CUBO A Mathematical Journal
Vol.11, N$^{\underline{o}}$02, (85–105). May 2009

Dynamic Oligopolies and Intertemporal Demand Interaction

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ABSTRACT

Dynamic oligopolies are examined with continuous time scales and under the assumption that the demand at each time period is affected by earlier demands and consumptions. After the mathematical model is introduced the local asymptotical stability of the equilibrium is examined, and then we will discuss how information delays alter the stability conditions. We will also investigate the occurrence of a Hopf bifurcation giving the possibility of the birth of limit cycles. Numerical examples will be shown to illustrate the theoretical results.

RESUMEN

Dinámica de oligopolios son examinados en escala de tiempo continuo y bajo la suposición que la demanda en todo tiempo periodico es afectada por la demanda e consumo temprano. Es presentado el modelo matemático y examinada la estabilidad asintótica local y entonces discutiremos como la información de retrazo altera las condiciones de estabilidad. También investigamos el acontecimiento de bifurcación de Hofp dando la posibilidad de nacimiento de ciclos limites. Ejemplos números son exhibidos para ilustrar los resultados teóricos.

Key words and phrases: *Noncooperative games, dynamic systems, time delay, stability.*

Math. Subj. Class.: *91A20, 91A06.*

1 Introduction

Oligopoly models are the most frequently studied subjects in the literature of mathematical economics. The pioneering work of [3] is the basis of this field. His classical model has been extended by many authors, including models with product differentiation, multi-product, labor-managed oligopolies, rent-seeking games to mention a few. A comprehensive summary of single-product models is given in [6] and their multi-product extensions are presented and discussed in [7] including the existence and uniqueness of static equilibria and the asymptotic behavior of dynamic oligopolies. In both static and dynamic models the inverse demand function relates the demand and price of the same time period, however in many cases the demand for a good in one period will have an effect on the demand and price of the goods in later time periods. In the case of durable goods the market becomes saturated, and even in the case of non-durable goods the demand for and consumption of the goods in earlier periods will lead to taste or habit formation of consumers that will affect future demands. Intertemporal demand interaction has been considered by many authors in analyzing international trade (see for example, [4] and [9]). More recently [8] have developed a two-stage oligopoly and examined the existence and uniqueness of the Nash equilibrium.

In this paper we will examine the continuous counterpart of the model of [8]. After the dynamic model is introduced, the asymptotic behavior of the equilibrium will be analyzed. We will show that under realistic conditions the equilibrium is always locally asymptotically stable. We will also show that this stability might be lost when the firms have only delayed information about the outputs of the competitors and about the demand interaction.

2 The Mathematical Model

An n-firm single-product oligopoly is considered without product differentiation. Let x_k be the output of firm k and let $C_k(x_k)$ be the cost of this firm. It is assumed that the effect of market saturation, taste and habit formation, etc. of earlier time periods are represented by a timedependent variable Q, which is assumed to be driven by the dynamic rule

$$\dot{Q} = H\left(\sum_{k=1}^{n} x_k, Q\right), \tag{2.1}$$

where H is a given bivariate function. It is also assumed that the price function depends on both the total production level of the industry and Q. Hence the profit of firm k can be formulated as

$$\Pi_k = x_k f\left(x_k + S_k, Q\right) - C_k(x_k) \tag{2.2}$$

where $S_k = \sum_{l \neq k} x_l$. If L_k denotes the capacity limit of firm k, then $0 \leq x_k \leq L_k$ and $0 \leq S_k \leq \sum_{l \neq k} L_l$.

The common domain of functions H and f is $[0, \sum_{k=1}^{n} L_k] \times \mathcal{R}_+$, and the domain of the cost function C_k is $[0, L_k]$. Assume that H is continuously differentiable, f and C_k are twice continuously differentiable on their entire domains.

With any fixed values of $S_k \in [0, \sum_{l \neq k} L_l]$ and $Q \geq 0$, the best response of firm k is

$$\mathcal{R}_k(S_k, Q) = \arg \max_{0 \leq x_k \leq L_k} \left\{ x_k f(x_k + S_k, Q) - C_k(x_k) \right\}, \tag{2.3}$$

which exists since Π_k is continuous in x_k and the feasible set for x_k is a compact set.

As it is usual in oligopoly theory we make the following additional assumptions:

(A) $f_x^{'} < 0$;

(B) $f_x^{'} + x_k f_{xx}^{''} \leq 0$;

(C) $f_x^{'} - C_k^{''} < 0$

for all k and feasible x_k, S_k and Q.

Under these assumptions Π_k is strictly concave in x_k, so the best response of each firm is unique and can be obtained as follows:

$$\mathcal{R}_k(S_k, Q) = \begin{cases} 0 & \text{if} \quad f(S_k, Q) - C_k^{'}(0) \leq 0 \\ L_k & \text{if} \quad L_k f_x^{'}(L_k + S_k, Q) + f(L_k + S_k, Q) - C_k^{'}(L_k) \geq 0 \\ x_k^* & \text{otherwise,} \end{cases} \tag{2.4}$$

where x_k^* is the unique solution of equation

$$f(x_k + S_k, Q) + x_k f_x^{'}(x_k + S_k, Q) - C_k^{'}(x_k) = 0 \tag{2.5}$$

in interval $(0, L_k)$.

If we assume that at each time period each firm adjusts its output into the direction towards its best response, then the resulting dynamism becomes

$$\dot{x}_k = K_k \cdot (\mathcal{R}_k(S_k, Q) - x_k) \qquad (k = 1, 2, \cdots, n), \tag{2.6}$$

where K_k is the speed of adjustment of firm k, and if we add the dynamic equation (2.1) to these differential equations, an $(n+1)$-dimensional dynamic system is obtained. Clearly, $\bar{x_1}, \cdots, \bar{x_n}, \bar{Q}$ is a steady state of this system if and only if

$$H\left(\sum_{k=1}^{n} \bar{x_k}, \bar{Q}\right) = 0 \tag{2.7}$$

and

$$\bar{x_k} = \mathcal{R}_k\left(\sum_{l \neq k} \bar{x_l}, \bar{Q}\right) \tag{2.8}$$

for all k. The asymptotic behavior of the steady states will be examined in the next section.

3 Stability Analysis

Assume that $\bar{x_1}, \cdots, \bar{x_N}, \bar{Q}$ is an interior steady state, that is, both

$$f(\sum_{l \neq k} \bar{x_l}, \bar{Q}) - C_k^{'}(0)$$

and

$$L_k f_x^{'}(L_k + \sum_{l \neq k} \bar{x_l}, \bar{Q}) + f(L_k + \sum_{l \neq k} \bar{x_l}, Q) - C_k^{'}(L_k)$$

are nonzero for all k. If $\bar{x_k} = 0$ or $\bar{x_k} = L_k$, then clearly both $\frac{\partial \mathcal{R}_k}{\partial S_k}$ and $\frac{\partial \mathcal{R}_k}{\partial Q}$ are equal to zero. Otherwise these derivatives can be obtained by implicitly differentiating equation (2.5) with respect to S_k and Q:

$$r_k = \frac{\partial \mathcal{R}_k}{\partial S_k} = -\frac{f_x^{'} + x_k f_{xx}^{''}}{2f_x^{'} + x_k f_{xx}^{''} - C_k^{''}} \tag{3.1}$$

and

$$\bar{r}_k = \frac{\partial \mathcal{R}_k}{\partial Q} = -\frac{f_Q^{'} + x_k f_{xQ}^{''}}{2f_x^{'} + x_k f_{xx}^{''} - C_k^{''}}. \tag{3.2}$$

Assumptions (B) and (C) imply that

$$-1 < r_k \leq 0, \tag{3.3}$$

and if we assume that

(D) $\quad f_Q^{'} + x_k f_{xQ}^{''} \leq 0$

for all k and feasible x_k, S_k and Q, then

$$\bar{r}_k \leq 0. \tag{3.4}$$

Notice that relations (3.3) and (3.4) hold for all interior steady states.

The Jacobian of system given by the differential equations (2.1) and (2.6) has the special form

$$\mathbf{J} = \begin{pmatrix} -K_1 & K_1 r_1 & \cdots & K_1 r_1 & K_1 \bar{r}_1 \\ K_2 r_2 & -K_2 & \cdots & K_2 r_2 & K_2 \bar{r}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ K_n r_n & K_n r_n & \cdots & -K_n & K_n \bar{r}_n \\ h & h & \cdots & h & \bar{h} \end{pmatrix}$$

where

$$h = \frac{\partial H}{\partial x} \quad \text{and} \quad \bar{h} = \frac{\partial H}{\partial Q}.$$

The main result of this section is the following.

Theorem 1 *Assume that at the steady state, $\bar{h} \leq 0$ and $r_k \bar{h} - \bar{r}_k h > 0$ for all k. Then the steady state is locally asymptotically stable.*

Proof. We will prove that the eigenvalues of $\mathbf{J}$ have negative real parts at the steady state. The eigenvalue equation of $\mathbf{J}$ can be written as

$$-K_k u_k + K_k r_k \sum_{l \neq k} u_l + K_k \bar{r}_k v = \lambda u_k \quad (1 \leq k \leq n) \tag{3.5}$$

and

$$h \sum_{k=1}^{n} u_k + \bar{h} v = \lambda v. \tag{3.6}$$

Let $U = \sum_{k=1}^{n} u_k$, then from (3.6),

$$hU = (\lambda - \bar{h}) v.$$

Assume first that $h = 0$. Then the eigenvalues of $\mathbf{J}$ are $\bar{h}$ and the eigenvalues of matrix

$$\begin{pmatrix} -K_1 & K_1 r_1 & \cdots & K_1 r_1 \\ K_2 r_2 & -K_2 & \cdots & K_2 r_2 \\ \vdots & \vdots & & \vdots \\ K_n r_n & K_n r_n & \cdots & -K_n \end{pmatrix}.$$

Notice that this matrix is the Jacobian of continuous classical Cournot dynamics and it is well-known that its eigenvalues have negative real parts if $K_k > 0$ and $-1 < r_k \leq 0$ for all k (see for example, [1]). In this case the assumptions of the theorem imply that $\bar{h} < 0$, so all eigenvalues of $\mathbf{J}$ have negative real parts.

Assume next that $h \neq 0$. Then from (3.6),

$$U = \frac{\lambda - \bar{h}}{h} v, \tag{3.7}$$

and from (3.5),

$$u_k = \frac{K_k r_k U + K_k \bar{r}_k v}{\lambda + K_k(1 + r_k)} = \frac{K_k r_k (\lambda - \bar{h}) + K_k \bar{r}_k h}{(\lambda + K_k(1 + r_k))h} v. \tag{3.8}$$

By adding these equations for all values of k and using (3.7) we have

$$\sum_{k=1}^{n} \frac{K_k r_k \lambda + K_k(\bar{r}_k h - r_k \bar{h})}{\lambda + K_k(1 + r_k)} = \lambda - \bar{h}\,, \tag{3.9}$$

since $v \neq 0$, otherwise (3.8) would imply that $u_k = 0$ for all k.

Assume next that $\lambda = A + iB$ is a root of equation (3.9) with $A \geq 0$. Let $g(\lambda)$ denote the left hand side of this equation. Then

$$\begin{aligned} Re\, g(\lambda) &= Re \frac{K_k(r_k A + \bar{r}_k h - r_k \bar{h}) + i K_k r_k B}{A + K_k(1 + r_k) + iB} \\ &= \frac{K_k(r_k A + \bar{r}_k h - r_k \bar{h})(A + K_k(1 + r_k)) + K_k r_k B^2}{(A + K_k(1 + r_k))^2 + B^2} < 0 \end{aligned}$$

and

$$Re\,(\lambda - \bar{h}) = A - \bar{h} \geq 0,$$

which is an obvious contradiction. □

The conditions of the theorem are satisfied in the special model of [7, section 5.4] on oligopolies with saturated markets, where the dynamic rule of Q is assumed to be linear,

$$\dot{Q} = \sum_{k=1}^{n} x_k - C \cdot Q$$

with some $C > 0$. In this case $h = 1$ and $\bar{h} = -C$, so $h \neq 0$ and $r_k \bar{h} - \bar{r}_k h = -C r_k - \bar{r}_k > 0$ unless $r_k = \bar{r}_k = 0$.

4 The Effect of Delayed Information

In this section, we assume that the conditions of Theorem 1 hold, and the firms have only delayed information on their own outputs as well as on the output of the rest of the industry. It is also assumed that they have also delayed information on the value of parameter Q. A similar situation occurs when the firms react to average past information rather than reacting to sudden market changes. As in [2] we assume continuously distributed time lags, and will use weighting functions of the form

$$w(t-s,T,m) = \begin{cases} \frac{1}{T} e^{-\frac{t-s}{T}} & \text{if} \quad m=0 \\ \frac{1}{m!} \left(\frac{m}{T}\right)^{m+1} (t-s)^m e^{-\frac{(t-s)m}{T}} & \text{if} \quad m \geq 1 \end{cases} \tag{4.1}$$

where $T > 0$ is a real and $m \geq 0$ is an integer parameter. The main properties and the applications of such weighting functions are discussed in detail in [2]. By replacing the delayed quantities by their expectations, equations (2.6) become a set of Volterra-type integro-differential equations:

$$\begin{aligned} \dot{x}_k(t) = K_k \cdot \Bigg(R_k \left(\int_0^t w(t-s,T_k,m_k) \sum_{l \neq k} x_l(s) ds, \int_0^t w(t-s,U_k,p_k) Q(s) ds \right) \\ - \int_0^t w(t-s,V_k,l_k) x_k(s) ds \Bigg) \qquad (1 \leq k \leq n) \end{aligned} \tag{4.2}$$

accompanied by equation (2.1). It is well-known that equations (4.2) are equivalent to a higher dimensional system of ordinary differential equations, so all tools known from the stability theory of ordinary differential equations can be used here. Linearizing equation (4.2) around the steady state we have

$$\begin{aligned} \dot{x}_{k\delta} = K_k \cdot \Bigg(r_k \cdot \int_0^t w(t-s,T_k,m_k) \sum_{l \neq k} x_{l\delta}(s) ds \\ + \bar{r}_k \cdot \int_0^t w(t-s,U_k,p_k) Q_\delta(s) ds - \int_0^t w(t-s,V_k,l_k) x_{k\delta}(s) ds \Bigg) \end{aligned} \tag{4.3}$$

where $x_{k\delta}$ and Q_δ are deviations of x_k and Q from their steady state levels. We seek the solution in the form $x_{k\delta}(t) = u_k e^{\lambda t}$ and $Q_\delta = v e^{\lambda t}$, then we substitute these into equation (4.3) and let $t \to \infty$. The resulting equation will have the form

$$\begin{aligned} -\left(\lambda + K_k \left(1 + \frac{\lambda V_k}{c_k}\right)^{-(l_k+1)} \right) u_k + K_k r_k \left(1 + \frac{\lambda T_k}{a_k}\right)^{-(m_k+1)} \sum_{l \neq k} u_l \\ + K_k \bar{r}_k \left(1 + \frac{\lambda U_k}{b_k}\right)^{-(p_k+1)} v = 0 \end{aligned} \tag{4.4}$$

where

$$a_k = \begin{cases} 1 & \text{if} \quad m_k = 0 \\ m_k & \text{otherwise,} \end{cases}$$

$$b_k = \begin{cases} 1 & \text{if} \quad p_k = 0 \\ p_k & \text{otherwise,} \end{cases}$$

$$c_k = \begin{cases} 1 & \text{if} \quad l_k = 0 \\ l_k & \text{otherwise,} \end{cases}$$

and we use the identity

$$\int_0^\infty w(s,T,m)e^{-\lambda s}ds = \begin{cases} (1+\lambda T)^{-1} & \text{if} \quad m=0 \\ \left(1+\frac{\lambda T}{m}\right)^{-(m+1)} & \text{if} \quad m \geq 1. \end{cases}$$

Linearizing equation (2.1) around the steady state we have

$$\dot{Q}_\delta(t) = h \cdot \sum_{k=1}^n x_{k\delta}(t) + \bar{h}Q_\delta(t) \tag{4.5}$$

and by substituting $x_{k\delta}(t) = u_k e^{\lambda t}$ and $Q_\delta = ve^{\lambda t}$ into this equation we have

$$h\sum_{k=1}^n u_k + (\bar{h}-\lambda)v = 0. \tag{4.6}$$

For the sake of simplicity introduce the notation

$$A_k(\lambda) = \lambda + K_k\left(1+\frac{\lambda V_k}{c_k}\right)^{-(l_k+1)},$$

$$B_k(\lambda) = K_k r_k\left(1+\frac{\lambda T_k}{a_k}\right)^{-(m_k+1)},$$

and

$$C_k(\lambda) = K_k\bar{r}_k\left(1+\frac{\lambda U_k}{b_k}\right)^{-(p_k+1)},$$

then equation (4.4) can be rewritten as

$$-A_k(\lambda)u_k + B_k(\lambda)\sum_{l\neq k} u_l + C_k(\lambda)v = 0 \qquad (1 \leq k \leq n). \tag{4.7}$$

Equations (4.7),(4.6) have nontrival solution if and only if

$$det\begin{pmatrix} -A_1(\lambda) & B_1(\lambda) & \cdots & B_1(\lambda) & C_1(\lambda) \\ B_2(\lambda) & -A_2(\lambda) & \cdots & B_2(\lambda) & C_2(\lambda) \\ \vdots & \vdots & & \vdots & \vdots \\ B_n(\lambda) & B_n(\lambda) & \cdots & -A_n(\lambda) & C_n(\lambda) \\ h & h & & h & \bar{h}-\lambda \end{pmatrix} = 0. \tag{4.8}$$

Assume first that $h = 0$. Then the conditions of Theorem 1 imply that $\bar{h} < 0$. The eigenvalues are therefore $\lambda = \bar{h}$, which is negative, and the roots of equation

$$det\begin{pmatrix} -A_1(\lambda) & B_1(\lambda) & \cdots & B_1(\lambda) \\ B_2(\lambda) & -A_2(\lambda) & \cdots & B_2(\lambda) \\ \vdots & \vdots & & \vdots \\ B_n(\lambda) & B_n(\lambda) & \cdots & -A_n(\lambda) \end{pmatrix} = 0. \tag{4.9}$$

By introducing

$$\mathbf{D} = diag\big(- A_1(\lambda) - B_1(\lambda), \cdots, -A_n(\lambda) - B_n(\lambda)\big), \mathbf{1}^\top = (1, \cdots, 1)$$

and

$$\mathbf{b} = \big(B_1(\lambda), \ldots, B_n(\lambda)\big)^\top,$$

this equation can be rewritten as

$$\begin{aligned} det(\mathbf{D} + \mathbf{b} \cdot \mathbf{1}^\top) &= det(\mathbf{D})\, det(\mathbf{I} + \mathbf{D}^{-1}\mathbf{b} \cdot \mathbf{1}^\top) \\ &= \prod_{k=1}^{n} \Big(- A_k(\lambda) - B_k(\lambda)\Big) \cdot \left[1 - \sum_{k=1}^{n} \frac{B_k(\lambda)}{A_k(\lambda) + B_k(\lambda)}\right] = 0\,. \end{aligned} \tag{4.10}$$

Therefore in this case we have to examine the locations of the roots of equations

$$A_k(\lambda) + B_k(\lambda) = 0 \quad (1 \leqq k \leqq n) \tag{4.11}$$

and

$$\sum_{k=1}^{n} \frac{B_k(\lambda)}{A_k(\lambda) + B_k(\lambda)} = 1\,. \tag{4.12}$$

Assume next that $h \neq 0$ and $A_k(\lambda) + B_k(\lambda) \neq 0$. By introducing the new variable $U = \sum_{k=1}^{n} u_k$, equation (4.7) can be rewritten as

$$B_k(\lambda)U = \Big(A_k(\lambda) + B_k(\lambda)\Big)u_k - C_k(\lambda)v\,.$$

By combining this equation with (4.6) we have

$$u_k = \frac{hC_k(\lambda) + B_k(\lambda)(\lambda - \bar{h})}{h\big(A_k(\lambda) + B_k(\lambda)\big)} v\,, \tag{4.13}$$

where we assume that the denominator is nonzero. Adding this equation for all values of k and using (4.6) again we see that

$$\sum_{k=1}^{n} \frac{hC_k(\lambda) + B_k(\lambda)(\lambda - \bar{h})}{A_k(\lambda) + B_k(\lambda)} = \lambda - \bar{h}\,. \tag{4.14}$$

Here we also used the fact that $v \neq 0$, since otherwise u_k would be zero for all k from equation (4.13), and eigenvectors must be nonzero.

Notice first that in the absence of information lags $T_k = U_k = V_k = 0$ for all k, and in this special case equation (4.14) reduces to (3.9) as it should. The analysis of the roots of equations (4.11), (4.12) and (4.14) in the general case requires the use of computer methods, however in the case of symmetric firms and special lag structures analytic results can be obtained. Assume symmetric firms with identical cost functions, same initial outputs, identical time lags and speeds of adjustment. Then the firms also have identical trajectories, $A_k(\lambda) \equiv A(\lambda)$, $B_k(\lambda) \equiv B(\lambda)$, $C_k(\lambda) \equiv C(\lambda)$ and therefore $u_k \equiv u$. Because of this symmetry equations (4.7) and (4.6) become

$$\big(- A(\lambda) + (n-1)B(\lambda)\big)u + C(\lambda)v = 0 \tag{4.15}$$

and

$$nhu + (\bar{h} - \lambda)v = 0\,. \tag{4.16}$$

If $h = 0$, then from the conditions of Theorem 1 we know that $\bar{h} < 0$. Equation (4.16) implies that in this case either $\lambda = \bar{h}$ or $v = 0$. In the first case this eigenvalue is negative, which cannot destroy stability. In the second case $u \neq 0$ and equation (4.15) implies that

$$-A(\lambda) + (n-1)B(\lambda) = 0\,. \tag{4.17}$$

If $h \neq 0$, then

$$u = -\frac{\bar{h} - \lambda}{nh}\, v \tag{4.18}$$

and by substituting this relation into equation (4.15) we get a single equation for v:

$$\left(\Big(- A(\lambda) + (n-1)B(\lambda)\Big)\Big(- \frac{\bar{h} - \lambda}{nh}\Big) + C(\lambda)\right)v = 0\,. \tag{4.19}$$

Notice that $v \neq 0$, otherwise from (4.18) $u = 0$ would follow and eigenvectors must be nonzero. Therefore we have the following equation:

$$nhC(\lambda) + (n-1)B(\lambda)(\lambda - \bar{h}) - A(\lambda)(\lambda - \bar{h}) = 0\,.$$

Notice first that in the case of $h = 0$ this equation reduces to (4.17) and $\lambda = \bar{h}$. Note next that this equation can be rewritten as the polynomial equation

$$\begin{aligned}
\lambda(\lambda - \bar{h})&\left(1 + \frac{\lambda T}{a}\right)^{m+1}\left(1 + \frac{\lambda U}{b}\right)^{p+1}\left(1 + \frac{\lambda V}{c}\right)^{l+1} \\
&+ (\lambda - \bar{h})K\left(1 + \frac{\lambda T}{a}\right)^{m+1}\left(1 + \frac{\lambda U}{b}\right)^{p+1} \\
&- nhK\bar{r}\left(1 + \frac{\lambda T}{a}\right)^{m+1}\left(1 + \frac{\lambda V}{c}\right)^{l+1} \\
&- (n-1)(\lambda - \bar{h})Kr\left(1 + \frac{\lambda U}{b}\right)^{p+1}\left(1 + \frac{\lambda V}{c}\right)^{l+1} = 0\,.
\end{aligned} \tag{4.20}$$

4.1 No information lag

Assume first that there is no information lag. Then $T = U = V = 0$, and equation (4.20) specializes as

$$\lambda(\lambda - \bar{h}) + (\lambda - \bar{h})K - nhK\bar{r} - (n-1)(\lambda - \bar{h})Kr = 0$$

which is quadratic,

$$\lambda^2 + \lambda\big(- \bar{h} + K - (n-1)Kr\big) + \big(- \bar{h}K - nhK\bar{r} + (n-1)\bar{h}Kr\big) = 0\,. \tag{4.21}$$

Under the conditions of Theorem 1, all coefficients are positive. Therefore the roots have negative real parts. We have already proved this fact in Theorem 1 in the more general case.

4.2 Time lag in Q

Assume next that there is no time lag in the outputs but there is time lag in assessing the value of Q. Then $T = V = 0$, and if $p = 0$, then equation (4.20) becomes

$$\lambda(\lambda - \bar{h})(1 + \lambda U) + (\lambda - \bar{h})K(1 + \lambda U) - nhK\bar{r} - (n-1)(\lambda - \bar{h})Kr(1 + \lambda U) = 0$$

or

$$\begin{aligned} &\lambda^3 U + \lambda^2\Big(1 + U(K - \bar{h} - nKr + Kr)\Big) \\ &\quad + \lambda\Big(- \bar{h} + K - nKr + Kr + U(-K\bar{h} + nKr\bar{h} - Kr\bar{h})\Big) \\ &\quad + \Big(- \bar{h}K(r+1) + Kn(r\bar{h} - \bar{r}h)\Big) = 0\,. \end{aligned} \tag{4.22}$$

Under the conditions of Theorem 1, all coefficients are positive. By applying the Routh–Hurwitz criterion all roots have negative real parts if and only if

$$\begin{aligned} &\Big[- \bar{h} + K(1 - U\bar{h})\big(1 - (n-1)r\big)\Big]\Big[1 - U\bar{h} + UK\big(1 - (n-1)r\big)\Big] \\ &\qquad > U\Big(- \bar{h}K\big(1 - (n-1)r\big) - Knh\bar{r}\Big), \end{aligned} \tag{4.23}$$

which is a quadratic inequality in U. By introducing the notation $z = 1 - (n-1)r > 0$ it can be written as

$$U^2 K\bar{h}z(\bar{h} - Kz) + U\Big((\bar{h} - Kz)^2 + Knh\bar{r}\Big) + (-\bar{h} + Kz) > 0\,, \tag{4.24}$$

where the leading coefficient is nonnegative and the constant term is positive.

We have to consider next the following cases:

Case 1. If $h \leqq 0$, then (4.24) holds for all $U > 0$, since all coefficients are positive.

Case 2. Assume next that $h > 0$. If

$$\bar{r} \geqq -\frac{(\bar{h} - Kz)^2}{Knh},$$

then the linear coefficient is also nonnegative, so (4.24) holds for all $U > 0$. If

$$\bar{r} < -\frac{(\bar{h} - Kz)^2}{Knh},$$

then the linear coefficient of (4.24) is negative. We have now the following subcases.

(A) Assume first that $\bar{h} = 0$. Then the conditions of Theorem 1 imply that $\bar{r} < 0$ and $h > 0$. In this special case (4.24) becomes linear, so it holds if and only if

$$U < \frac{-Kz}{(Kz)^2 + Knh\bar{r}}. \tag{4.25}$$

The stability region in the $(\bar{r}, U)$ space is illustrated in Figure 1.

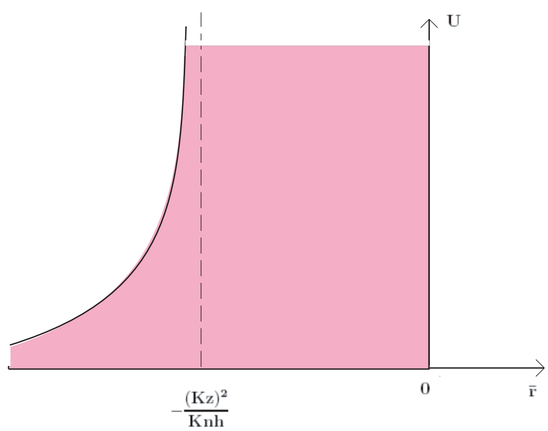

Figure 1: Stability region in the $(\bar{r}, U)$ space for $\bar{h} = 0$

(B) Assume next that $\bar{h} \neq 0$, then $\bar{h} < 0$. The discriminant of (4.24) is zero, if

$$\bar{r} = \bar{r}^* = \frac{-(Kz - \bar{h})^2 - 2(Kz - \bar{h})\sqrt{-K\bar{h}z}}{Knh}. \tag{4.26}$$

In this case (4.24) has a real positive root U^*, and the equilibrium is locally asymptotically stable if $U \neq U^*$.

If $\bar{r} > \bar{r}^*$, then the discriminant is negative, (4.24) has no real roots, so it is satisfied for all U, that is, the equilibrium is locally asymptotically stable.

If $\bar{r} < \bar{r}^*$, then the discriminant is positive, there are two real positive roots $U_1^* < U_2^*$, and the equilibrium is locally asymptotically stable if $U < U_1^*$ or $U > U_2^*$. The stability region in the $(\bar{r}, U)$ space is shown in Figure 2.

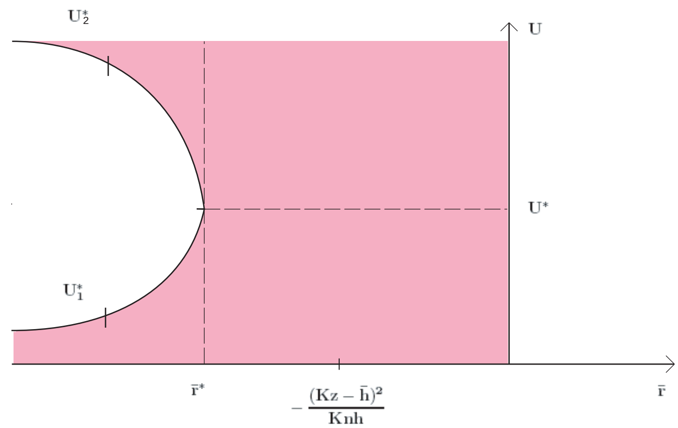

Figure 2: Stability region in the $(\bar{r}, U)$ space for $\bar{h} < 0$

Returning to Case (A), assume that $\bar{r} < -\frac{(Kz)^2}{Knh}$. Starting from a very small value of U, increase its value gradually. Until reaching the critical value

$$U = \frac{-Kz}{(Kz)^2 + Knh\bar{r}}\,, \tag{4.27}$$

the equilibrium is locally asymptotically stable. This stability is lost after crossing the critical value. We will next prove that at the critical value a Hopf bifurcation occurs giving the possibility of the birth of limit cycles. Notice first that since $\bar{h} = 0$, equation (4.22) has the special form

$$\lambda^3 U + \lambda^2(1 + UKz) + \lambda Kz + (-Knh\bar{r}) = 0\,, \tag{4.28}$$

and (4.23) specializes as

$$Kz(1 + UKz) > -UKnh\bar{r}\,. \tag{4.29}$$

At the critical value this inequality becomes equality, so at the critical value (4.28) can be rewritten as

$$\begin{aligned} 0 &= \lambda^3 U + \lambda^2(1 + UKz) + \lambda\frac{-UKnh\bar{r}}{(1 + UKz)} - Knh\bar{r} \\ &= \left(\lambda^2 - \frac{Knh\bar{r}}{1 + UKz}\right)\Big(U\lambda + (1 + UKz)\Big) \end{aligned}$$

so the roots are

$$\lambda_1 = -\frac{1+UKz}{U} \qquad \text{and} \qquad \lambda_{23} = \pm i\alpha$$

with

$$\alpha^2 = \frac{-Knh\bar{r}}{1+UKz} \left(= \frac{Kz}{U}\right).$$

So we have a negative real root and a pair of pure complex roots. Select now U as the bifurcation parameter and consider the eigenvalues as functions of U. By implicitly differentiating equation (4.28) with respect to U, with the notation $\dot{\lambda} = \frac{d\lambda}{dU}$ we have

$$3\lambda^2\dot{\lambda}U + \lambda^3 + 2\lambda\dot{\lambda}(1+UKz) + \lambda^2 Kz + \dot{\lambda}Kz = 0$$

implying that

$$\dot{\lambda} = \frac{-\lambda^3 - \lambda^2 Kz}{3\lambda^2 U + 2\lambda(1+UKz) + Kz}.$$

At the critical values $\lambda = \pm i\alpha$, so

$$\dot{\lambda} = \frac{\pm i\alpha^3 + \alpha^2 Kz}{-2\alpha^2 U \pm 2\alpha i(1+UKz)}$$

with real part

$$Re\dot{\lambda} = \frac{2\alpha^4}{(4\alpha^4 U^2) + 4\alpha^2(1+UKz)^2} > 0 \tag{4.30}$$

so all conditions of Hopf bifurcation are satisfied.

The other case of $\bar{h} \neq 0$ can be examined in a similar way. The details are omitted. We will however illustrate this case later in a numerical study.

4.3 Time lag in S_k

Assume next that the firms have instantaneous information about their own outputs and parameter Q, but have only delayed information about the output of the rest of the industry. In this case $U = V = 0$, and $T > 0$. By assuming $m = 0$, equation (4.20) becomes

$$\begin{aligned} \lambda(\lambda-\bar{h})(1+\lambda T) + (\lambda-\bar{h})K(1+\lambda T) - nhK\bar{r}(1+\lambda T) \\ - (n-1)(\lambda-\bar{h})Kr = 0\,, \end{aligned} \tag{4.31}$$

which is again a cubic equation:

$$\begin{aligned} \lambda^3 T + \lambda^2(1-\bar{h}T+TK) + \lambda\big(-\bar{h}+K-K\bar{h}T-nhKT\bar{r}-(n-1)Kr\big) \\ + \big(-K\bar{h}-nhK\bar{r}+(n-1)Kr\bar{h}\big) = 0\,. \end{aligned}$$

However, in contrast to the previous case there is no guarantee that the coefficients are all positive. Under the conditions of Theorem 1, the cubic and quadratic coefficients are positive, the linear coefficient is positive if

$$nhKT\bar{r} < -\bar{h} + Kz - K\bar{h}T \tag{4.32}$$

and the constant term is positive if

$$nhK\bar{r} < -K\bar{h}z \,. \tag{4.33}$$

If these relations hold, then the Routh-Hurwitz stability criterion implies that the eigenvalues have negative real parts if and only if

$$(1 - \bar{h}T + TK)(-\bar{h} + Kz - K\bar{h}T - nhKT\bar{r}) > T(-K\bar{h}z - nhK\bar{r}) \tag{4.34}$$

which can be written as a quadratic inequality of T:

$$T^2K(\bar{h} - K)(\bar{h} + nh\bar{r}) + T\Big((\bar{h} - K)^2 - (n-1)K^2r\Big) + (Kz - \bar{h}) > 0\,. \tag{4.35}$$

We have to consider now two cases:

Case 1. If $\bar{h} = 0$, then $\bar{r} < 0$ and $h > 0$, so all coefficients of (4.35) are positive, so it holds for all $T > 0$. Notice that in this case (4.32) and (4.33) are also satisfied, so the equilibrium is locally asymptotically stable.

Case 2. If $\bar{h} \neq 0$, then $\bar{h} < 0$. The linear coefficient and the constant term of (4.35) are both positive, however the sign of the quadratic coefficient is indeterminate. Therefore we have to consider two subcases:

(A) Assume first that $\bar{h} + nh\bar{r} \leqq 0$. Then the quadratic coefficient is nonnegative, so (4.35) holds for all $T > 0$. In this case $nhKT\bar{r} \leqq -\bar{h}KT$, so (4.32) also holds. Assume first that $h \geqq 0$, then (4.33) is also satisfied, consequently the equilibrium is locally asymptotically stable. Assume next that $h < 0$, then the conditions of Theorem 1 imply that both r and $\bar{h}$ are negative. Therefore

$$nhK\bar{r} \leqq -K\bar{h} < -K\bar{h} + K\bar{h}(n-1)r = -K\bar{h}z\,,$$

so (4.33) also holds, and the equilibrium is locally asymptotically stable again.

(B) Assume next that $\bar{h} + nh\bar{r} > 0$. In this case both h and $\bar{r}$ must be negative. In this case

$$\bar{r} < \frac{-\bar{h}}{nh}\,, \tag{4.36}$$

the quadratic coefficient of (4.35) is negative, therefore (4.35) has two real roots, one is positive, T^*, and the other is negative. Clearly, (4.35) holds if $T < T^*$.

Relation (4.33) holds if

$$\bar{r} > \frac{-\bar{h}z}{nh}\,, \tag{4.37}$$

and under conditions (4.36) and (4.37), relation (4.32) holds if

$$T < \frac{Kz - \bar{h}}{nhK\bar{r} + K\bar{h}} = T^{**}\,.$$

We will next prove that $T^* < T^{**}$, so this last condition is irrelevant for the local asymptotic stability of the equilibrium. Let $p(T)$ denote the left hand side of (4.35), then $T^* < T^{**}$ if $p(T^{**}) < 0$. This inequality is the following:

$$\frac{(Kz-\bar{h})^2(\bar{h}-K)}{K(nh\bar{r}+\bar{h})} + \frac{(Kz-\bar{h})\big[(\bar{h}-K)^2-(n-1)K^2r\big]}{K(nh\bar{r}+\bar{h})} + (Kz-\bar{h}) < 0$$

which can be simplified as

$$(Kz-\bar{h})(\bar{h}-K) + (\bar{h}-K)^2 - (n-1)K^2r + K(nh\bar{r}+\bar{h}) < 0\,.$$

This relation is equivalent to the following:

$$\begin{aligned} 0 > z\bar{h} + nh\bar{r} &= \bar{h} - n\bar{h}r + \bar{h}r + nh\bar{r} \\ &= \bar{h}(1+r) - n(\bar{h}r - h\bar{r}) \end{aligned}$$

where both terms are negative, which completes the proof.

Figure 3 shows the stability region in the $(\bar{r}, T)$ space. The occurrence of Hopf bifurcation at the critical value T^* can be examined in the same way as shown before, the details are omitted, however a numerical study of his case will be presented in the next section.

5 Numerical Examples

We will examine next a special case of the oligopoly model of [7] with saturated markets. It is assumed that

$$\dot{Q} = \sum_{k=1}^{n} x_k - \alpha Q$$

with some $0 < \alpha < 1$, and the market price and the cost functions are linear:

$$f\left(\sum_{k=1}^{n} x_k, Q\right) = A - \left(\sum_{k=1}^{n} x_k + \beta Q\right)$$

and

$$C_k(x_k) = c_k x_k \qquad (k = 1, 2, \ldots, n)\,.$$

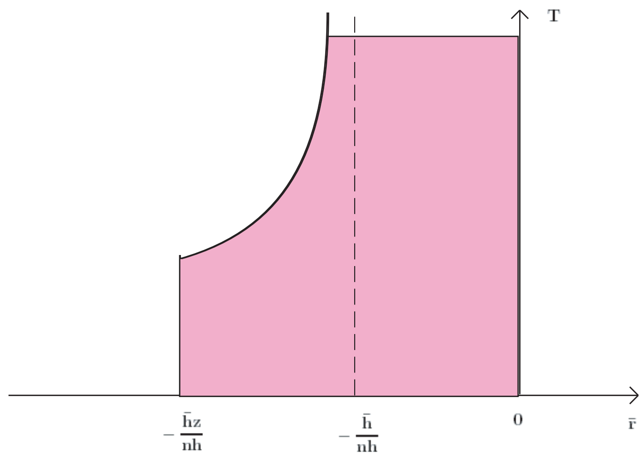

Figure 3: Stability region in the $(\bar{r}, T)$ space for $h < 0$

In this case the best response of firm k is

$$\mathcal{R}_k(S_k, Q) = \frac{-S_k - \beta Q + A - c_k}{2}.$$

We also assume that $K_k \equiv K$ and $c_k \equiv c$. It is easy to see that

$$h = 1, \bar{h} = -\alpha, r = -\frac{1}{2}, \bar{r} = -\frac{\beta}{2}$$

and

$$z = 1 - (n-1)(-\frac{1}{2}) = \frac{n+1}{2}.$$

Assume $U > 0$, $T = V = 0$ as in subsection 4.2. From case 2(B) we know that the equilibrium is locally asymptotically stable if

$$\bar{r} > \bar{r}^* = -\frac{\left(\frac{K(n+1)}{2} + \alpha\right)^2 + 2\left(\frac{K(n+1)}{2} + \alpha\right)\sqrt{\frac{K\alpha(n+1)}{2}}}{Kn}.$$

If $\bar{r} = \bar{r}^*$, then it is locally asymptotically stable if

$$U \neq U^* = \frac{\sqrt{\frac{K\alpha(n+1)}{2}}}{K\alpha z} = \sqrt{\frac{2}{K\alpha(n+1)}},$$

and if $\bar{r} < \bar{r}^*$, then the equilibrium is locally asymptotically stable if $U < U_1^*$ or $U > U_2^*$, where U_1^* and U_2^* are the positive roots of the left hand side of (4.24).

We have selected the numerical values $A = 25, c = 5, \alpha = K = \frac{1}{100}$ and $n = 3$. In this case $z = 2$ and

$$\bar{r}^* = -\frac{3 + 2\sqrt{2}}{100} \approx -0.0583,$$

so if we select

$$\beta > \frac{3+2\sqrt{2}}{50} \approx 0.1166$$

then we have two real roots of the left hand side of (4.24). So if $\beta = 0.5$, then it has the form

$$U^2 \frac{6}{100^3} - U \frac{66}{100^2} + \frac{3}{100} = 0$$

with the roots

$$U^*_{1,2} = 50(11 \pm \sqrt{119})\,,$$

so the critical values are

$$U^*_1 \approx 4.564 \quad \text{and} \quad U^*_2 \approx 1095.436\,.$$

In Figure 4, Figure 5 and Figure 6 we have illustrated this phenomenon. Figure 4 shows a shrinking cycle with $U = 3$. Figure 5 shows the complete limit cycle with $U = U^*_1$, and Figure 6 illustrates an expanding cycle with $U = 6$.

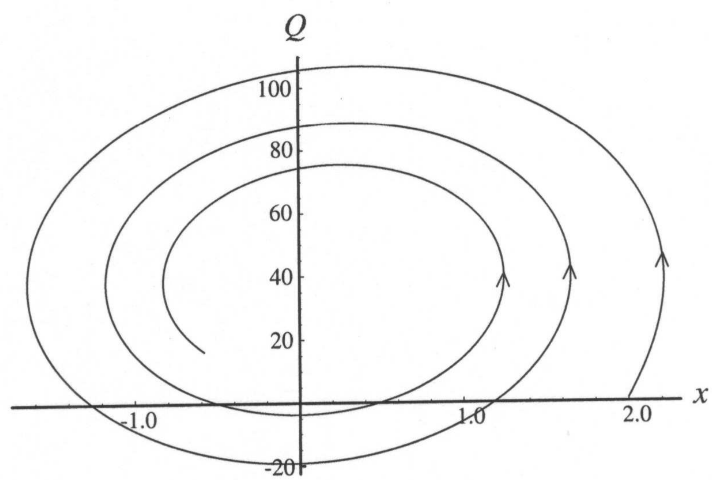

Figure 4: Shrinking cycle

We will next illustrate a model with delay in the output of the rest of the industry as in subsection 4.3. Assume now the dynamic equation

$$\dot{Q} = -\gamma \sum_{k=1}^{n} x_k - \alpha Q$$

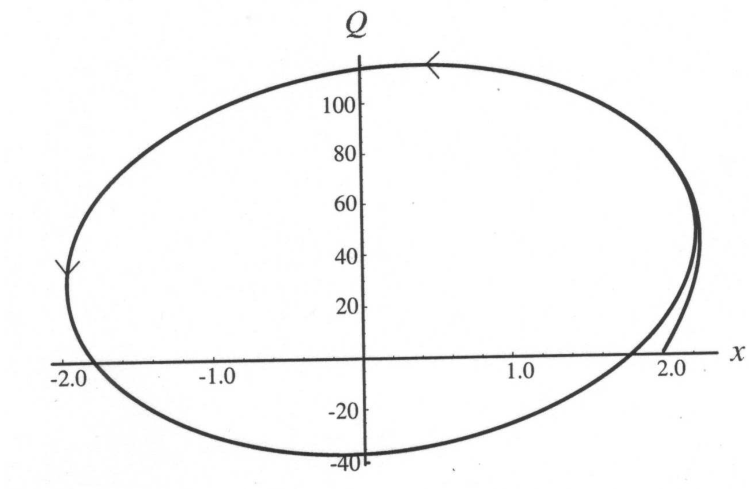

Figure 5: Complete cycle

with positive γ and α. Assume also that $K_k \equiv K$, the price function and the cost functions are the same as in the previous case, so we have

$$h = -\gamma, \bar{h} = -\alpha, r = -\frac{1}{2} \quad \text{and} \quad \bar{r} = -\frac{\beta}{2}\,,$$

and so $z = \frac{n+1}{2}$ as before.

We now assume $T > 0$, $U = V = 0$. Conditions (4.36) and (4.37) are satisfied if

$$-\frac{\beta}{2} < \frac{\alpha}{-n\gamma} \quad \text{and} \quad -\frac{\beta}{2} > \frac{\alpha \cdot \frac{n+1}{2}}{-n\gamma}\,,$$

that is,

$$\frac{2\alpha}{n\gamma} < \beta < \frac{\alpha(n+1)}{n\gamma}\,.$$

In this case T^* is the positive solution of equation

$$-T^2K(\alpha+K)(n\beta\gamma - 2\alpha) + T(2\alpha^2 + 4\alpha K + K^2(n+1)) + \big(2\alpha + K(n+1)\big) = 0\,.$$

We have selected the numerical values $A = 25, c_k = 5, \alpha = K = \gamma = \frac{1}{10}$ and $n = 3$. The value of β has to be between $\frac{2}{3}$ and $\frac{4}{3}$, so for the sake of simplicity we have chosen $\beta = 1$. Then the quadratic equation (4.35) for T^* has the form

$$-T^2\frac{2}{1000} + T \cdot \frac{10}{100} + \frac{6}{10} = 0$$

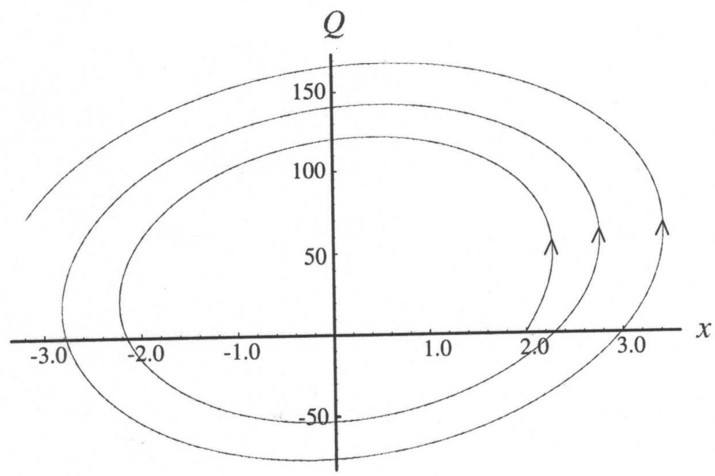

Figure 6: Expanding cycle

with positive root

$$T^* = 5(5 + \sqrt{37}) \approx 55.4138\,.$$

With this critical value a complete limit cycle is obtained, if $T < T^*$ then the limit cycle changes to a shrinking one and if $T > T^*$ it becomes an expanding cycle. The resulting figures are very similar to those presented in the previous case, so they are not shown here.

6 Conclusions

In this paper dynamic oligopolies were examined with continuous time scales and intertemporal demand interaction. The effect of earlier demands and consumptions were modelled by introducing an additional state variable, and in an n-firm oligopoly this concept resulted in an $(n+1)$-dimensional continuous system.

The local asymptotical stability of the equilibrium has been proved under general conditions. This stability however might be lost if the firms have only delayed information on the demand interaction, on their own outputs and also on the outputs of the competitors. By assuming continuously distributed time lags stability conditions were derived for important special cases and in cases when instability occurs the occurrence of Hopf bifurcation was investigated giving the possibility of the birth of limit cycles. The theoretical results have been illustrated by computer studies.

Received: April 21, 2008. Revised: May 09, 2008.

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