A Remark on the Enclosure Method for a Body with an Unknown Homogeneous Background Conductivity

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ABSTRACT

Previous applications of the enclosure method with a *finite* set of observation data to a mathematical model of electrical impedance tomography are based on the assumption that the conductivity of the background body is homogeneous and *known*. This paper considers the case when the conductivity is homogeneous and *unknown*. It is shown that, in two dimensions if the domain occupied by the background body is enclosed by an *ellipse*, then it is still possible to extract some information about the location of unknown cavities or inclusions embedded in the body without knowing the background conductivity provided the Fourier series expansion of the voltage on the boundary does not contain high frequency parts (band limited) and satisfies a non vanishing condition of a quantity involving the Fourier coefficients.

RESUMEN

Previas aplicaciones del método de cercamiento con un conjunto finito de datos de observación para un modelo matemático de tomografia electrica son basados en la suposición de que la conductividad del cuerpo es homogenea y conocida. Este artículo

considera el caso cuando la conductividad es homogenea y desconocida. Es demostrado en dos dimensiones que si el domínio ocupado por el cuerpo es encerrado por una elipse, entonces es aún posible extraer alguna información acerca de la localización de las cavidades desconocidas o inclusiones inmersas en el cuerpo sin conocimiento de la conductividad con talque la expansión en series de Fourier del voltaje sobre la frontera no contenga frecuencias altas (fajas acotadas) y satisfaga una condición de no nulidad de una cierta cantidad envolviendo los coeficientes de Fourier.

1   Introduction

The aim of this paper is to reconsider previous applications [5, 6] of the *enclosure method* with a *finite* set of observation data to inverse boundary value problems related to a continuum model of electrical impedance tomography [1, 2]. The point is: those applications are based on the assumption that the conductivity of the background body is homogeneous and *known*. However, from a mathematical point of view, the problem whether or not one can still extract some information about unknown discontinuity from the finite set of observation data without knowing the exact value of the conductivity is quite interesting. Proofs of some previous known uniqueness results that employ a finite set of observation data, for example, [4] for cracks and [3, 9] for inclusions are based on the assumption that the conductivity of the background body is known. This is because they start with applying the uniqueness of the Cauchy problem for elliptic equations.

Besides needless to say, we cannot know the exact value of the conductivity of the background body. The inaccurate value causes an error on the observation data and therefore on the indicator function in the enclosure method.

In order to describe the problem more precisely let us start with recalling a typical application of the enclosure method with a single set of observation data.

Let Ω be a bounded domain of $\mathbf{R}^2$ with Lipschitz boundary. Let D be an open subset with Lipschitz boundary of Ω such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. Consider a non constant solution of the elliptic problem:

$$\triangle u = 0 \text{ in } \Omega \setminus \overline{D},$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D. \tag{1.1}$$

Here $\nu = (\nu_1, \nu_2)$ denotes the unit outward normal vector field on $\partial(\Omega \setminus \overline{D})$. The D is a mathematical model of the union of cavities inside the body.

In [5] we considered the problem of extracting information about the location and shape of D in two dimensions from the observation data that is a single set of Cauchy data of u on $\partial\Omega$. Assuming that D is given by the inside of a polygon with an additional condition on the diameter, we established an extraction formula of the *convex hull* of D from the data. The method uses a special exponential solution of the Laplace equation. The solution takes the form $e^{-\tau t}e^{\tau x \cdot (\omega + i\omega^\perp)}$ where $\tau(> 0)$ and t are parameters; both ω and $\omega^\perp$ are unit vectors and satisfy $\omega \cdot \omega^\perp = 0$. The solution divides the whole plane into two half ones which have the line $\{x \,|\, x \cdot \omega = t\}$ as the common boundary. In one part $\{x \,|\, x \cdot \omega > t\}$ the solution is growing as $\tau \longrightarrow \infty$ and in another part $\{x \,|\, x \cdot \omega < t\}$ decaying. Using this solution, we define the so-called indicator function $I_{\omega, \omega^\perp}(\tau, t)$ of the independent variable τ with parameter t:

$$I_{\omega, \omega^\perp}(\tau, t) = e^{-\tau t} \int_{\partial\Omega} \left\{ -\frac{\partial}{\partial\nu} e^{\tau x \cdot (\omega + i\omega^\perp)} u + \frac{\partial u}{\partial\nu} e^{\tau x \cdot (\omega + i\omega^\perp)} \right\} ds.$$

The enclosure method gives us information about the position of half plane $x \cdot \omega > t$ relative to D by checking the asymptotic behaviour of the indicator function as $\tau \longrightarrow \infty$. For the description of the behaviour we recall the support function $h_D(\omega) = \sup_{x \in D} x \cdot \omega$. Moreover we say that ω is *regular* if the set $\{x \,|\, x \cdot \omega = h_D(\omega)\} \cap \partial D$ consists of only one point.

What we established in [5] is: for regular ω there exist positive constants A and $\mu(> 1/2)$ such that, as $\tau \longrightarrow \infty$

$$|I_{\omega, \omega^\perp}(\tau, 0)| \sim \frac{A}{\tau^\mu} e^{\tau h_D(\omega)} \tag{1.2}$$

provided

$$\operatorname{diam} D < \operatorname{dis}(D, \partial\Omega). \tag{1.3}$$

This fact is the core of the enclosure method. Since we have the trivial identity

$$I_{\omega, \omega^\perp}(\tau, t) = e^{-\tau t} I_{\omega, \omega^\perp}(\tau, 0),$$

from (1.2) one could conclude that: if $t > h_D(\omega)$, then the indicator function is decaying exponentially; if $t = h_D(\omega)$, then the indicator function is decaying truly algebraically; if $t < h_D(\omega)$, then the indicator function is growing exponentially. Moreover from (1.2), we immediately obtain also the *one line* formula

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log |I_{\omega, \omega^\perp}(\tau, 0)| = h_D(\omega).$$

However this is the case when the background conductivity is *known*.

Consider the case when the background conductivity is given by a positive constant γ. In this case the indicator function should be replaced with

$$I_{\omega, \omega^\perp}(\tau, t) = e^{-\tau t} \int_{\partial\Omega} \left\{ -\gamma \frac{\partial}{\partial\nu} e^{\tau x \cdot (\omega + i\omega^\perp)} u + \gamma \frac{\partial u}{\partial\nu} e^{\tau x \cdot (\omega + i\omega^\perp)} \right\} ds.$$

Needless to say we obtain the same result as above if γ is *known*. However, if γ is *unknown*, then the term

$$e^{-\tau t} \int_{\partial\Omega} \gamma \frac{\partial}{\partial\nu} e^{\tau x \cdot (\omega + i\omega^\perp)} u \, ds$$

becomes unknown and therefore one can use only the term

$$e^{-\tau t} \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} e^{\tau x \cdot (\omega + i\omega^\perp)} ds \qquad (1.4)$$

if $u = f$ on $\partial\Omega$ is given.

The purpose of this paper is to give a remark on the problem: can one still extract information about the location and shape of D from the quantity (1.4) in the case when f is given?

In this paper we show that, in two dimensions if the domain occupied by the background body is enclosed by an *ellipse*, then it is still possible to extract some information about the location of unknown cavities or inclusions embedded in the body without knowing the background conductivity provided the Fourier series expansion of the voltage on the boundary does not contain high frequency parts (band limited) and satisfies a non vanishing condition of a quantity involving the Fourier coefficients.

2 Extraction formulae

Let Ω be the domain enclosed by an ellipse. By choosing a suitable system of orthogonal coordinates one can write

$$\Omega = \left\{ (x_1, x_2) \mid \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 < 1 \right\}$$

where $a \geq b > 0$. In what follows we always use this coordinates system.

Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. Then $x \cdot (\omega + i\omega^\perp) = (x_1 - ix_2)(\omega_1 + i\omega_2)$. Let $v = e^{\tau x \cdot (\omega + i\omega^\perp)}$.

2.1 Preliminary computation

In this subsection first given $f = u|_{\partial\Omega}$ we study the asymptotic behaviour of the integral

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds.$$

However, integration by parts yields

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds = \gamma \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} ds - \gamma \int_{\partial D} u \frac{\partial v}{\partial \nu} ds \qquad (2.1)$$

and we have already studied the asymptotic behaviour of the second term as described in Introduction (see (1.2)). Therefore it suffices to study that of the first term. Since

$$\int_{\partial\Omega} u \frac{\partial v}{\partial \nu} ds = \tau(\omega_1 + i\omega_2) \int_{\partial\Omega} u \, v \, (\nu_1 - i\nu_2) ds, \qquad (2.2)$$

we compute the integral in the right hand side.

Write

$$f(\theta) = f(a\cos\theta, b\sin\theta) = \frac{1}{2}\alpha_0 + \sum_{m=1}^{\infty}(\alpha_m\cos m\theta + \beta_m\sin m\theta)$$

where

$$\alpha_m = \frac{1}{\pi}\int_0^{2\pi}f(a\cos\theta, b\sin\theta)\cos m\theta d\theta, \ \beta_m = \frac{1}{\pi}\int_0^{2\pi}f(a\cos\theta, b\sin\theta)\sin m\theta d\theta.$$

Define

$$\gamma_0 = \alpha_0/2, \ \gamma_m = (\alpha_m - i\beta_m)/2, \gamma_{-m} = \overline{\gamma_m}, m \geq 1.$$

Lemma 2.1. *We have: if $a = b$, then*

$$\int_{\partial\Omega} u\,v\,(\nu_1 - i\nu_2)ds = 2\pi a^2 \sum_{m=0}^{\infty}\frac{\{a\tau(\omega_1 + i\omega_2)\}^m}{m!}\gamma_{m+1}; \tag{2.3}$$

if $a > b$, then

$$\int_{\partial\Omega} u\,v\,(\nu_1 - i\nu_2)ds = 2\pi ab \sum_{m=0}^{\infty}i^m J_m(-i\sqrt{a^2 - b^2}\tau(\omega_1 + i\omega_2))C_m(f) \tag{2.4}$$

where $C_0(f) = A_-\overline{\gamma_1} + A_+\gamma_1$, for $m = 1, 2, \cdots$

$$C_m(f) = (A_-\gamma_{m-1} + A_+\gamma_{m+1})\left(\sqrt{\frac{a+b}{a-b}}\right)^m + (A_-\overline{\gamma_{m+1}} + A_+\overline{\gamma_{m-1}})\left(\sqrt{\frac{a-b}{a+b}}\right)^m$$

and

$$A_\pm = \frac{1}{2}\left(\frac{1}{a} \pm \frac{1}{b}\right).$$

Proof. Set $z = e^{i\theta}$. Since

$$\nu(a\cos\theta, b\sin\theta) = \frac{1}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2}}\left(\frac{\cos\theta}{a}, \frac{\sin\theta}{b}\right)$$

and

$$ds = ab\sqrt{\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2}\,d\theta,$$

we have

$$(\nu_1 - i\nu_2)ds = ab(A_- z + A_+ z^{-1})\frac{dz}{iz}.$$

Note also that

$$f(a\cos\theta, b\sin\theta) = \sum_m \gamma_m z^m$$

and

$$x_1 - ix_2 = B_- z + B_+ z^{-1}$$

where

$$B_\pm = \frac{a \pm b}{2}.$$

Using those expressions, we can write

$$\int_{\partial\Omega} u\, v\, (\nu_1 - i\nu_2) ds$$

$$= \frac{ab}{i} \sum_m \gamma_m \int_{|z|=1} (A_- z + A_+ z^{-1}) z^{m-1} \exp\left\{\tau(B_- z + B_+ z^{-1})(\omega_1 + i\omega_2)\right\} dz.$$

Define

$$I_l(\tau) = \int_{|z|=1} z^l \exp\left\{\tau(B_- z + B_+ z^{-1})(\omega_1 + i\omega_2)\right\} dz.$$

Consider the case when $a > b$. Using the generating function of the Bessel functions, we have

$$\exp\left\{\tau(B_- z + B_+ z^{-1})(\omega_1 + i\omega_2)\right\} = \sum_n J_n\left(-i\sqrt{a^2 - b^2}\tau(\omega_1 + i\omega_2)\right)\left(i\sqrt{\frac{a-b}{a+b}}\right)^n z^n$$

and therefore

$$I_l(\tau) = 2\pi i(-1)^{l+1} J_{l+1}\left(-i\sqrt{a^2 - b^2}\tau(\omega_1 + i\omega_2)\right)\left(-i\sqrt{\frac{a+b}{a-b}}\right)^{l+1}.$$

If $a = b$, then

$$I_l(\tau) = 0, \ l \le -2; \ I_l(\tau) = 2\pi i \frac{\{a\tau(\omega_1 + i\omega_2)\}^{l+1}}{(l+1)!}, \ l \ge -1.$$

Since

$$\int_{\partial\Omega} u\, v\, (\nu_1 - i\nu_2) ds = \frac{ab}{i} \sum_m \gamma_m \left(A_- I_m(\tau) + A_+ I_{m-2}(\tau)\right),$$

we obtain the desired conclusion. □

2.2   Main result

We denote by $E(\Omega)$ the set of all points on the segment that connects the focal points $(-\sqrt{a^2 - b^2}, 0)$ and $(\sqrt{a^2 - b^2}, 0)$ of Ω. It is easy to see that the support function of the set $E(\Omega)$ is given by the formula $h_{E(\Omega)}(\omega) = \sqrt{a^2 - b^2}|\omega_1|$.

We say that a function $f(\theta) = f(a\cos\theta, b\sin\theta)$ of θ is *band limited* if there exists a natural number $N \ge 1$ such that, for all $m \ge N+1$ the m-th Fourier coefficients α_m and β_m of the function vanish. Then we know that $C_m(f) = 0$ for all $m \ge N+2$.

Now we state the main result of this paper.

Theorem 2.1. *Let γ be a positive constant. Assume that (1.3) is satisfied. Let ω be regular with respect to D. Let f be band limited and u be the solution of (1.1) with $u = f$ on $\partial\Omega$.*

(1) Let $a > b$. Let ω satisfy $\omega_1 \neq 0$. Let f satisfy

$$\sum_{m=1}^{\infty} (sgn\ \omega_1)^m m^2 C_m(f) \neq 0. \tag{2.5}$$

The formula

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds \right| = \max \left(h_D(\omega), h_{E(\Omega)}(\omega) \right), \tag{2.6}$$

is valid.

(2) Let $a = b$. Let f satisfy: for some $N \geq 1$ $\alpha_m = \beta_m = 0$ for all m with $m \geq N + 1$ and $\alpha_N^2 + \beta_N^2 \neq 0$. The formula

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds \right| = \max \left(h_D(\omega), 0 \right), \tag{2.7}$$

is valid.

- We say that a D is *behind* the line $x \cdot \omega = t$ from the direction ω if the D is contained in the half plane $x \cdot \omega < t$. One important consequence of the formula (2.6) is: one can know whether the unknown cavity D is behind the line $x \cdot \omega = h_{E(\Omega)}(\omega)$ from the direction ω, however, in that case one cannot know the line $x \cdot \omega = h_D(\omega)$ itself from the formula. This shows the limit to extract the whole convex hull of D without an additional assumption.

- The assumption that f is band limited is just for a simplicity of the computation and can be relaxed. It is possible to apply directly the *saddle point method* to study the asymptotic behaviour of the integrals in Lemma 2.1 for a f that is not band limited. Moreover we want to point out that in a practical situation, one cannot produce highly oscillatory voltages on the boundary. This is due to the limit of numbers of electrodes attached on the boundary of the body.

- A typical example of a band-limited f that satisfies (2.5) for all ω with $\omega_1 \neq 0$ is the f given by

$$f(\theta) = A \cos N\theta + B \sin N\theta$$

where $N \geq 1$ and $A^2 + B^2 \neq 0$. See Remark 2.1 below for this explanation. In general we have to choose two fs corresponding to whether $\omega_1 > 0$ or $\omega_1 < 0$.

Proof of Theorem 2.1. When $a = b$, the (2.7) is an easy consequence of (1.2), (2.1), (2.2) and (2.3). The problem is the case when $a > b$. We employ the *compound asymptotic expansion* (see

page 118 of [8] for the notion of the compound asymptotic expansion) of the Bessel function due to Hankel(see (9.09) and 9.3 of page 133 in [8]):

$$J_m(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \times$$
$$\left\{ \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{A_{2s}(m)}{z^{2s}} - \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{A_{2s+1}(m)}{z^{2s+1}} \right\} \quad (2.8)$$

as $z \longrightarrow \infty$ in $|\arg z| \le \pi - \delta$ for each fixed $\delta \in]0, \pi[$ where $A_0(m) = 1$ and, for $s = 1, 2, \cdots$

$$A_s(m) = \frac{1}{s!8^s}(4m^2 - 1^2)(4m^2 - 3^2)\cdots(4m^2 - (2s-1)^2).$$

First we consider the case when $\omega_1 > 0$. From (2.8) in the case when $z = -i\sqrt{a^2 - b^2}\tau(\omega_1 + i\omega_2)$ we obtain

$$J_m(z) = \left(\frac{1}{2\pi z}\right)^{1/2} e^{iz}(-i)^m e^{-i\pi/4} \left\{ 1 - \frac{4m^2 - 1}{8iz} + O\left(\frac{1}{\tau^2}\right) \right\} \quad (2.9)$$

as $\tau \longrightarrow \infty$. Since f is band limited, one can find $N \ge 1$ such that, for all $m \ge N + 1$ the m-th Fourier coefficients α_m and β_m of f vanish. Then $C_m(f) = 0$ for $m \ge N + 2$ and from (2.4) and (2.9) we obtain

$$\int_{\partial\Omega} uv(\nu_1 - i\nu_2)ds = 2\pi ab \left(\frac{1}{2\pi z}\right)^{1/2} e^{iz}e^{-i\pi/4}$$

$$\times \left\{ \left(1 + \frac{1}{8iz}\right) \sum_{m=0}^{N+1} C_m(f) + i\frac{1}{2z} \sum_{m=1}^{N+1} m^2 C_m(f) + O\left(\frac{1}{\tau^2}\right) \right\}. \quad (2.10)$$

Here we claim that

$$\sum_{m=0}^{N+1} C_m(f) = 0. \quad (2.11)$$

It suffices to prove the claim in the case when

$$f(a\cos\theta, b\sin\theta) = \alpha_j \cos j\theta + \beta_j \sin j\theta \quad (2.12)$$

for each fixed $j = 1, 2, \cdots, N$. Since $\sum_{m=0}^{\infty} C_m(f) = C_{j-1}(f) + C_j(f) + C_{j+1}(f)$ and we have

$$C_{j+1}(f) = A_-\gamma_j \left(\sqrt{\frac{a+b}{a-b}}\right)^{j+1} + A_+\overline{\gamma_j} \left(\sqrt{\frac{a-b}{a+b}}\right)^{j+1},$$

$$C_j(f) = 0,$$

$$C_{j-1}(f) = A_+\gamma_j \left(\sqrt{\frac{a+b}{a-b}}\right)^{j-1} + A_-\overline{\gamma_j} \left(\sqrt{\frac{a-b}{a+b}}\right)^{j-1},$$

we get

$$\sum_{m=0}^{\infty} C_m(f) = \left\{ A_+ + A_- \left(\frac{a+b}{a-b} \right) \right\} \left\{ \gamma_j \left(\sqrt{\frac{a+b}{a-b}} \right)^{j-1} + \overline{\gamma_j} \left(\sqrt{\frac{a-b}{a+b}} \right)^{j+1} \right\}.$$

Since

$$A_+ + A_- \left(\frac{a+b}{a-b} \right) = 0,$$

we see that the claim (2.11) is valid. Therefore (2.10) becomes

$$\int_{\partial\Omega} uv(\nu_1 - i\nu_2)ds = i\pi abz^{-1} \left(\frac{1}{2\pi z} \right)^{1/2} e^{iz} e^{-i\pi/4} \left\{ \sum_{m=1}^{N+1} m^2 C_m(f) + O\left(\frac{1}{\tau} \right) \right\}. \qquad (2.13)$$

Set $\omega_1 + i\omega_2 = e^{i\vartheta}$ with $-\pi/2 < \vartheta < \pi/2$. Then $z^{1/2} = \sqrt{\tau}(a^2 - b^2)^{1/4}e^{i(\vartheta-\pi/2)/2}$. Since $e^{iz} = e^{\tau h_{E(\omega)}(\omega)}e^{i\tau\sqrt{a^2-b^2}\omega_2}$, from (1.2), (2.1), (2.2) and (2.13) we obtain the compound asymptotic formula:

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds$$

$$\sim -\gamma\sqrt{\frac{\pi}{2}} ab(a^2 - b^2)^{-3/4} e^{-i\vartheta/2}\tau^{-1/2} e^{\tau h_{E(\Omega)}(\omega)}e^{i\tau\sqrt{a^2-b^2}\omega_2} \sum_{m=1}^{N+1} m^2 C_m(f) - \gamma e^{\tau h_D(\omega)} \frac{A}{\tau^\mu}.$$

From this we know that the quantity

$$\exp\left\{ -\tau \max\left(h_D(\omega), h_{E(\Omega)}(\omega) \right) \right\} \left| \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds \right|$$

is truly *algebraic* decaying as $\tau \longrightarrow \infty$. Note that we have used the lower bound of μ: $\mu > 1/2$. Therefore we obtain the formula (2.6). Next consider the case when $\omega_1 < 0$. Write $R_\omega(\tau; f) = \int_{\partial\Omega} fv(\nu_1 - i\nu_2)ds$. Then we have $R_\omega(\tau; f) = -R_{-\omega}(\tau; f^*)$ where $f^*(x) = f(-x)$. Since the m-th Fourier coefficients of f^* are given by $(-1)^m$ times those of f and the first component of $-\omega$ is positive, we can derive the corresponding result in the case when $\omega_1 < 0$ from the result in the case when $\omega_1 > 0$ by replacing $C_m(f)$ in the condition (2.5) with $-(-1)^m C_m(f)$. $\square$

Remark 2.1. Fix $j = 1, 2, \cdots, N$ and let f be given by (2.12). Then a direct computation similar to the proof of the claim (2.11) yields

$$\sum_{m=1}^{\infty} m^2 C_m(f) = (j-1)^2 C_{j-1}(f) + (j+1)^2 C_{j+1}(f)$$

$$= -\frac{2}{ab} j(a^2 - b^2)^{-(j-1)/2} \{(a+b)^j \gamma_j - (a-b)^j \overline{\gamma_j}\}.$$

This yields also

$$\sum_{m=1}^{\infty}(-1)^m m^2 C_m(f) = (-1)^{j-1}\sum_{m=1}^{\infty} m^2 C_m(f)$$

$$= (-1)^j \frac{2}{ab} j(a^2-b^2)^{-(j-1)/2}\{(a+b)^j\gamma_j - (a-b)^j\overline{\gamma_j}\}.$$

These yield: a f whose Fourier coefficients α_j and β_j vanish for all $j \geq N+1$ with some $N \geq 1$, satisfies the condition (2.5) if and only if

$$\sum_{j=1}^{N}(\operatorname{sgn}\omega_1)^j j(a^2-b^2)^{-(j-1)/2}\{(a+b)^j\gamma_j - (a-b)^j\overline{\gamma_j}\} \neq 0. \tag{2.14}$$

It is clear that there are many fs satisfying the condition (2.14).

Remark 2.2. In (1) the case when $\omega_1 = 0$ is not treated. In this case $\omega_2 = \pm 1$. If $\omega_2 = 1$, then from (2.8) we have

$$J_m(z) = \left(\frac{1}{2\pi\sqrt{a^2-b^2}\tau}\right)^{1/2} \times$$

$$\left\{e^{i\tau\sqrt{a^2-b^2}}(-i)^m e^{-i\pi/4}\left(1 + i\frac{4m^2-1}{8z}\right) + e^{-i\tau\sqrt{a^2-b^2}} i^m e^{i\pi/4}\left(1 - i\frac{4m^2-1}{8z}\right)\right\} + O(\tau^{-5/2})$$

where $z = -i\sqrt{a^2-b^2}\,\tau(\omega_1 + i\omega_2)$. Then from (1.2), (2.1), (2.2) and (2.4) the problem can be reduced to the study of the asymptotic behaviour of the quantity

$$\sum_{m=0}^{N+1}\left\{e^{i\tau\sqrt{a^2-b^2}}e^{-i\pi/4}\left(1 + i\frac{4m^2-1}{8z}\right) + (-1)^m e^{-i\tau\sqrt{a^2-b^2}}e^{i\pi/4}\left(1 - i\frac{4m^2-1}{8z}\right)\right\} C_m(f) \tag{2.15}$$

as $\tau \longrightarrow \infty$. This seems very complicated for general τ. However, if we choose

$$\tau = \frac{l\pi}{\sqrt{a^2-b^2}}, \quad l = 1, 2, \cdots, \tag{2.16}$$

then (2.15) becomes

$$(-1)^l e^{-i\pi/4}\left\{\sum_{m=0}^{N+1}\left(1 + i\frac{4m^2-1}{8z}\right) C_m(f) + i\sum_{m=0}^{N+1}(-1)^m\left(1 - i\frac{4m^2-1}{8z}\right) C_m(f)\right\}$$

$$= \frac{(-1)^l e^{-i\pi/4} i}{2z}\sum_{m=1}^{N+1} m^2\{C_m(f) - iC_m(f^*)\}.$$

Note that we have used the claim (2.11) for f and f^*. Therefore if f satisfies the condition

$$\sum_{m=1}^{\infty} m^2\{C_m(f) - iC_m(f^*)\} \neq 0 \tag{2.17}$$

instead of (2.5), then for τ given by (2.16), the formula

$$\lim_{l \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v ds \right| = \max\left(h_D(\omega), 0\right),$$

is valid. By replacing f with f^*, we know also that: if $\omega_2 = -1$, then the same formula is valid provided

$$\sum_{m=1}^{\infty} m^2 \{C_m(f) + iC_m(f^*)\} \neq 0 \tag{2.18}$$

instead of (2.17). From the computation in Remark 2.1 one can sum the conditions (2.17) and (2.18) up in the single form:

$$\sum_{j=1}^{N} \left\{1 + (-1)^j (\operatorname{sgn} \omega_2) i\right\} j (a^2 - b^2)^{-(j-1)/2} \{(a+b)^j \gamma_j - (a-b)^j \overline{\gamma_j}\} \neq 0$$

where $N \geq 1$ and chosen in such a way that, for all $m \geq N + 1$ the m-th Fourier coefficients of f vanish.

2.3   Uniqueness

As a corollary of Theorem 2.1 we obtain a uniqueness theorem.

Corollary 2.1. *Let γ be a positive constant. Assume that D satisfies (1.3).*

(1) Let Ω be a domain enclosed by an ellipse. Let f_+ and f_- be band limited and satisfy

$$\sum_{m=1}^{\infty} (\pm)^m m^2 C_m(f_{\pm}) \neq 0.$$

Let $u_{\pm}$ be the solution of (1.1) with $u_{\pm} = f_{\pm}$ on $\partial\Omega$. Then the Neumann data $\gamma \partial u_+/\partial \nu$ and $\gamma \partial u_-/\partial \nu$ on $\partial\Omega$ uniquely determine the convex hull of $D \cup E(\Omega)$.

(2) Let Ω be a domain enclosed by a circle. Let f be band limited and non constant. Let u be the solution of (1.1) with $u = f$ on $\partial\Omega$. Then the Neumann data $\gamma \partial u/\partial \nu$ uniquely determines the convex hull of $D \cup \{0\}$.

We emphasize that γ is *unknown*. This makes the situation difficult definitely. Assume that we have two unknowns $(D, \gamma) = (D_1, \gamma_1), (D_2, \gamma_2)$ and solutions u_1 and u_2 both satisfying (1.1) and the boundary condition $u = f$ on $\partial\Omega$. The key point of a standard and traditional approach is to prove that if $\gamma_1 \partial u_1/\partial \nu = \gamma_2 \partial u_2/\partial \nu$ on $\partial\Omega$, then $u_1 = u_2$ in a neighbourhood of $\partial\Omega$. If $\gamma_1 = \gamma_2$, then the conclusion is true because of the uniqueness of the Cauchy problem for the Laplace equation. However, if γ is unknown, i.e., the assumption $\gamma_1 = \gamma_2$ is dropped, one can not immediately get the conclusion (note that we are considering a finite set of observation data not the full Dirichlet-to-Neumann map). Our approach skips this point by using an analytical formula that directly connects the data with unknown discontinuity.

The proof of Corollary 2.1 is based on: given D the set of all directions that are not regular with respect to D is a finite set; the formulae (2.6) are valid for $f = f_\pm$ in (1); the formula (2.7) is valid for f in (2). Therefore, for example, in (1) we see that the Neumann data uniquely determine the values of $\max(h_D(\omega), h_{E(\Omega)}(\omega))$ which is the support function of the convex hull of $D \cup E(\Omega)$ at the directions ω except for a finite set of directions. Since the support function h_D and $h_{E(\Omega)}$ are continues on the unit circle and so is $\max(h_D(\,\cdot\,), h_{E(\Omega)}(\,\cdot\,))$. A density argument yields the desired uniqueness.

Remark 2.3. If ∂D is *smooth*, then (2) of Corollary 2.1 does not hold. Let Ω be the unit open disc centered at the origin of the coordinates system and for $0 < R < 1$ let $D(R)$ be the open disc centered at the origin with the radius R. Let $0 < R_1, R_2 < 1$. Fix an integer $m \geq 1$. For each $j = 1, 2$ let u_j be the weak solution of the problem (1.1) with $D = D(R_j)$ and the Dirichlet data $u_j(r, \theta)|_{r=1} = \cos m\theta$ where (r, θ) denotes the usual polar coordinates centered at the origin. Then we know that

$$u_1(r, \theta) = \frac{1}{1 + R_1^{2m}}(r^m + R_1^{2m}r^{-m}) \cos m\theta, \ \ u_2(r, \theta) = \frac{1}{1 + R_2^{2m}}(r^m + R_2^{2m}r^{-m}) \cos m\theta.$$

This yields

$$\frac{1 + R_2^{2m}}{1 - R_2^{2m}} \frac{\partial u_2}{\partial \nu} = m \cos m\theta = \frac{1 + R_1^{2m}}{1 - R_1^{2m}} \frac{\partial u_1}{\partial \nu} \ \text{on}\, \partial\Omega.$$

Since R_1 and R_2 are arbitrary chosen, this means that one cannot uniquely determine $D(R)$ from the single set of the Dirichlet and Neumann data $f(\theta) = \cos m\theta$ and $\gamma \partial u/\partial \nu$ on $\partial\Omega$ in the case when $\gamma = (1 + R^{2m})(1 - R^{2m})$. This suggests that the *singularity* of ∂D is essential for the validity of (2) in Corollary 2.1.

3   An application to the inverse conductivity problem

The idea in the proof of Theorem 2.1 can be applied to the case when the unknown domain D is a model of an *inclusion*.

We assume that the conductivity $k = k(x)$ of the body that occupies Ω is given by $k(x) = \gamma$ if $x \in \Omega \setminus D$; $k(x) = \tilde{\gamma}$ if $x \in D$. It is assumed that the γ and $\tilde{\gamma}$ are positive constants and satisfy $\gamma \neq \tilde{\gamma}$. The voltage u inside the body satisfies the equation $\nabla \cdot k\nabla u = 0$ in Ω. Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. Let $\tau > 0$ and $v = e^{\tau x \cdot (\omega + i\omega^\perp)}$.

In [6] we have already proved that if u is not a constant function and D is polygonal and satisfies the condition (1.3), then for a given direction ω that is regular with respect to D the formula

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} \left(\gamma \frac{\partial u}{\partial \nu} v - \gamma \frac{\partial v}{\partial \nu} u \right) ds \right| = h_D(\omega),$$

is valid. Note that $k = \gamma$ on $\partial\Omega$ and we do not assume that the conductivity $\tilde{\gamma}$ of D is known.

Here we propose the same question as that of Introduction. Assume that we do not know k in the whole domain. Given a non constant voltage $f = u|_{\partial\Omega}$ on $\partial\Omega$ is it possible to extract some information about the location of D from the corresponding current density $k\partial u/\partial\nu$ on $\partial\Omega$?

The answer is yes in the case when the Ω is enclosed by an ellipse. It starts with recalling the equation

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial\nu} v \, ds = \int_{\partial\Omega} \gamma \frac{\partial v}{\partial\nu} u \, ds - (\gamma - \tilde{\gamma}) \int_{\partial D} u \frac{\partial v}{\partial\nu} ds. \tag{3.1}$$

Recall Key Lemma in [6]: there exist positive constants B and $\lambda(> 1/2)$ such that, as $\tau \longrightarrow \infty$

$$\left| \int_{\partial D} u \frac{\partial v}{\partial\nu} ds \right| \sim \frac{B}{\tau^\lambda} e^{\tau h_D(\omega)}. \tag{3.2}$$

Then from (2.2), (3.1), (3.2) and Lemma 2.1 we see that the completely same statements as those in Theorem 2.1, Corollary 2.1 and Remarks 2.1 and 2.2 are valid.

Remark 3.1. In [7] we employed the *difference* of the values of the voltage at arbitrary fixed two points on the boundary of a *general* two-dimensional bounded domain Ω with smooth boundary. More precisely we introduced the operator

$$\Lambda_k(P, Q) : g \longmapsto u(P) - u(Q)$$

where P and Q are two arbitrary points on $\partial\Omega$; g satisfies $\int_{\partial\Omega} g \, ds = 0$; the u is a solution of the equation $\nabla \cdot k\nabla u = 0$ in Ω and satisfies the Neumann boundary condition $k\partial u/\partial\nu = g$ on $\partial\Omega$.

Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. Let $\tau > 0$ and $v = e^{\tau x \cdot (\omega + i\omega^\perp)}$. What we have proved is: if $g = \partial v/\partial\nu$ on $\partial\Omega$ and D is polygonal and satisfies the condition (1.3), then for a given direction ω that is regular with respect to D the formula

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log | \{\Lambda_k(P, Q) - \Lambda_\gamma(P, Q)\} (g)| = h_D(\omega), \tag{3.3}$$

is valid. Note that we have used the relationship

$$\{\Lambda_k(P, Q) - \Lambda_\gamma(P, Q)\} (g) = \frac{1}{\gamma} \{\Lambda_{k/\gamma}(P, Q) - \Lambda_1(P, Q)\} (g).$$

If γ is unknown, then one cannot use the term $\Lambda_\gamma(P, Q)(g)$ in (3.3). However, that has the simple form

$$\Lambda_\gamma(P, Q)(g) = \frac{1}{\gamma} \{v(P) - v(Q)\}$$

for $g = \partial v/\partial\nu$ on $\partial\Omega$. Using this form, Proposition 3.1 and Lemma 3.1 in [7], one immediately gets the following formulae provided D is polygonal and satisfies the condition (1.3) and ω is regular with respect to D:

- if ω is not perpendicular to the line passing through P and Q, then

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log |\Lambda_k(P, Q)(g)| = \max \left(h_D(\omega), h_{\{P, Q\}}(\omega) \right);$$

- if ω is perpendicular to the line passing through P and Q, choose, for example,

$$\tau = \frac{\pi}{|P - Q|} \left(\frac{1}{2} + 2l \right), \; l = 0, 1, 2, \cdots,$$

then

$$\lim_{l \longrightarrow \infty} \frac{1}{\tau} \log |\Lambda_k(P, Q)(g)| = \max \left(h_D(\omega), h_{\{P, Q\}}(\omega) \right).$$

4　Conclusion

We confirmed that: in the case when the background conductivity is homogeneous and *unknown* the enclosure method still works provided:

- the domain that is occupied by a background body has a simple geometry;

- the Fourier series expansion of the voltage on the boundary does not contain high frequency parts (band limited) and satisfies a non vanishing condition of a quantity involving the Fourier coefficients.

However, the method yields a less information about the location and shape of unknown cavity or inclusion compared with the case when the conductivity is *known*. We found an explicit *obstruction* that depends on the geometry of the background body.

Acknowledgements

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