CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$01, (77–92). March 2008

Regular and Strongly Regular Time and Norm Optimal Controls

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ABSTRACT

Pontryagin's maximum principle in its infinite dimensional version provides (separate) necessary and sufficient conditions for both time and norm optimality for the system $y' = Ay + u$ (A the infinitesimal generator of a strongly continuous semigroup). Among controls that satisfy the maximum principle, a smoothness distinction can be defined in terms of smoothness of the final value of the costate. This paper addresses some issues related to this distinction.

RESUMEN

El principio del máximo de Pontryagin, en su version de dimension infinita, proporciona condiciones necesarias y suficientes (separadamente) para optimalidad en el tiempo y en la norma para el sistema $y' = Ay + u$ (A el generador infinitesimal de un semigrupo fuertemente contínuo). Entre los controles que satisfacen el principio del máximo se puede establecer una jerarquía de regularidad en términos de la regularidad del valor final del co-estado. Este artículo considera algunas cuestiones relacionadas con ésta jerarquía.

Key words and phrases: *linear control systems in Banach spaces, time optimal problem, norm optimal problem*

Math. Subj. Class.: *93E20, 93E25.*

1 Introduction.

We consider the control system

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0, T; E)$, where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ in a Banach space E. In the *norm optimal* problem we drive the initial point ζ to a point target,

$$y(T\,) = \bar{y}$$

in a fixed time interval $0 \le t \le T$ minimizing $\|u(\cdot)\|_{L^\infty(0,T;E)}$, while in the *time optimal* problem we drive to the target with a bound on the norm of the control (say $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$) in optimal time T. Solutions or *trajectories*

$$y(t) = S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma$$

of the initial value problem (1.1) are continuous and denoted by $y(t) = y(t, \zeta, u)$. For the time optimal problem, controls in $L^\infty(0, T; E)$ with norm $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$ are named *admissible.*

Separate necessary and sufficient conditions for both norm and time optimality can be given in terms of the maximum principle, which requires the construction of spaces of multipliers (final values of the costates). We summarize [5] or [7, **2.3**]. When the infinitesimal generator A has a bounded inverse, we define the space E^*_{-1} as the completion of E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^* y^*\|_{E^*}\,.$$

Each $S(t)^*$ can be extended to an (equally named) operator $S(t)^* : E^*_{-1} \to E^*_{-1}$, and the space $Z^1(T)$ consists of all $z \in E^*_{-1}$ such that $S(t)^* z \in E^*$ and[1]

$$\|z\|_{Z^1(T)} = \int_0^T \|S(t)^* z\| dt < \infty \,. \tag{1.2}$$

Equipped with $\|\cdot\|_{Z^1(T)}$, $Z^1(T)$ is a Banach space. All spaces $Z^1(T)$ coincide and all norms $\|\cdot\|_{Z^1(T)}$ are equivalent for $T > 0$. $Z^1(T)$ is an example of a *multiplier space,* defined as an arbitrary linear space $\mathcal{Z} \supseteq E^*$ to which $S(t)^*$ can be extended in such a way that $S(t)^* \mathcal{Z} \subseteq E^*$. When A does not have a bounded inverse, the construction of the spaces above is modified as follows. Since A is a semigroup generator, $(\lambda I - A)^{-1}$ exists for $\lambda > \omega$ and E^*_{-1} is the completion of E^* in any of the equivalent norms

$$\|y^*\|_{E^*_{-1}, \lambda} = \|((\lambda I - A)^{-1})^* y^*\|_{E^*} \,, \quad (\lambda > \omega) \,.$$

The definition of $Z^1(T)$ (and of multiplier spaces) is the same. See [8, **2.3**] for details.

A control $u(\cdot) \in L^\infty(0, T; E)$ satisfies *Pontryagin's maximum principle* if

$$\langle S(T-t)^* z, \bar{u}(t) \rangle = \max_{\|u\| \le \rho} \langle S(T-t)^* z, u \rangle \quad \text{a. e. in } 0 \le t < T \,, \tag{1.3}$$

$\langle \cdot\,, \cdot \rangle$ the duality of the space E and the dual E^*, with $\rho = \|u(\cdot)\|_{L^\infty(0,T;E)}$ and z in some multiplier space $\mathcal{Z}$. We call z the *multiplier* and $S(T-t)^* z$ the *costate* corresponding to the control $\bar{u}(t)$. We work under the standing assumption that (1.3) is nonempty; this means $S(T-t)^* z$ is not identically zero in the interval $0 \le t < T$, although we don't mind $S(T-t)^*$ vanishing in part of the interval (in which part (1.3) provides no information on $\bar{u}(t)$). The assumption that (1.3) is nonempty implies in particular that $z \neq 0$. The maximum principle takes a simple form when E is a Hilbert space; if fact, it reduces to

$$\bar{u}(t) = \rho \, \frac{S(T-t)^* z}{\|S(T-t)^* z\|} \quad \text{a. e. in } \ 0 \le t < T \,, \tag{1.4}$$

where $S(T-t)^* z \neq 0$ in $0 \le t < T$.

A large part of the theory of optimal controls for the system (1.1) deals with the relation between optimality and the maximum principle (1.3), a relation which is elementary in finite dimension but becomes rather involved in an infinite dimensional space E. All one has (at present) are separate necessary and sufficient conditions for optimality based on the maximum principle (Theorem 1.1 below). We call an optimal control $\bar{u}(t)$ *regular* if it satisfies (1.3) with $z \in Z^1(T)$.

[1] At this level of generality, the semigroup $S(t)^*$ may not be strongly continuous, or even strongly measurable (consider, for instance, the translation semigroup $S(t)y(x) = y(x-t)$ in $E = L^1(\infty, \infty)$). However, $S(t)^*$ is always E-weakly continuous, which guarantees that $\|S(t)^*\|$ is lower semicontinuous, hence measurable. This gives sense to the integral (1.2). Note also that in existing literature (for instance, [7]) $Z^1(T)$ is called $Z(T)$ (sometimes $Z_w(T)$ for "weak" to emphasize that $S(t)^*$ may not be strongly continuous). We use the superindex 1 since spaces $Z^p(T)$ $(p \neq 1)$ will be introduced later.

Theorem 1.1. *Assume $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ time or norm optimally in the interval $0 \le t \le T$ and that*

$$\bar{y} - S(T)\zeta \in D(A)\,. \tag{1.5}$$

Then $u(t)$ is regular. Conversely, let $\bar{u}(t)$ be a regular control. Then $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ norm optimally in the interval $0 \le t \le T$; if $\rho = 1$ the drive is time optimal.

For the proof see [5, Theorem 5.1], [7, Theorem 2.5.1]; we note that in the sufficiency half of Theorem 1.1 no conditions of the type of (1.5) are put on the initial value ζ or the target $\bar{y}$.[2] A control $u(\cdot)$ is called *strongly regular* if it satisfies (1.3) with $z \in E^*$. The notion of strongly regular control adds nothing to the two implications in Theorem 1.1, but it is of interest in applications. In fact, if E^* is a Hilbert space then (1.4) shows that a strongly regular control is (at least) continuous in $0 \le t \le T$, whereas a merely regular control may "oscillate" at the endpoint T of the control interval. This makes a difference, for instance, in numerical approximations of the optimal control.[3]

The question addressed in this paper is, *characterize the control systems* (1.1) *for which all* (*time, norm*) *optimal controls are strongly regular.* Part of the answer to this question is known; a sufficient condition for all optimal controls being strongly regular is

$$S(t)E = E \quad (t > 0)\,. \tag{1.6}$$

This condition is valid in any Banach space (Theorem 2.1 below). The main contribution of this paper is the opposite implication, which we only prove under special assumptions on E (Corollary 4.8). We also show (Remark 4.9) that if these special assumptions are dropped, the implication ceases to be true.

2 Reversible semigroups.

Semigroups satisfying (1.6) we call *reversible.* In this section, no restrictions are placed upon the Banach space E.

Theorem 2.1. *Let $S(t)$ be a reversible semigroup. Then all optimal controls for* (1.1) *are strongly regular, that is, they satisfy* (1.3) *with $z \in E^*$.*

[2] The statement on time optimality, however, needs additional assumptions on the initial condition ζ and the target $\bar{y}$. These conditions are satisfied if either $\zeta = 0$ or $\bar{y} = 0$ [6], [7, Theorem 2.5.7]. We point out that the conditions are on the "size" of ζ $\bar{y}$, not on their smoothness like (1.5); for instance, for $\zeta = 0$, $\bar{y}$ may be an arbitrary element of E. We also need to assume that $S(t)^* z \ne 0$ in the entire interval $0 \le t \le T$.

[3] Piermarco Cannarsa has pointed out situations involving optimal controls for semilinear equations, where strong regularity of (linear) optimal controls is actually needed; plain regularity is not enough.

The proof of Theorem 2.1 requires some auxiliary results.

Lemma 2.2. *Let the E^*-valued, E-weakly continuous function $f(t)$ satisfy*

$$\int_0^T \langle f(t), u(t)\rangle dt \le C\bigg(\int_0^T \|u(t)\|^p\bigg)^{1/p} dt \quad (u(\cdot) \in L^\infty(0,T;E)) \tag{2.1}$$

for some p, $1 \le p < \infty$. Then, if $p > 1$ and $1/p + 1/q = 1$ we have

$$\bigg(\int_0^T \|f(t)\|^q\bigg)^{1/q} \le C\,, \tag{2.2}$$

with equality in (2.2) *if C is the smallest constant satisfying* (2.1). *If $p = 1$,*

$$\|f(t)\| \le C \quad (0 \le t \le T)\,, \tag{2.3}$$

with equality in (2.3) *if C is the smallest constant satisfying* (2.1).

For $p > 1$ the proof of Lemma 2.2 is essentially similar to that of [7, Lemma 2.2.1 and Lemma 2.2.10] thus we omit it. For $p = \infty$, assume (2.3) fails. Then there exists $y \in E$ and a nontrivial interval e such that $\langle f(t), y\rangle \ge (C+\epsilon)\|y\|$. Setting

$$u(t) = \begin{cases} y & t \in e \\ 0 & t \notin e \end{cases}$$

we obtain

$$\int_0^T \langle f(t), u(t)\rangle dt = \int_e \langle f(t), y\rangle dt \ge (C+\epsilon)|e|\|y\| = (C+\epsilon)\int_e \|u(t)\| dt\,,$$

contradicting (2.1). This completes the proof.

Given $T > 0$ and $1 \le p \le \infty$, the *reachable space* $R^p(T)$ (at time T) of the system (1.1) consists of all

$$y = y(t,0,u) = \int_0^T S(T-\sigma)u(\sigma)d\sigma \quad u(\cdot) \in L^p(0,T;E)\,,$$

and is equipped with the norm

$$\|y\|_{R^p(T)} = \inf\left\{\|u\|_{L^p(0,T;E)};\int_0^T S(T-\sigma)u(\sigma)d\sigma = y\right\}\,,$$

which makes $R^p(T)$ a Banach space, isometrically isomorphic to the quotient space

$$L^p(0,T;E)/\mathcal{N}^p$$

where $\mathcal{N}^p$ is the closed subspace of $L^p(0,T;E)$ of all $u(\cdot)$ with

$$\int_0^T S(T-\sigma)u(\sigma)d\sigma = 0\,.$$

We note in passing that all spaces $R^\infty(T)$ coincide (with equivalent norms) for $T>0$. This is proved in [2], [7, **2.1**] and can be extended to $p<\infty$, but is not particularly relevant here. If $r>p$ Hölder's inequality gives

$$\int_0^T \|u(\sigma)\|^p d\sigma = \int_0^T 1\cdot(\|u(\sigma)\|^r)^{p/r}dt \le T^{(r-p)/r}\biggl(\int_0^T \|u(\sigma)\|^r dr\biggr)^{p/r},$$

thus

$$\|y\|_{R^p(T)} \le T^{(r-p)/pr}\|y\|_{R^r(T)}\,,$$

and it follows that $R^r(T)\hookrightarrow R^p(T)$. (the symbol $\hookrightarrow$ means "is imbedded in". Another application of Hölder's inequality produces

$$\|y\| \le \|S(T-\cdot)\|_{L^{p/(p-1)}(0,T)}\|y\|_{R^p(T)} \le T^{(p-1)/p}\|S(T-\sigma)\|_{L^\infty(0,T)}\|y\|_{R^p(T)}\,,$$

so that $R^p(T)\hookrightarrow E$. Finally, if $y\in D(A)$, integration by parts gives

$$y = \int_0^T S(T-\sigma)\frac{y-\sigma Ay}{T}d\sigma$$

thus, if we equip $D(A)$ with its customary graph norm, we have $D(A)\hookrightarrow R^\infty(T)$. Putting all the imbeddings together,

$$D(A)\hookrightarrow R^\infty(T)\hookrightarrow R^r(T)\hookrightarrow R^p(T)\hookrightarrow E \quad (p<r)\,. \tag{2.4}$$

All imbeddings except the first are dense in the norm of the bigger space. For the imbeddings $R^\infty(T)\hookrightarrow R^r(T)\hookrightarrow R^p(T)$ this follows from denseness of $L^\infty(0,T;E)$ (thus of $L^r(0,T;E)$) in $L^p(0,T;E)$, and for $R^p(T)\hookrightarrow E$ from denseness of $D(A)$ in E (for these results and more details about function spaces of E-valued functions see [1, Chapter III] or [9, Chapter III]. Whether or not $D(A)$ is dense in $R^\infty(T)$ in the norm of the latter space is one of the main themes of [7, Chapters 2 and 3]. The following result follows immediately from denseness of $R^\infty(T)$ in E and from the open mapping principle.

Lemma 2.3. *We have $R^\infty(T)=E$ (with equivalent norms) if and only if*

$$\|y\|_{R^\infty(T)} \le C\|y\| \quad (y\in R^\infty(T))\,.$$

Lemma 2.4 below is proved in [7, Theorem 2.2.3 and beginning of **2.2**]:

Lemma 2.4. $R^\infty(T)=E$ *(with equivalent norms) if and only if $S(t)$ is reversible.*

Theorem 2.5. *Assume $S(t)$ is reversible. Then*

$$Z^1(T) = E^* \quad (T > 0) \tag{2.5}$$

with equivalent norms.

Proof. Assume $S(t)$ is reversible. Then, by Lemma 2.4, $R^\infty(T) = E$ with equivalent norms. It follows that $R^\infty(T)^* = E^*$ with equivalent norms as well.

Let $z \in Z^1(T)$. We can define a bounded linear functional ξ_z on $R^\infty(T)$ by

$$\langle \xi_z, y\rangle = \left\langle \xi_z\,, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle = \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma\,. \tag{2.6}$$

It can be easily seen that (2.6) pays heed to the equivalence relation in $R^\infty(T) = L^\infty(0,T;E)$ $/\mathcal{N}^\infty$ [5], [7, Lemma 2.3.5] and it follows from (1.2) that ξ_z is bounded in the norm of $R^\infty(T)$, precisely

$$\|\xi_z\| = \int_0^T \|S(T-\sigma)^* z\| d\sigma = \int_0^T \|S(\sigma)^* z\| d\sigma\,. \tag{2.7}$$

The inequality $\leq$ in (2.7) is obvious; for the equality, see [5] or [7, **2.3**]. By Lemma 2.4, ξ_z is as well bounded in the norm of E. Accordingly,

$$\int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \leq C\left\| \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\|.$$

This implies

$$\int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \leq C\int_0^T \|u(\sigma)\| d\sigma \quad (u(\cdot) \in L^\infty(0,T;E))$$

and Lemma 2.2 shows that

$$\|S(t)^* z\| \leq C \quad (0 < t \leq T)\,. \tag{2.8}$$

We show that (2.8) implies $z \in E^*$. Let $\{t_n\}$ be a positive decreasing sequence with $t_n \to 0$. Then

$$|\langle S(t_n)^* z - S(t_m)z,\, y\rangle| = |\langle S(t_m)^* z,\, S(t_n - t_m)y - y\rangle| \leq C\|S(t_n - t_m)y - y\|$$

for $n < m$, so that $\{S(t_n)z\}$ is Cauchy in the E-weak topology of E^*. Accordingly, it converges E-weakly to $y^* \in E^*$, and we have

$$\langle S(t)^* z, y\rangle = \langle S(t-t_n)^* S(t_n)^* z,\, y\rangle = \langle S(t_n)^* z,\, S(t-t_n)y\rangle \to \langle S(t)^* y^*, y\rangle\,,$$

thus

$$S(t)^* z = S(t)^* y^* \quad (t > 0)\,. \tag{2.9}$$

The operator $(A^{-1})^* : E^*_{-1} \to E^*$ is 1-1 (and onto). Applying $(A^{-1})^*$ to both sides of (2.9) and applying the functionals on both sides to an element $y \in E$ we obtain

$$\langle (A^{-1})^* z, S(t)y\rangle = \langle (A^{-1})^* y^*, S(t)y\rangle \quad (t > 0)\,.$$

Letting $t \to 0$ we obtain $(A^{-1})^* z = (A^{-1})^* y^*$, thus $z = y^*$ as claimed. This ends the proof.

We note the following interesting byproduct of the proof of Theorem 2.5 (in particular, of the lines following (2.8)). Define $Z^\infty(T)$ as the space of all $z \in E^*_{-1}$ such that $S(t)^* z$ is bounded in $0 \le t \le T$ equipped with the norm

$$\|z\|_{Z^\infty(T)} = \max_{0\le t\le T} \|S(t)^* z\|\,.$$

Then (with no conditions on the space E or the semigroup $S(t)$),

Lemma 2.6. *We have*

$$Z^\infty(T) = E^*$$

with equivalent norms.

3 Regular implies strongly regular, I.

The first question is this. Assume that (2.5) fails, that is, that the inclusion

$$Z^1(T) \supset E^* \tag{3.1}$$

is strict. *Does this mean that there are regular controls which are not strongly regular?* To attempt to answer this question is complicated by lack of uniqueness of z in the maximum principle (1.3) as in the following example, which is taken from [8].

Example 3.1. Consider the space $E = \ell^0$ consisting of all numerical sequences $y = \{y_n\} = \{y_1, y_2, \dots\}$ such that $\lim_{n\to\infty} y_n = 0$, equipped with the norm $\|y\|_0 = \max_{n\ge1} |y_n|$. The dual is $E^* = \ell^1$, the space of all numerical sequences $y^* = \{y^*_n\}$ such that $\|y^*\|_1 = \sum_{n=1}^\infty |y^*_n| < \infty$, the duality of both spaces given by $\langle y^*, y\rangle = \sum_{n=1}^\infty y^*_n y_n$. The semigroup and generator are

$$S(t)\{y_n\} = \{e^{-nt} y_n\}\,, \quad A\{y_n\} = -\{n y_n\}\,, \tag{3.2}$$

A with maximal domain ($\lim_{n\to\infty} n|y_n| = 0$). The space E^*_{-1} consists of all sequences $\{y_n\}$ with

$$\|(A^{-1})^*\{y^*_n\}\| = \sum_{n=1}^\infty \frac{|y^*_n|}{n} < \infty\,. \tag{3.3}$$

If $\{y_n^*\} \in E_{-1}^*$ we have

$$\int_0^T \|S(t)^*z\|dt = \left\|\left\{ \int_0^T e^{-nt}y_n^*\right\}\right\| = \sum_{n=1}^{\infty} |y^*|\frac{1-e^{-nT}}{n} \le \|(A^{-1})^*\{y_n^*\}\| \, ,$$

thus $E_{-1}^* = Z^1(T)$ and the inclusion (3.1) is strict. Due to existence requirements for optimal controls for (1.1) with this choice of space and generator, controls are taken in $L_w^\infty(0,T;\ell^\infty)$ rather than in $L^\infty(0,T;\ell^0)$, where ℓ^∞ is the space of all bounded numerical sequences $y = \{y_n\}$ equipped with the norm $\|y\|_\infty = \max_{n\ge 1}|y_n|$. This means the u in the maximum principle (1.3) belongs to ℓ^∞ rather than in ℓ^0. See [8] for additional details. We also take the following result from [8].

Theorem 3.2. *An admissible control $\bar u(t) = \{\bar u_n(t)\}$ satisfies the maximum principle* (1.3) *with $z = \{z_n\}$ in any multiplier space if and only if $\bar u_m(t) = 1$ $(0 \le t \le T)$ or $u_m(t) = -1$ $(0 \le t \le T)$ for at least one $m \ge 1$.*

Proof. We take $\rho = 1$. The maximum principle for this space and generator is

$$\begin{aligned}
\langle S(T-t)^*\{z_n\}, \{\bar u_n(t)\}\rangle &= \sum_{n=1}^{\infty} e^{-n(T-t)} z_n \bar u_n(t) \\
&= \max_{\|\{u_n\}\|_{\ell^\infty}\le 1} \langle S(T-t)^*\{z_n\}, \{u_n\}\rangle \\
&= \max_{|u_n|\le 1} \sum_{n=1}^{\infty} e^{-n(T-t)} z_n u_n \\
&= \sum_{n=1}^{\infty} e^{-n(T-t)} |z_n| \, ,
\end{aligned}$$

so that we must have $\bar u_m(t) = \operatorname{sign} z_m$ whenever $z_m \ne 0$. Conversely, if the assumptions of Theorem 3.2 are satisfied for $\{\bar u_n(t)\}$ we obtain the maximum principle (1.3) with $\{z_n\} = \delta_{mn}$ (δ_{mn} the Kronecker delta). This ends the proof.

Strictness of the inclusion (3.1) *and* uniqueness of z in the maximum principle (1.3)[4] do imply the existence of optimal controls that are regular but nor strongly regular. We just take $z \in Z^1(T) \setminus E^*$ and use the sufficiency statement in Theorem 1.1.

Uniqueness of z holds (for instance) in Hilbert spaces. If $\bar u(t)$ satisfies the maximum principle with two different $z, \zeta \in Z^1(T)$ then, assuming (as always) that the maximum principle is nonempty and taking $\rho = 1$ for simplicity,

$$\bar u(t) = \frac{S(T-t)^*z}{\|S(T-t)^*z\|} = \frac{S(T-t)^*\zeta}{\|S(T-t)^*\zeta\|}$$

[4] "Uniqueness of z" obviously means "uniqueness up to multiplication by a constant"; if $\bar u(t)$ satisfies the maximum principle (1.3) with two different z, ζ then $\zeta = \alpha z$, $\alpha \ne 0$. The condition that $\alpha \ne 0$ is required by the assumed nontriviality of (1.3).

in some interval $T-\epsilon \leq t \leq T$. Multiplying by the product of the denominators and applying $(A^{-1})^*$ to both sides we obtain

$$\|S(T-t)^*\zeta\|S(T-t)^*(A^{-1})^*z = \|S(T-t)^*z\|S(T-t)^*(A^{-1})^*\zeta$$

where $(A^{-1})^*z, (A^{-1})^*\zeta \in E^*$, thus, if $\{t_n\}$ is a decreasing sequence with $t_n \to 0$ we have

$$S(T-t_n)^*(A^{-1})^*\zeta = \alpha_n S(T-t_n)^*(A^{-1})^*z\,. \tag{3.4}$$

Now, if $y \in E$ is such that $\langle y, (A^{-1})^*z\rangle \neq 0$ we apply the functionals on both sides of (3.4) to y and obtain

$$\langle (A^{-1})^*\zeta\,,\, S(T-t_n)y\rangle = \alpha_n\langle (A^{-1})^*z\,,\, S(T-t_n)y\rangle\,, \tag{3.5}$$

which shows that $\alpha_n \to \alpha$, thus we can take limits in (3.5), now written for arbitrary $y \in E$, obtaining

$$\langle (A^{-1})^*\zeta\,, y\rangle = \alpha\langle (A^{-1})^*z\,, y\rangle\,,$$

thus

$$\zeta = \alpha z \tag{3.6}$$

where $\alpha \neq 0$ due to the requirement that (1.3) be nonempty (see comments after (1.3)).

4 Regular implies strongly regular, II.

We show in this section the converse of Theorem 2.5. The first result is on one of the imbeddings in (2.4),

$$D(A) \hookrightarrow R^p(T)\,. \tag{4.1}$$

Lemma 4.1. *If $1 \leq p < \infty$ the imbedding (4.1) is dense.*

Proof. Let $\{\lambda_n\}$ be an increasing sequence with $\lambda_n \to \infty$. It follows from the dominated convergence theorem that if $u(\cdot) \in L^p(0,T;E)$ then $\lambda_n R(\lambda_n; A)u(\cdot) \to u(\cdot)$ in the norm of $L^p(0,T;E)$ thus

$$\lambda_n R(\lambda_n; A)\int_0^T S(T-\sigma)u(\sigma)d\sigma \to \int_0^T S(T-\sigma)u(\sigma)d\sigma$$

in the norm of $R^p(T)$. This ends the proof.

Given $1 \leq q < \infty$, the space $Z^q(T) \subseteq Z^1(T)$ consists of all $z \in Z^1(T)$ such that

$$\int_0^T \|S(t)^*z\|^q dt < \infty \tag{4.2}$$

equipped with the norm $\|S(\cdot)^* z\|_{L^q(0,T)}$. For $q = \infty$, the space was defined at the end of **2** (and shown to coincide with E^*).

Theorem 4.2. *The dual space $R^p(T)^*$, $1 \le p < \infty$ is algebraically and metrically isomorphic to $Z^q(T)$, $1/q + 1/p = 1$.*

The proof is based on the calculation of the dual for $p = \infty$, which we outline below. Bounded functionals ξ_z on $R^\infty(T)$ of the form (2.6) are called *regular,* and $\mathcal{R}(T) \subseteq R^\infty(T)^*$ is the subspace of all regular functionals. Bounded functionals ξ_s on $R^\infty(T)$ that vanish in $D(A) \subseteq R^\infty(T)$ are called *singular;* the space of all such functionals is $\mathcal{S}(T) \subseteq R^\infty(T)^*$. Application of the Hahn - Banach theorem gives

$$\mathcal{S}(T) = \{0\} \iff D(A) \textit{ is dense in } R^\infty(T) \ (\textit{in the norm of } R^\infty(T))\,.$$

Theorem 4.3. [7, Theorem 2.4.1]. *We have*[5]

$$R^\infty(T)^* = \mathcal{R}(T) \oplus \mathcal{S}(T) \quad (\textit{Banach direct sum})\,.$$

Proof of Theorem 4.2. Let ξ be a bounded linear functional in $R^p(T)$. Then (due to the second imbedding (2.4)) ξ is a bounded linear functional in $R^\infty(T)$ as well, hence, due to Theorem 4.3. we have $\xi = \xi_z + \xi_s$ with ξ_z regular and ξ_s singular. If $u(\cdot) \in L^\infty(0,T;E)$ and

$$\int_0^T S(T-\sigma)u(\sigma)d\sigma \in D(A) \tag{4.3}$$

we have

$$\begin{aligned}\left\langle \xi\,, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle &= \left\langle \xi_z\,, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle \\ &= \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma. \end{aligned} \tag{4.4}$$

Now, $D(A)$ is dense in $R^p(T)$ and $R^\infty(T)$ is dense in $R^p(T)$, thus (4.4) can be extended to all elements (4.3) of $R^p(T)$ whether or not they belong to $D(A)$. Since ξ is bounded in $R^p(T)$ we have

$$\int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \le \|\xi\|_{R^p(T)^*} \left\| \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\|_{R^p(T)}. \tag{4.5}$$

For the case $p > 1$ this implies

$$\int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \le \|\xi\|_{R^p(T)^*} \left(\int_0^T \|u(\sigma)\|^p d\sigma \right)^{1/p} \quad (u(\cdot) \in L^p(0,T;E))$$

[5] "Banach direct sum" means algebraic direct sum plus bounded projections from the space into each of the two subspaces.

and it follows from Lemma 2.2 that

$$S(\cdot)^* z \in Z^q(T)\,, \quad \|z\|_{Z^q(T)} = \left(\int_0^T \|S(T-\sigma)\|^q d\sigma \right)^{1/q} = \|\xi\|_{R^p(T)^*}\,,$$

equality coming from the fact that $\|\xi\|_{R^p(T)^*}$ is the least constant that does the job in (4.5). That an element of $Z^q(T)$ produces a functional in $R^p(T)$ through (2.6) is a consequence of Hölder's inequality. In the case $p = 1$, (4.5) implies

$$S(\cdot)^* z \in Z^\infty(T)\,, \quad \|z\|_{Z^\infty(T)} = \sup_{0\le t\le T} \|S(t)^* z\| = \|\xi\|_{R^p(T)^*}\,.$$

This ends the proof of Theorem 4.2.

We note that a proof of Theorem 4.2 which is independent of the (rather involved) identification of $R^\infty(T)^*$ can be given using the equality

$$L^p_w(0,T;E)^* = L^q(0,T;E^*) \tag{4.6}$$

for $1/p + 1/q = 1$ ([10], [4]) where the subindex w means "E-weakly measurable". However, the description of the spaces $L^p_w(0,T;E)^*$ (and the definition of the norms) is rather involved as well.

Corollary 4.4. *We have*

$$R^1(T) = E \tag{4.7}$$

with equivalent norms.

Proof. We have $R^1(T) \hookrightarrow E$ and $R^1(T)$ is dense in E. On the other hand, by Lemma 2.6 we have $R^1(T)^* = Z^\infty(T) = E^*$ with equivalent norms. This is easily seen to imply equivalence of the norms of $R^1(T)$ and E. In fact, it suffices to note that, as a consequence of the Hahn - Banach theorem we have

$$\|y\| = \sup_{\|y^*\|\le 1} \langle y^*, y\rangle\,. \tag{4.8}$$

for any Banach space E and its dual E^*. Equivalence of the norms and the fact that $R^1(T)$ is dense in E implies (4.7).

So far, all results in this section have been proved for an arbitrary Banach space E and strongly continuous semigroup $S(t)$. Our objective below is the proof of the converse of Theorem 2.5, thus we assume $Z^1(T) = E^*$. By intercession of Lemma 2.6, this is the same as $Z^1(T) = Z^\infty(T)$, which in turn is equivalent to

$$\int_0^T \|S(t)^* z\| dt < \infty \implies \|S(t)^* z\| \text{ is bounded in } 0 \le t \le T\,. \tag{4.9}$$

It follows that all $Z^p(T)$ coincide, $1 \le p \le \infty$.

Lemma 4.5. *Under* (4.9) *all* $Z^p(T)$ *norms are equivalent to the norm in* $Z^1(T)$*, with constants that do not depend on* p*.*

Proof. Independently of (4.9) we have

$$\|z\|_{Z^p(T)} = \left(\int_0^T \|S(t)^* z\|^p dt \right)^{1/p} \le T^{1/p} \|S(\cdot)z\|_{L^\infty(0,T)} = T^{1/p}\|z\|_{Z^\infty(T)} . \qquad (4.10)$$

If (4.9) holds then, since $Z^1(T) = Z^\infty(T)$ is a Banach space under the two norms, by the open mapping principle these norms have to be equivalent: hence

$$\|z\|_{Z^\infty(T)} \le C\|z\|_{Z^1(T)}$$

which, combined with (4.10) gives

$$\|z\|_{Z^p(T)} \le CT^{1/p}\|z\|^1(T) .$$

On the other hand, and independently of (4.9),

$$\begin{aligned} \|z\|_{Z^1(T)} &= \int_0^T \|S(t)^* z\| dt \\ &\le T^{(p-1)/p} \left(\int_0^T \|S(t)^* z\|^p dt \right)^{1/p} = CT^{(p-1)/p}\|z\|_{Z^p(T)} . \end{aligned}$$

Theorem 4.6. *Assume* (4.9) *holds. Then*

$$R^p(T) = E \quad (1 \le p < \infty) ,$$

all norms equivalent to the norm of E *with constants that do not depend on* p*.*

Proof. Since $R^p(T)^* = Z^{p/(p-1)}(T)$ algebraically and metrically, Lemma 4.5 says that $R^p(T)^* = Z^\infty(T) = E^*$, all norms equivalent to the norm of E^* with constants than do not depend of p. The corresponding statement for $R^p(T)$, E is a consequence of denseness of $R^p(T)$ in E and (4.8). This completes the proof.

Theorem 4.7. *Let* E *be reflexive and separable. If*

$$Z^1(T) = E^* .$$

then $S(t)$ *is reversible, that is,* (1.6) *holds.*

Proof. We shall show that $S(t)$ is reversible by proving that $R^\infty(t) = E$ and using Lemma 2.4. If E is reflexive and separable then $X = E^*$ is reflexive and separable as well and

$$L^q(0, T; E) = L^p(0, T; X)^*$$

$(1/p + 1/q = 1)$ where, due to the assumptions (and unlike in the generality of (4.6)) the space on the left is described exactly in the same form as the space on the right. Since X is separable, the space $L^p(0,T;X)$ is separable as well for $1 \le p < \infty$. This implies that the $L^p(0,T;X)$-weak topology in any bounded subset of $L^q(0,T;E)$ is defined by a metric ([1, Theorem 3, p. 434]), which justifies the "passing to a subsequence" arguments below.

Under the assumptions, given $y \in E$ we may avail ourselves of Theorem 4.6, and construct a sequence $\{u_n(\cdot)\}$, $u_n(\cdot) \in L^n(0,T;E)$ such that

$$\int_0^T S(T-\sigma)u_n(\sigma)d\sigma = y\,, \quad (\|u_n(\cdot)\|_{L^n(0,T;E)} \le C\|y\|\,,\ n = 2,3,4,\dots) \tag{4.11}$$

where C does not depend on n. Since

$$L^2(0,T;E) = L^2(0,T;X)^*$$

we can select a subsequence of $\{u_n(\cdot)\}$ $L^2(0,T;X)$-weakly convergent in $L^2(0,T;E)$; since

$$L^3(0,T;X) = L^{3/2}(0,T;X)^*$$

we can select a subsequence of the previous subsequence that is $L^{3/2}(0,T,X)$-weakly convergent in $L^3(0,T,X)$ (thus $L^2(0,T;X)$-weakly convergent in $L^2(0,T;E)$); since

$$L^4(0,T;X) = L^{4/3}(0,T;X)^*$$

we can select a subsequence of the previous subsequence that is $L^{4/3}(0,T;X)$-weakly convergent in $L^4(0,T;X)$ (thus $L^{3/2}(0,T;X)$-weakly convergent in $L^3(0,T;E)$), $L^2(0,T;X)$-weakly convergent in $L^2(0,T;E)$), ... and so on. Picking the diagonal sequence, we finally obtain a sequence $\{u_n(\cdot)\}$ such that, eventually, it belongs to every $L^m(0,T;E)$ and such that

$$u_n(\cdot) \to \bar{u}(\cdot) \in L^m(0,T;E) \quad L^{m/(m-1)}(0,T;X)\text{-weakly in } L^m(0,T;E)\,.$$

(the fact that the limit $\bar{u}(\cdot)$ is the same in all spaces is elementary). Now, it follows from the norm estimation in (4.11) that

$$\|\bar{u}(\cdot)\|_{L^m(0,T;E)} \le C \quad (2,3,\dots)$$

with C independent of m, hence $\bar{u}(\cdot) \in L^\infty(0,T;E)$. The first relation (4.11) implies

$$\left\langle y^*, \int_0^T S(T-\sigma)u_n(\sigma)d\sigma \right\rangle = \int_0^T \langle S(T-\sigma)^*y^*, u_n(\sigma)\rangle d\sigma$$
$$\to \int_0^T \langle S(T-\sigma)^*y^*, \bar{u}(\sigma)\rangle d\sigma = \left\langle y^*, \int_0^T S(T-\sigma)\bar{u}(\sigma)d\sigma \right\rangle \quad (y \in E^*)$$

so that

$$y = \int_0^T S(T-\sigma)\bar{u}(\sigma)d\sigma$$

and the proof of Theorem 4.7 is finished.

Corollary 4.8. *Let E be reflexive and separable. Assume all regular controls for* (1.1) *are strongly regular and that z in the maximum principle* (1.3) *depends uniquely on $\bar{u}(t)$* (*as in the comments preceding* (3.6)). *Then $S(t)$ is reversible.*

Proof. If the inclusion $Z^1(T) \supset E^*$ is strict, taking $z \in Z^1(T) \setminus E^*$ and using the sufficiency statement in Theorem 1.1 we can construct a regular $\bar{u}(t)$ which is not strongly regular. Accordingly, we must have $Z^1(T) = E^*$ and Theorem 4.7 applies.

Remark 4.9. Theorem 3.2. shows that the conclusion of Corollary 4.8 collapses if we drop the assumptions that E be reflexive and that z be unique. In fact, if $E = \ell^1$ and A is given by (3.2) every control $\bar{u}(t)$ that satisfies the maximum principle (1.3) with any multiplier $z = \{z_n\} \neq 0$ satisfies (1.3) as well with $\{z_n\} = \{\delta_{mn}\} \in \ell^1 = E^*$, thus it qualifies as strongly regular. However, the semigroup (3.1) is far from reversible.

Received: January 2007. Revised: September 2007.

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