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Prime Factorization of Entire Functions

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ABSTRACT

Let n be a prime number and let $f(z)$ be a transcendental entire function. Then it is proved that both $[f(z)+cz]^n$ and $[f(z)+cz]^{-n}$ are uniquely factorizable for any complex number c, except for a countable set in $\mathbb{C}$.

RESUMEN

Sea n un número primo y $f(z)$ una función entera transcendental. Entonces ambos $[f(z)+cz]^n$ y $[f(z)+cz]^{-n}$ se factorizan de manera única para cualquier número complejo c, excepto para un conjunto numerable en $\mathbb{C}$.

Key words and phrases: *entire function, unique factorization.*

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1 Introduction

The fundamental theorem of elementary number theory states that every integer $n \geq 2$ can be expressed uniquely as the product of primes in the form

$$n = p_1^{m_1} \cdots p_k^{m_k}, \qquad \text{for} \quad k \geq 1,$$

with distinct prime factors $p_1, \ldots, p_k$ and corresponding exponents $m_1 \geq 1, \ldots, m_k \geq 1$ uniquely determined by n. For example, $2700 = 2^2 3^3 5^2$.

In 1922, Ritt ([14]) generalized this theorem to polynomials. To state his result, we introduce the following concepts.

Let $F(z)$ be a nonconstant meromorphic function. A decomposition

$$F(z) = f(g(z)) = f \circ g(z) \tag{1}$$

will be called a factorization of $F(z)$ with $f(z)$ and $g(z)$ being the left and right factors of $F(z)$, respectively, where $f(z)$ is meromorphic and $g(z)$ is entire ($g(z)$ may be meromorphic when $f(z)$ is rational) (see [2], [4], [19]).

A function $F(z)$ is said to be prime (pseudo-prime) if $F(z)$ is nonlinear and every factorization of the form (1) implies that either $f(z)$ is fractional linear or $g(z)$ is linear (either $f(z)$ is rational or $g(z)$ is a polynomial).

Example 1 *$e^z + z$ is prime.*

This is stated by Rosenbloom [15] and proved by Gross [3].

Example 2 *$(\cos z)e^{az+b} + p(z)$ is prime, where a $(\neq 0)$ and b are constants, and $p(z)$ is a nonconstant polynomial.*

This was conjectured by Gross–Yang [5] and proved by Hua [7].

Suppose that a function $F(z)$ has two prime factorizations

$$F(z) = f_1 \circ \cdots \circ f_m(z) = g_1 \circ \cdots \circ g_n(z),$$

i.e., f_i $(i = 1, \ldots, m)$ and g_j $(j = 1, \ldots, n)$ are prime functions. If $m = n$ and if there exist linear functions L_j $(j = 1, \ldots, n-1)$ such that

$$f_1(z) = g_1 \circ L_1^{-1}, \quad f_2(z) = L_1 \circ g_2 \circ L_2^{-1}, \quad \ldots, \quad f_n(z) = L_n^{-1} \circ g_n(z),$$

then the two factorizations are called equivalent. If any two prime factorizations of $F(z)$ are equivalent, then $F(z)$ is called uniquely factorizable. In particular, for an entire function

$F(z)$, if any two prime entire factorizations of $F(z)$ are equivalent, then $F(z)$ is called uniquely factorizable in the entire sense.

Ritt [14] proved the following result.

Proposition 1 *Let $p(z)$ be a nonlinear polynomial. If $p(z)$ has two prime factorizations*

$$p(z) = p_1 \circ \cdots \circ p_m(z) = q_1 \circ \cdots \circ q_n(z),$$

where p_i $(i = 1, \ldots, m)$ and q_j $(j = 1, \ldots, n)$ are polynomials, then $m = n$. Moreover, one factorization can be changed to another one by a sequence of applications of any of the following three ways:

1. *replace p_i and p_{i+1} by $p_i \circ L$ and $L^{-1} \circ p_{i+1}$, respectively;*
2. *alternate p_i and p_{i+1} when both are Chebychev polynomials;*
3. *replace z^k and $z^s h(z^k)$ by $z^s h(z)^k$ and z^k, respectively, where $h(z)$ is a polynomial, and s and k are natural numbers.*

Example 3 $z^{10} + 1 = (z^5 + 1) \circ z^2 = (z^2 + 1) \circ z^5.$

However, Ritt's result cannot be extended to rational functions.

Example 4 $z^3 \circ \frac{z^2-4}{z-1} \circ \frac{z^2+2}{z+1} = \frac{z(z-8)^3}{(z+1)^3} \circ z^3.$

This example was given by Michael Zieve (see [1]).

For transcendental functions, the diverse cases are very complex. For example, e^z can have infinitely many nonlinear factors.

Example 5 *For any integer n,*

$$e^z = z^2 \circ z^3 \circ \cdots \circ z^n \circ e^{z/n!}.$$

The following example shows that transcendental entire functions can have non-equivalent prime factorizations (see [10]).

Example 6

$$z^2 \circ \left(ze^{z^2}\right) = \left(ze^{2z}\right) \circ z^2.$$

Of course, there are functions which are uniquely factorizable. The following example is given by Urabe [17].

Example 7 *For any two nonconstant polynomials $p(z)$ and $q(z)$,*

$$(z + e^{p(e^z)}) \circ (z + q(e^z))$$

is uniquely factorizable.

The following result, proved by Hua [6], shows that, for a given function, we can construct uncountably many uniquely factorizable functions.

Proposition 2 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then both $f(z^n) - cz^n$ and $(z^n - c)f(z^n)$ are uniquely factorizable for any complex number c except for a countable set.*

In this paper, we prove the following two results.

Theorem 1 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then $[f(z) - cz]^n$ is uniquely factorizable for any complex number c except for a countable set.*

Theorem 2 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then $[f(z) - cz]^{-n}$ is uniquely factorizable for any complex number c except for a countable set.*

2 Some Lemmas

The following lemmas will be used in the proof of the theorems.

Lemma 1 ([4]) *Suppose that $p(z)$ is a nonconstant polynomial and $g(z)$ is entire. Then $p(g(z))$ is periodic if and only if $g(z)$ is periodic.*

Lemma 2 ([11]) *Let $f(z)$ be a transcendental entire function. Then for any complex number c except for a countable set, $f(z) - cz$ is prime.*

Remark. So far, there is no example with countably infinite exceptions. In [13], it is proved that there is at most one exception for $f(z) = g(e^z)$, where $g(z)$ is an entire function satisfying $\max_{|z|=r} |g(z)| \leq e^{Kr}$ for a positive constant K. In [8] and [18], some other functions $f(z)$ are studied.

Lemma 3 ([12]) *Let $f(z)$ be a transcendental entire function. We denote by $\nu(a,f)$ the least order of almost all zeros of $f(z)-a$, where "almost all" means all with possibly finite exceptions. Then*

$$\sum_{a\neq\infty}\left(1-\frac{1}{\nu(a,f)}\right)\leq 1.$$

Lemma 4 ([16]) *Let $f(z)$ and $g(z)$ be prime entire functions. Assume that both $f(z)$ and $F(z)=f(g(z))$ are non-periodic. Then $F(z)$ is uniquely factorizable if and only if $F(z)$ is uniquely factorizable in the entire sense.*

Lemma 5 *Let $f(z)$ be a nonconstant meromorphic function. Then $f(z)-cz$ is non-periodic for any complex number c with at most one exception.*

Proof of Lemma 5. Suppose there exist two different numbers c and d such that $f(z)-cz$ and $f(z)-dz$ are periodic with period u and v, respectively. Then $f'(z)$ is periodic and $f'(z+u)=f'(z)=f'(z+v)$. Let w be the period of $f'(z)$. Then there exist two nonzero integers m and k such that $u=mw$ and $v=kw$. This implies that $u=\frac{m}{k}v$. Hence

$$\begin{aligned}
f(z)-cz &= f(z+ku)-c(z+ku)\\
&= f(z+mv)-c(z+ku)\\
&= f(z+mv)-d(z+mv)+d(z+mv)-c(z+ku)\\
&= f(z)-dz+d(z+mv)-c(z+ku)\\
&= f(z)-cz+dmv-cku.
\end{aligned}$$

Therefore $dmv=cku$, and so, $d=c$, which is a contradiction. $\square$

The following lemma is a simple version of the so-called Borel Unicity Theorem which can be found in [2] and [4].

Lemma 6 *Let $h_0(z),\ldots,h_n(z)$ be rational functions and let $g_1(z),\ldots,g_n(z)$ be nonconstant entire functions such that*

$$\sum_{j=1}^{n}h_j(z)e^{g_j(z)}=h_0(z).$$

Then $h_0=0$.

Lemma 7 *Let $f(z)$ be a transcendental entire function. Then*

$$f(z)-cz\neq P(z)e^{f_1(z)}$$

for all $c\in\mathbb{C}$ with at most one exception, where $P(z)$ is a polynomial and $f_1(z)$ is a nonconstant entire function.

Proof of Lemma 7. Suppose to the contrary that there exist two different constants c and d, two polynomials $P_1(z)$ and $P_2(z)$, and two nonconstant entire functions $f_1(z)$ and $f_2(z)$ such that

$$f(z) - cz = P_1(z)e^{f_1(z)}$$

and

$$f(z) - dz = P_2(z)e^{f_2(z)}.$$

Then

$$cz - dz = P_2(z)e^{f_2(z)} - P_1(z)e^{f_1(z)}.$$

By Lemma 6, $cz - dz = 0$; thus $d = c$ which is a contradiction. $\square$

3 Proof of Theorem 1

Let

$$F(z) = [f(z) - cz]^n = z^n \circ (f(z) - cz).$$

Obviously, z^n is non-periodic.

Let

$$Z(f) = \{f(z) : \; f''(z) = 0\}.$$

Then $Z(f)$ is a countable set, and for any $c \notin Z(f)$, $f'(z) - c$ has only simple zeros ([9, Theorem F]). We combine $Z(f)$ and all the exceptions (if any) in Lemmas 1, 2, 5 and 7 to form an exceptional set E. Then E is a countable set which may be empty. For any $c \in \mathbb{C} - E$, we have the following properties:

(P1) The function $F(z)$ is non-periodic;

(P2) The function $f(z) - cz$ is prime;

(P3) $f'(z) - c$ has only simple zeros.

(P4) $f(z) - cz \neq P(z)e^{f_1(z)}$ for any polynomial $P(z)$ and nonconstant entire function $f_1(z)$.

Next we assume $c \in \mathbb{C} - E$.

By Lemma 4, we need only prove that $F(z)$ is uniquely factorizable in the entire sense, which means, we just need to consider entire factors. Assume that

$$F(z) = g(z) \circ h(z), \tag{2}$$

where $g(z)$ and $h(z)$ are nonconstant entire functions. We consider three cases.

Case 1. $g(z)$ has at least two zeros, z_1 and z_2, of order m_1 and m_2, respectively, such that $(n, m_1) = (n, m_2) = 1$, that is, n and m_i $(i = 1, 2)$ have no common factors other than 1. Then by (2) and the fact that n is prime, the order of any zero of $h(z) - z_i$ $(i = 1, 2)$ should be a multiple of n. Hence

$$\nu(z_i, h) \geq n \geq 3 \quad (i = 1, 2),$$

which implies that

$$\sum_{a \neq \infty} \left(1 - \frac{1}{\nu(a, f)}\right) \geq 1 - \frac{1}{3} + 1 - \frac{1}{3} > 1.$$

This is a contradiction to Lemma 3.

Case 2. $g(z)$ has one zero, z_0, of order m such that $(n, m) = 1$. Then by (2) and the fact that n is prime, $g(z)$ and $h(z)$ can be written as

$$g(z) = (z - z_0)^r g_1(z)^n, \quad h(z) = z_0 + h_1(z)^n, \quad r = m \ (\mathrm{mod}\ n), \tag{3}$$

where $g_1(z)$ and $h_1(z)$ are entire functions. Obviously, $1 \leq r < n$. Substituting (3) into (2) we have

$$F(z) = h_1(z)^{rn}[g_1(z_0 + h_1(z)^n)]^n,$$

which implies that

$$\begin{aligned} f(z) - cz &= uh_1(z)^r g_1(z_0 + h_1(z)^n) \\ &= [uz^r g_1(z_0 + z^n)] \circ h_1(z), \end{aligned} \tag{4}$$

where u is an n-th root of unity. Since $f(z) - cz$ is prime, we have two subcases as follows.

Case 2.1. Since the left factor $uz^r g_1(z_0 + z^n)$ is linear, then $r = 1$ and g_1 is a constant. It follows from (3) that $g(z)$ is linear. This is a trivial case.

Case 2.2. The right factor $h_1(z)$ is linear. Let $h_1(z) = az + b$ $(a, b \in \mathbb{C}, a \neq 0)$. By (4),

$$f(z) - cz = u(az + b)^r g_1[z_0 + (az + b)^n)]. \tag{5}$$

If $g_1(z)$ has a zero, then by differentiating (5) we see that $f'(z) - c$ has a zero of order $n - 1 \geq 2$, which is a multiple zero of $f'(z) - c$. This contradicts (P3). Therefore $g_1(z)$ has no zero. This implies that there exists a nonconstant entire function $g_2(z)$ such that $g_1(z) = e^{g_2(z)}$. By (5),

$$f(z) - cz = u(az + b)^r e^{g_2[z_0 + (az+b)^n)]},$$

which contradicts (P4).

Case 3. The order of any zero of $g(z)$ is a multiple of n. Then there exists an entire function $g_2(z)$ such that

$$g(z) = g_2(z)^n. \tag{6}$$

It follows from (2) that

$$[f(z) - cz]^n = [g_2 \circ h(z)]^n,$$

and so,

$$f(z) - cz = ug_2(z) \circ h(z)$$

for an n-th root of unity, u. Since $f(z) - cz$ is prime, we have two subcases.

Case 3.1. The left factor $ug_2(z)$ is linear. It follows from (6) that $g(z) = z^n \circ L(z)$ for a linear function $L(z)$. Therefore we get an equivalent factorization.

Case 3.2. The right factor $h(z)$ is linear. This is a trivial case.

The proof is complete. □

4 Proof of Theorem 2

Assume that

$$[f(z) - cz]^{-n} = g(z) \circ h(z),$$

where $g(z)$ is a nonconstant meromorphic function and $h(z)$ is a nonconstant entire function. Then we have

$$[f(z) - cz]^n = \frac{1}{g(z)} \circ h(z).$$

Now, since the left-hand side is entire, the conclusion follows from Lemma 4 and Theorem 1. □

5 Open Questions

Question 1 *Can n be 2 in Theorems 1 and 2?*

Question 2 *What kind of rational functions are uniquely factorizable?*

Question 3 *Is $(z + e^{e^z}) \circ (z + e^{e^z})$ uniquely factorizable?*

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