq W_2(|X|)$, with $W_1(r) \to \infty$ as $r \to \infty$,

(ii) $V'_{(2)}(t, X(t)) \leq -W_3(|X(t)|) + M, \ M > 0,$.

and

(iii) $W_3(U) > M$, for some $U > 0$.

Then the solutions of (2) are U.B. and U.U.B.

The parallel results of Theorem 1.1 and Theorem 1.2 for delay equations may be found in [15;p.190; p.202], and stated as the following.

Theorem 1.3. *Let* $V : \mathbf{R}_+ \times \mathbf{C_H} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(\|\phi\|) \le V(t,\phi) \le W_2(\|\phi\|)$,

and

(ii) $V'_{(2)} \le -W_3(\|\phi\|)$.

Then the zero solution of (1) is uniformly asymptotically stable.

Theorem 1.4. *Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(\|\phi\|)$,

(ii) $V'_{(1)}(t,X_t) \le 0$, *for* $|X(t)|$ *large,*

(iii) $W_1(r) - W_3(r) \to \infty$ *as* $r \to \infty$.

Then solutions of (1) are U.B.

Theorem 1.3 is a direct parallel result of Theorem 1.1 for delay equations. It has not proved to be useful. For applications, investigators gave the next theorem [15; p.192].

Theorem 1.5. *Suppose that* $F(t,\phi)$ *is bounded for* ϕ *bounded. Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(\|\phi\|_2)$, *where* $\|\cdot\|_2$ *denotes the* L^2*-norm;*

(ii) $V'_{(1)}(t,X_t) \le -W_3(|X(t)|)$.

Then $X \equiv 0$ *is uniformly asymptotically stable.*

In 1978, Burton [1] eliminated the condition that $F(t,\phi)$ is bounded for ϕ bounded in Theorem 1.5. Since then, stability theory of this type has been developed very much. In 1989, Burton and Hatvani [4] gave the following quite general results. Concepts of **PIM** and **IP** used below will be defined in the next section.

Theorem 1.6. *Suppose that* $D, V : \mathbf{R}_+ \times \mathbf{C_H} \to \mathbf{R}_+$ *are continuous,* $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ *is* **PIM**, *and the following conditions are satisfied.*

(i) $W_1(|X(t)|) \le V(t,X_t) \le W_2(|X(t)|) + W_3(\int_{t-h}^t D(s,X_s)ds)$;

(ii) $V'_{(1)}(t,X_t) \le -\eta(t)W_4(D(t,X_t))$;

(iii) $D(t,\phi) \le W_5(\|\phi\|)$;

(iv) for some $K \in (0,H)$ *there is a wedge* W_k *such that* $[t \in \mathbf{R}_+,\ u : [-2h,0] \to \mathbf{R^n}$ *is continuous,* $|u(s)| < K$ *for* $s \in [-2h,0]]$ *imply*

$$W_k(\inf\{|u(r)| : -h \le r \le 0\}) \le \int_{-h}^0 D(t+s,u_s)ds.$$

Then $X = 0$ *is uniformly asymptotically stable.*

Theorem 1.7. *Let* $V : \mathbf{R}_+ \times \mathbf{C_H} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(\|\phi\|)$;

and

(ii) $V'_{(1)}(t, X_t) \le -\eta_1 W_3(|X'(t)|) - \eta_2(t) W_4(|X(t)|)$, *where* $\eta_1 > 0$ *is a constant,* $\lim_{S\to\infty} \int_{t_*}^{t_*+S} \eta_2(s)ds = \infty$ *uniformly with respect to* t_*, *and there are* $\alpha > 0$, $r_0 > 0$ *such that* $r > r_0$ *implies* $W_3(r) \ge \alpha r$.

Then X=0 is uniformly asymptotically stable.

In 1991, Wang [10, 11] generalized and unified these two theorems and gave the following general and yet clean theorem.

Theorem 1.8. *Let* $D, V : \mathbf{R}_+ \times \mathbf{C_H} \to \mathbf{R}_+$ *with* V *continuous and* D *continuous along the solutions of (1). Suppose that there are continuous functions* η_1, $\eta_2 : \mathbf{R}_+ \to \mathbf{R}_+$ *and that the following conditions hold:*

(i) $W_1(|X(t)|) \le V(t, X_t) \le W_2(|X(t)| + \int_{t-h}^{t} D(s, X_s)ds)$;
(ii) $V'_{(1)}(t, X_t) \le -\gamma_1(t) W_3(m(X_t)) - \gamma_2(t) W_4(D(t, X_t))$; *where* $\gamma_1 \epsilon\, \mathbf{IP}(S)$ *for some* $S > 0$, $\gamma_2 \epsilon\, \mathbf{PIM}$, *and* $m(\phi) = \min_{-h\le s\le 0} |\phi(s)|$;
(iii) $D(t,\phi) \le W_5(\|\phi\|)$.

Then $X = 0$ *is uniformly asymptotically stable.*

The research on stability of this type continues. In 1994, Wang [12] improved Theorem 1.8 with weaker decrescentness. But comparing with stability theory, boundedness theory develops much more slowly than stability theory does. For U.B. although Theorem 1.4 had been proved before 1966, a parallel result like Theorem 1.5 without the condition that F is bounded for ϕ bounded had not been given until 1986. In 1986, Burton and Zhang [5] showed

Theorem 1.9. *Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(\int_{-h}^{0} W_4(|\phi(s)|)ds)$,

(ii) $V'_{(1)}(t, X_t) \le -W_4(|X(t)|) + M, M > 0$,

(iii) $W_1(r),\ W_4(r) \to \infty$, *as* $r \to \infty$.

Then solutions of (1) is U.B. and U.U.B.

In 1990, Burton [3] showed

Theorem 1.10. *Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *continuous with*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(\|\phi\|_2)$,

(ii) $V'_{(1)}(t, X_t) \le -W_4(\|X_t\|_2) + M,\ M > 0$,

(iii) $W_1(r) \to \infty$, *as* $r \to \infty$, $W_4(U/2) \ge 12M$ *for some* $U > 0$.

Then solutions of (1) are U.B. and U.U.B.

In this paper, we are going to give some general theorems like Theorem 1.8, generalize theorems like Theorem 1.10, and investigate some examples. One application of uniform boundedness and ultimate uniform boundedness is to prove the existence of periodic solutions. [2] gives much discussion on periodic solutions. Makay [8] discussed dissipativeness, which is weaker than U.U.B, and gave an interesting result on periodic solutions. Based on this paper and [13], the author examines many common functional differential equations, and obtains not only U.B. and U.U.B., but also the existence of periodic solutions. For more examples or applications of this paper and [13], please see [14].

2 Preliminaries

Definition 2.1. A measurable function $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be *integrally positive with parameter* $\alpha > 0$ ($\mathbf{IP}(\alpha)$) if $\lim_{t\to\infty} \inf \int_t^{t+\alpha} \eta(s)ds > 0$.

That is, $\eta \in \mathbf{IP}(\alpha)$ implies that there exist $T > 0$, and $\Gamma > 0$ such that for each $t > T$, $\int_t^{t+\alpha} \eta(s)ds \geq \Gamma$. Thus we also denote $\mathbf{IP}(\alpha, \Gamma) = \mathbf{IP}(\alpha)$. This definition is equivalent to the original one, which can be found in [4]. Now we give a weaker definition than the last one.

Definition 2.2. A measurable function $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be *partially integrally positive with parameters* $\alpha > 0$, $\beta > 0$, and $\Gamma > 0$ ($\mathbf{PIP}(\alpha, \beta, \mathbf{\Gamma})$) if there is a sequence $\{t_n\}_1^\infty$ with $\alpha \leq t_{n+1} - t_n \leq \beta$ such that $\int_{t_n}^{t_n+\alpha} \eta(s)ds \geq \Gamma$.

Clearly $\eta \in \mathbf{IP}(\alpha, \mathbf{\Gamma})$ implies $\eta \in \mathbf{PIP}(\alpha, \beta, \Gamma)$ for any $\beta \geq \alpha$.

Lemma 2.1. *Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be continuous and $G(t) = \int_{t-h}^t f(s)ds$. If $G(t_1) \geq \varepsilon$ for some $t_1 \geq 2h$ and $\varepsilon > 0$, then there is a closed interval $[a, b]$ of length h containing t_1 in which $G(t) \geq \varepsilon/2$.*

The proof of this lemma, which was originally proved by T. Krisztin, can be found in [9].

3 Main Results

Definition 3.1. A functional $D : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ is said to be *continuous along solutions of* (1) if $D(t, X_t)$ is continuous on $[t_0, \infty)$ for each solution $X(t, t_0, \phi)$ of (1) defined on $[t_0, \infty)$.

Denote

$$m(\phi) = \min_{-h \leq s \leq 0} |\phi(s)| \; for \; each \; \phi \in \mathbf{C}.$$

Theorem 3.1. *Let $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ be continuous and $D : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ be continuous along solutions of (1). Let $\gamma_1 \in \mathbf{PIP}(\alpha, \beta, \mathbf{\Gamma_1})$ and $\gamma_2 \in \mathbf{IP}(\mathbf{h}, \mathbf{\Gamma_2})$. Denote $C = \max\{\beta, h\}$. Let $W_1,\ W_2,\ W_3,\ W_4,\ W_5$ be wedges with $W_1(r) \to \infty$, as $r \to \infty$. Assume that*

(i) $W_1(|X(t)|) \leq V(t, X_t) \leq W_2(|X(t)| + \int_{t-h}^t D(u, X_u)du)$;

(ii) $V(t, \phi) \leq W_3(\|\phi\|)$;

(iii) $V'_{(1)}(t,X_t) \le -\gamma_1(t)W_4(m(X_t)) - \gamma_2(t)W_5(\int_{t-h}^t D(u,X_u)du) + M$;

(iv) There is a $\xi > 0$ *such that*

$$W_4(\xi)\Gamma_1 > 10MC, \text{ and } W_5(\xi/2)\Gamma_2 > 10MC.$$

Then solutions of (1) are U.B. and U.U.B.

Proof. $\gamma_1 \in \mathbf{PIP}(\alpha,\beta,\mathbf{\Gamma_1})$ implies that there is a sequence $\{t_n\}_1^\infty$ with $\alpha \le t_{n+1} - t_n \le \beta$ such that $\int_{t_n}^{t_n+\alpha} \gamma_1(u)du \ge \Gamma_1$. $\gamma_2 \in \mathbf{IP}(h,\Gamma_2)$ implies that there is $T_1 > 0$ such that for each $t > T_1$, $\int_t^{t+h} \gamma_2(u)du \ge \Gamma_2$.

First, we want to show U.B. That is, for each $B_1 > 0$, there is a $B_2 > 0$ such that $[t_0 \ge 0, \phi \in \mathbf{C}, \|\phi\| < B_1$, and $t \ge t_0]$ imply $|X(t,t_0,\phi)| < B_2$. Denote $X(t) = X(t,t_0,\phi)$.

Let $T_2 = \max\{t_1, T_1\}$, $U = W_2(2\xi)$, and $\Delta = \max\{2W_3(B_1) + (T_2+5C)M, U\}$. Let $\mathbf{I_0} = [t_0, t_0+T_2+5C]$, $\mathbf{I_k} = [t_0+T_2+5kC, t_0+T_2+5(k+1)C]$, $k = 1,\ 2,\ 3,\ \cdots$.

Claim I. For each $k = 0,\ 1,\ 2,\ \cdots$, there is a $q_k \in \mathbf{I_k}$ such that $V(q_k, X_{q_k}) < \Delta$.

We use mathematical induction to prove it.

For $k = 0$, integrating (iii) from t_0 to $t \in \mathbf{I_0}$, we have

$$\begin{aligned} V(t,X_t) &\le V(t_0,X_{t_0}) + (T_2+5C)M \\ &\le W_3(B_1) + (T_2+5C)M < \Delta. \end{aligned} \tag{3}$$

Clearly, there is a $q_0 \in \mathbf{I_0}$ such that $V(q_0, X_{q_0}) < \Delta$. In fact, q_0 can be any number in $\mathbf{I_0}$. Particularly, we take $q_0 = t_0 + T_2 + 5C$.

For $k = n$, assume there is a $q_n \in \mathbf{I_n}$ such that $V(q_n, X_{q_n}) < \Delta$. We want to show there is a $q_{n+1} \in \mathbf{I_{n+1}}$ such that $V(q_{n+1}, X_{q_{n+1}}) < \Delta$.

If this is false, then $V(t,X_t) \ge \Delta$ on $\mathbf{I_{n+1}}$. It is clear that there is a $\overline{q}_n \epsilon \mathbf{I_n}$ such that

$$V(\overline{q}_n, X_{\overline{q}_n}) = \Delta, \text{ and } V(t,X_t) \ge \Delta \text{ on } [\overline{q}_n, t_0+T_2+5(n+1)C] \bigcup \mathbf{I_{n+1}}.$$

Then

$$W_2\left(|X(t)| + \int_{t-h}^t D(u,X_u)du\right) \ge V(t,X_t) \ge \Delta \ge U \tag{4}$$

on $[\overline{q}_n, t_0+T_2+5(n+1)C] \bigcup \mathbf{I_{n+1}}$. Particularly, consider the interval

$$\mathbf{\overline{I}_{n+1}} = [t_0+T_2+5(n+1)C+C, t_0+T_2+5(n+1)C+4C] \subset \mathbf{I_{n+1}}.$$

(4) implies that either there is some $t^* \in \mathbf{\overline{I}_{n+1}}$ with

$$\int_{t^*-h}^{t^*} D(u,X_u)du \ge \frac{1}{2}W_2^{-1}(U) = \xi$$

or $|X(t)| \geq \frac{1}{2}W_2^{-1}(U) = \xi$ for each $t \in \overline{\mathbf{I}}_{\mathbf{n+1}}$.

Case I. $\int_{t^*-h}^{t^*} D(u, X_u)du \geq \xi$.

By Lemma 2.1, there are a and b with $b - a = h$ and $t^* \in [a, b]$ such that for each $t \in [a, b]$

$$\int_{t-h}^{t} D(u, X_u)du \geq \frac{1}{2}\xi.$$

Clearly $[a, b] \subset \mathbf{I}_{\mathbf{n+1}}$.

Integrating (iii) from $\overline{q}_n$ to $t_0 + T_2 + 5(n + 2)C$, we have

$$\begin{aligned}
\Delta &\leq V(t_0 + T_2 + 5(n + 2)C, X_{t_0+T_2+5(n+2)C}) \\
&\leq V(\overline{q}_n, X_{\overline{q}_n}) - \int_{\overline{q}_n}^{t_0+T_2+5(n+2)C} \gamma_2(s)W_5\left(\int_{s-h}^{s} D(u, X_u)du\right) + 10MC \\
&\leq \Delta - W_5(\frac{1}{2}\xi)\int_{a}^{a+h} \gamma_2(s)ds + 10MC \\
&\leq \Delta - W_5(\frac{1}{2}\xi)\Gamma_2 + 10MC \\
&< \Delta,
\end{aligned}$$

a contradiction.

Case II. $|X(t)| \geq \xi$ for each $t \in \overline{\mathbf{I}}_{\mathbf{n+1}}$.

Note that $\overline{\mathbf{I}}_{\mathbf{n+1}}$ contains three subintervals of length of $\mathbf{C}$. Therefore $\overline{\mathbf{I}}_{\mathbf{n+1}}$ contains at least three members of $\{t_n\}$, say s_1, s_2, and s_3 with $s_1 < s_2 < s_3$. Then integrating (iii) from $\overline{q}_n$ to $t_0 + T_2 + 5(n + 2)C$, we have

$$\begin{aligned}
\Delta &\leq V(t_0 + T_2 + 5(n + 2)C, X_{t_0+T_2+5(n+2)C}) \\
&\leq V(\overline{q}_n, X_{\overline{q}_n}) - \int_{\overline{q}_n}^{t_0+T_2+5(n+2)C} \gamma_1(u)W_4(m(X_u))du + 10MC \\
&\leq \Delta - W_4(\xi)\int_{s_2}^{s_2+\alpha} \gamma_1(s)ds + 10MC \\
&\leq \Delta - W_4(\xi)\Gamma_1 + 10MC \\
&< \Delta,
\end{aligned}$$

a contradiction.

So there is a $q_{n+1} \in \mathbf{I}_{\mathbf{n+1}}$ such that $V(q_{n+1}, X_{q_{n+1}}) < \Delta$.

The mathematical induction is complete. Therefore for each $k = 0, 1, 2, \cdots$, there is a $q_k \in \mathbf{I}_{\mathbf{k}}$ such that $V(q_k, X_{q_k}) < \Delta$.

Now for each $t \geq t_0$,

$$V(t, X_t) \leq \Delta \text{ if } t \in \mathbf{I}_{\mathbf{0}} \text{ (see(3))},$$

or if $t \in \mathbf{I_k}$, $k = 1, 2, 3, \cdots$, (iii) implies

$$V(t, X_t) \leq V(q_k, X_{q_k}) + 5MC \leq \Delta + 5MC \text{ if } t \geq q_k;$$

or

$$V(t, X_t) \leq V(q_{k-1}, X_{q_{k-1}}) + 10MC \leq \Delta + 10MC \text{ if } t < q_k.$$

That is $W_1(|X(t)|) \leq V(t, X_t) \leq \Delta + 10MC$ for each $t \geq t_0$. Take $B_2 = W_1^{-1}(\Delta + 10MC)$. This proves U.B.

Next we are going to prove U.U.B. for bound B. B will be determined at the end of the proof. That is for each $B_3 > 0$, there exists a $T > 0$ such that $[t_0 \geq 0, \|\phi\| < B_3, t \geq t_0 + T]$ imply that $|X(t, t_0, \phi)| < B$. The proof is similar to that of U.B. U and ξ are the same as before.

Let

$$N = \max\left\{\left[\frac{W_3(B_3) + T_2M + 5MC}{W_5(\xi/2)\Gamma_2 - 10MC}\right], \left[\frac{W_3(B_3) + T_2M + 5MC}{W_4(\xi)\Gamma_1 - 10MC}\right]\right\} + 1,$$

and $T_3 = 10NC + T_2$, where $[x]$ denotes the greatest integer function.

Let $\mathbf{J_0} = [t_0 + T_2, t_0 + T_3 + 5C]$.

Claim II. There is a $p_0 \in \mathbf{J_0}$ such that $V(p_0, X_{p_0}) < U$.

We show the claim by contradiction. Assume $V(t, X_t) \geq U$ for each $t \in \mathbf{J_0}$. Then for each $t \in \mathbf{J_0}$,

$$W_2\left(|X(t)| + \int_{t-h}^{t} D(u, X_u)du\right) \geq U \tag{5}$$

Note that $\mathbf{J_0}$ can contain $2N + 1$ subintervals of length $3C$, say,

$$\mathbf{J_{0i}} = [t_0 + T_2 + 5iC + C, t_0 + T_2 + 5iC + 4C],\ i = 0, 1, 2, \cdots,\ 2N.$$

On each $\mathbf{J_{0i}}$, (5) implies that either there is a $u_i^* \in \mathbf{J_{0i}}$ such that $\int_{u_i^*-h}^{u_i^*} D(u, X_u)du \geq \xi$, or $|X(t)| \geq \xi$ on $\mathbf{J_{0i}}$.

If the number of these $\{u_i^*\}$ is more than $N + 1$, by Lemma 2.1, there are a_i and b_i with $b_i - a_i = h$ and $u_i^* \in [a_i, b_i]$ such that for each $t \in [a_i, b_i]$

$$\int_{t-h}^{t} D(u, X_u)du \geq \frac{1}{2}\xi.$$

Clearly $[a_i, b_i] \bigcap [a_{i+1}, b_{i+1}] = \emptyset$ and $[a_i, b_i] \subset \mathbf{J_0}$ for each such i.

Integrating (iii) from t_0 to t_0+T_3+5C, we have

$$\begin{aligned}
0 &\le V(t_0+T_3+5C, X_{t_0+T_3+5C)}) \\
&\le V(t_0, X_{t_0}) - \int_{t_0}^{t_0+T_3+5C} \gamma_2(s) W_5\left(\int_{s-h}^{s} D(u,X_u)du\right) ds + M(T_3+5C) \\
&\le W_3(\|X_{t_0}\|) - \sum_{i=1}^{N} \int_{a_i}^{b_i} \gamma_2(s) W_5\left(\int_{s-h}^{s} D(u,X_u)du\right) ds + M(T_3+5C) \\
&\le W_3(B_3) - W_5(\xi/2)\Gamma_2 N + 10NMC + T_2M + 5MC \\
&\le W_3(B_3) - N[W_5(\xi/2)\Gamma_2 - 10MC] + T_2M + 5MC \\
&< 0, \text{ by the choice of } N
\end{aligned}$$

a contradiction. This means that the number of $\{u_i^*\}$ is less than $N+1$. Suppose that there are more than $N+1$ intervals of $\mathbf{J_{0i}}$ on which $|X(t)| \ge \xi$, say these intervals are $\mathbf{J_{0i}},\ i=1,2,3,\cdots,$ $N+1$. Clearly, each of these intervals contains at least three members of $\{t_n\}$, say v_{1i}, v_{2i}, and v_{3i} with $v_{1i} < v_{2i} < v_{3i}$. Then integrating (iii) from t_0 to t_0+T_3+5C, we have

$$\begin{aligned}
0 &\le V(t_0+T_3+5C, X_{t_0+T_3+5C}) \\
&\le V(t_0, X_{t_0}) - \int_{t_0}^{t_0+T_3+5C} \gamma_1(s) W_4(m(X_s)) ds + M(T_3+5C) \\
&\le W_3(\|X_{t_0}\|) - \sum_{i=1}^{N} \int_{v_{2i}}^{v_{2i}+\alpha} \gamma_1(s) W_4(m(X_s)) ds + M(T_3+5C) \\
&\le W_3(B_3) - W_4(\xi) N\Gamma_1 + 10NMC + T_2M + 5MC \\
&\le W_3(B_3) - N[W_4(\xi)\Gamma_1 - 10MC] + T_2M + 5MC \\
&< 0, \text{ by the choice of } N
\end{aligned}$$

a contradiction.

Therefore there must be a $p_0 \in \mathbf{J_0}$ such that $V(p_0, X_{p_0}) < U$.

Now define $\mathbf{J_k} = [p_0 + 5(k-1)C, p_0 + 5kC]$ for $k = 1,2,3,\cdots$.

Claim III. For each $k=1,2,3,\cdots$, there is a $p_k \in \mathbf{J_k}$ such that $V(p_k, X_{p_k}) < U$.

We use mathematical induction, again. For $k=1, \mathbf{J_1} = [p_0, p_0+5C]$ and by Claim II, we obviously can take $p_1 = p_0$ with $V(p_1, X_{p_1}) < U$.

Assume that for $k=n$, there is a $p_n \in \mathbf{J_n}$ such that $V(p_n, X_{p_n}) < U$. We want to show for $k=n+1$, there is a $p_{n+1} \in \mathbf{J_{n+1}}$ such that $V(p_{n+1}, X_{p_{n+1}}) < U$. Assume for the sake of contradiction that $V(t, X_t) \ge U$ on J_{n+1}. It is clear that there is a $\overline{p}_n \in \mathbf{J_n}$ such that

$$V(\overline{p}_n, X_{\overline{p}_n}) = U,\ and\ V(t,X_t) \ge U\ on\ [\overline{p}_n, p_0+5nC] \bigcup \mathbf{J_{n+1}}.$$

Then

$$W_2\left(|X(t)| + \int_{t-h}^{t} D(u,X_u)du\right) \ge V(t,X_t) \ge U \tag{6}$$

on $[\overline{p}_n, p_0 + 5nC] \bigcup \mathbf{J_{n+1}}$. Particularly, consider the interval

$$\overline{\mathbf{J}}_{n+1} = [p_0 + 5nC + C, p_0 + 5nC + 4C] \subset \mathbf{J_{n+1}}.$$

(6) implies that either there is $t^* \in \overline{\mathbf{J}}_{\mathbf{n+1}}$ with $\int_{t^*-h}^{t^*} D(u, X_u)du \geq \xi$, or $|X(t)| \geq \xi$ for each $t \in \overline{\mathbf{J}}_{\mathbf{n+1}}$.

Case I. $\int_{t^*-h}^{t^*} D(u, X_u)du \geq \xi$.

By Lemma 2.1, there are a and b with $b - a = h$ and $t^* \in [a, b]$ such that for each $t \in [a, b]$, $\int_{t-h}^{t} D(u, X_u)du \geq \xi/2$. Clearly $[a, b] \in \mathbf{J_{n+1}}$. Integrating (iii) from $\overline{p}_n$ to $p_0 + 5(n+1)C$, we have

$$\begin{aligned} U &\leq V(p_0 + 5(n+1)C, X_{p_0+5(n+1)C}) \\ &\leq V(\overline{p}_n, X_{\overline{p}_n}) - \int_{\overline{p}_n}^{p_0+5(n+1)C} \gamma_2(s) W_5\left(\int_{s-h}^{s} D(u, X_u)du\right) ds + 10MC \\ &\leq U - W_5(\xi/2) \int_a^{a+h} \gamma_2(s)ds + 10MC \\ &\leq U - W_5(\xi/2)\Gamma_2 + 10MC \\ &< U, \end{aligned}$$

a contradiction.

Case II. $|X(t)| \geq \xi$ for each $t \in \overline{\mathbf{J}}_{\mathbf{n+1}}$.

Note that $\overline{\mathbf{J}}_{\mathbf{n+1}}$ contains three subintervals of length of C. Therefore $\overline{\mathbf{J}}_{\mathbf{n+1}}$ contains at least three members of $\{t_n\}$, say s_1, s_2, and s_3 with $s_1 < s_2 < s_3$. Then integrating (iii) from $\overline{p}_n$ to $p_0 + 5(n+1)C$, we have

$$\begin{aligned} U &\leq V(p_0 + 5(n+1)C, X_{p_0+5(n+1)C}) \\ &\leq V(\overline{p}_n, X_{\overline{p}_n}) - \int_{\overline{p}_n}^{p_0+5(n+1)C} \gamma_1(s) W_4(m(X_u))du + 10MC \\ &\leq U - W_4(\xi) \int_{s_2}^{s_2+\alpha} \gamma_1(s)ds + 10MC \\ &\leq U - W_4(\xi)\Gamma_1 + 10MC \\ &< U \end{aligned}$$

a contradiction.

So there is a $p_{n+1} \in \mathbf{J_{n+1}}$ such that $V(p_{n+1}, X_{p_{n+1}}) < U$.

The mathematical induction is complete. Therefore for each $k = 0, 1, 2, \cdots$, there is a $p_k \in \mathbf{J_k}$ such that $V(p_k, X_{p_k}) < U$.

Now for each $t \geq t_0 + T_3 + 5C$, there must be an integer $k > 0$ such that $t \in \mathbf{J_k}$. Then (iii) implies

$$W_1(|X(t)|) \leq V(t, X_t) \leq V(p_k, X_{p_k}) + 5MC \leq U + 5MC \text{ if } t \geq p_k;$$

or

$$W_1(|X(t)|) \leq V(t, X_t) \leq V(p_{k-1}, X_{p_{k-1}}) + 10MC \leq U + 10MC \text{ if } t < p_k;$$

The later case will not happen for $k = 1$ because of the choice of p_1.

Take $B = W_1^{-1}(U + 10MC)$ and $T = T_3 + 5C$. Then for each $t \geq t_0 + T$, $|X(t)| < B$. This proves U.U.B.

Corollary 3.1. *Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|X(t)|) \leq V(t, X_t) \leq W_2(|X(t)| + \int_{t-h}^t |X(s)|^p ds)$, *where* $W_1(r) \to \infty$, *as* $r \to \infty$*; and* $p > 0$ *is a constant;*

(ii) $V'_{(1)}(t, X_t) \leq -\gamma(t)W_6(\int_{t-h}^t |X(s)|^p ds) + M$, *where* $\gamma \in \mathbf{IP}(h, \Gamma)$, *and* $M > 0$ *is a constant;*

(iii) there is a $\xi > 0$ *such that* $\min\{W_6(\xi/2), W_6(h\xi^p)\}\Gamma > 20Mh$.

Then solutions of (1) are U.B. and U.U.B.

Proof. In Theorem 3.1, take $D(t, X_t) = |X(t)|^p$. Condition (ii) implies

$$\begin{aligned} V'_{(1)}(t, X_t) &\leq -\frac{1}{2}\gamma(t)W_6\left(\int_{t-h}^t |X(s)|^p ds\right) - \frac{1}{2}\gamma(t)W_6\left(\int_{t-h}^t |X(s)|^p ds\right) + M \\ &\leq -\frac{1}{2}\gamma(t)W_6\left(h(m(X_t))^p\right) - \frac{1}{2}\gamma(t)W_6\left(\int_{t-h}^t |X(s)|^p ds\right) + M. \end{aligned}$$

Clearly, $\gamma_1(t) = \frac{1}{2}\gamma(t) \in \mathbf{PIP}(h, h, \Gamma/2)$, $\gamma_2(t) = \frac{1}{2}\gamma(t) \in \mathbf{IP}(h, \Gamma/2)$. Take $W_4(r) = W_6(hr^p)$, and $W_5(r) = W_6(r)$. The other conditions of Theorem 3.1 can be verified easily.

With a little stronger condition, we can state a cleaner corollary.

Corollary 3.2. *Let* $V : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}_+$ *be continuous with*

(i) $W_1(|X(t)|) \leq V(t, X_t) \leq W_2(|X(t)| + \int_{t-h}^t |X(s)|^p ds)$, *where* $W_1(r) \to \infty$, *as* $r \to \infty$*; and* $p > 0$ *is a constant;*

(ii) $V'_{(1)}(t, X_t) \leq -\gamma(t)W_6(\int_{t-h}^t |X(s)|^p ds) + M$, *where* $\gamma \in \mathbf{IP}(h, \Gamma)$, $M > 0$ *a constant, and* $W_6(r) \to \infty$ *as* $r \to \infty$.

Then solutions of (1) are U.B. and U.U.B.

Remark. In application, the inequality

$$V(t, X_t) \leq W_2(|X(t)|) + W_7\left(\int_{t-h}^t D(u, X_u) du\right) \tag{7}$$

is more often seen. But Condition (i) of Theorem 3.1 looks cleaner and a little more convenient in proof. It can be shown that (7) and Condition (i) are equivalent.

Proposition 3.1. *(i) If* W_1 *and* W_2 *are two wedges on* $\mathbf{R}_+$, *then there are wedges* W_* *and* W^* *such that*

$$W_*(s + t) \leq W_1(s) + W_2(t) \leq W^*(s + t), \ for \ s, \ t \in \mathbf{R}_+.$$

(ii) If W is a wedge, then there are wedges W_1, W_2, W_3, and W_4 such that

$$W_1(s) + W_2(t) \leq W(s+t) \leq W_3(s) + W_4(t), \text{ for each } s,\ t \in \mathbf{R}_+.$$

Proposition 3.1(i) was proved in the both [9, Proposition 5] and [10, Lemma 2]. But Proposition 3.1(i) only shows that (7) implies Condition (ii) of Theorem 3.1. To show that Condition (ii) of Theorem 3.1 implies (7), we need Proposition 3.1(ii).

Proof of Proposition 3.1(ii). Obviously

$$W(s+t) = \frac{1}{2}W(s+t) + \frac{1}{2}W(s+t) \geq \frac{1}{2}W(s) + \frac{1}{2}W(t).$$

So take $W_1(s) = \frac{1}{2}W(s)$, and $W_2(t) = \frac{1}{2}W(t)$. It is also clear that

$$W(2s) + W(2t) \geq W(s+t),$$

since $s+t \leq \max\{2s, 2t\}$. Now take $W_3(s) = W(2s)$, and $W_4(t) = W(2t)$. This proves Proposition 3.1 (ii).

4 Examples

Example 4A. Consider the scalar equation

$$x'(t) = -a(t)x(t) + \int_{t-h}^{t} b(s)x(s)ds + f(t, x_t) \tag{8}$$

with $a : \mathbf{R}_+ \to \mathbf{R}_+$ and $b : [-h, \infty) \to R$ continuous, and $f(t, \phi) : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}$ continuous.

Theorem 4.1. *Suppose that the functions a and b of (8) satisfy:*

(a) there is a constant $\theta > 0$ with $0 < \theta h < 1$ such that $|b(t)| - \theta a(t) \leq 0$;

(b) $\int_{t-h}^{t} a(s)ds \leq B$ for some constant $B > 0$, and $\int_{t-h}^{t} a(s)ds$ is $\mathbf{PIP}(\alpha, \beta, \Gamma)$ for some constants $\alpha > 0$, $\beta > 0$, and $\Gamma > 0$;

(c) $|f(t, \phi)| \leq M$ for $(t, \phi) \in R_+ \times \mathbf{C}$, and $M > 0$ is a constant.

Then solutions of (8) are U.B. and U.U.B.

Proof. The conclusion follows Theorem 3.1. Find $\theta_0 > 0$ and $\delta > 0$ such that $\theta_0 = \theta + \delta$ and $0 < \theta_0 h < 1$. This can be done, for instance, by taking $\delta = \frac{1-\theta h}{2h}$. Then for $t \geq 0$, $|b(t)| - \theta_0 a(t) \leq -\delta a(t)$.

Define

$$V(t, x_t) = |x(t)| + \theta_0 \int_{-h}^{0} \int_{t+s}^{t} a(u)|x(u)|duds$$

and $D(t,x_t)=a(t)|x(t)|$. Then we have

$$\begin{aligned}|x(t)| &\le V(t,x_t)\le |x(t)|+\theta_0 h\int_{t-h}^{t} a(u)|x(u)|du\\ &\le |x(t)|+\theta_0 h\int_{t-h}^{t} D(s,x_s)ds.\end{aligned}$$

Therefore Condition (i) of Theorem 3.1 is satisfied. Clearly,

$$V(t,\phi)\le\left(1+\theta_0 h\int_{t-h}^{t} a(s)ds\right)\|\phi\|\le(1+\theta_0 hB)\|\phi\|$$

by Condition (b). Hence Condition (ii) of Theorem 3.1 is fulfilled.

We also have

$$\begin{aligned}V'(t,x_t) &\le -a(t)|x(t)|+\int_{t-h}^{t}|b(u)||x(u)|du+|f(t,x_t)|\\ &+ \theta_0 ha(t)|x(t)|-\theta_0\int_{t-h}^{t} a(u)|x(u)|du\\ &= (\theta_0 h-1)a(t)|x(t)|+\int_{t-h}^{t}[|b(u)|-\theta_0 a(u)]|x(u)|du+M\\ &\le (\theta_0 h-1)a(t)|x(t)|-\delta\int_{t-h}^{t} a(u)|x(u)|du+M \qquad (9)\\ &\le -\frac{1}{2}\delta\int_{t-h}^{t} a(u)du\ m(x_t)-\frac{1}{2}\delta\int_{t-h}^{t} D(u,x_u)du+M \qquad (10)\end{aligned}$$

This implies that Condition (iii) of Theorem 3.1 is satisfied.

Take $W_4(r)=r$, and $W_5(r)=r$. Then $W_4(r)\to\infty$ and $W_5(r)\to\infty$ as $r\to\infty$. Therefore Condition (iv) of Theorem 3.1 is also fulfilled. Now according to Theorem 3.1, solutions of (8) are U.B. and U.U.B.

Remark: If we use (9), we need to assume $a\in\mathbf{PIP}(\alpha,\beta,\Gamma)$ which is clearly stronger than $\int_{t-h}^{t} a(s)ds\in\mathbf{PIP}(\alpha,\beta,\Gamma)$. So Condition (iii) of Theorem 3.1 is weaker than

$$V'_{(1)}(t,X_t)\le-\gamma_1(t)W_4(|X(t)|)-\gamma_2(t)W_5\left(\int_{t-h}^{t} D(u,X_u)du\right)+M.$$

Theorem 4.2. *Consider Equation (8) again. Suppose that*

(a) there are constants $k>1$, $\alpha>0$, $\beta>0$, and $\Gamma>0$ such that

$$-a(t)+kh|b(t)|:=-\gamma(t)\le 0 \text{ and } \gamma\in\mathbf{PIP}(\alpha,\beta,\Gamma);$$

(b) $\int_{t-h}^{t}|b(s)|ds\le B$ for each $t\ge 0$ and some constant $B\ge 0$;

(c) $|f(t,\phi)|\le M$ for some constant $M\ge 0$ and each $(t,\phi)\in\mathbf{R}_+\times\mathbf{C}$.

Then solutions of (8) are U.B. and U.U.B.

Proof. The conclusion follows Theorem 3.1. Define $D(t,\phi) = |b(t)||\phi(0)|$ and

$$V(t,x_t) = |x(t)| + k\int_{-h}^{0}\int_{t+s}^{t} |b(u)||x(u)|duds.$$

Then

$$\begin{aligned} V'(t,x_t) &\leq -a(t)|x(t)| + \int_{t-h}^{t} |b(s)||x(s)|ds + |f(t,x+t)| \\ &+ k\int_{-h}^{0} |b(t)||x(t)|ds - k\int_{-h}^{0} |b(t+s)||x(t+s)|ds \\ &= (-a(t)+kh|b(t)|)|x(t)| + (1-k)\int_{t-h}^{t} |b(s)||x(s)|ds + M \\ &= -\gamma(t)|x(t)| - (k-1)\int_{t-h}^{t} D(s,x_s)ds + M. \end{aligned}$$

All the other conditions of Theorem 3.1 can be verified easily. Therefore the solutions of (4A) are U.B. and U.U.B.

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