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Recovering Higher-order Differential Operators on Star-type Graphs from Spectra

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ABSTRACT

We study an inverse problem of recovering arbitrary order ordinary differential operators on compact star-type graphs from a system of spectra. We establish properties of spectral characteristics, and provide a procedure for constructing the solution of the inverse problem of recovering coefficients of differential equations from the given spectra.

RESUMEN

Estudiamos un problema inverso de recuperar el orden de operadores diferenciales ordinarios sobre graficos compactos de tipo estrellado a partir de un sistema de espectro. Propiedades de la caracteristica espectral son establecidas y es dado un procecimiento para construir la solución del problema inverso de recuperar coeficientes de ecuaciones diferenciales a partir del espectro.

Key words and phrases: *Differential equations; Geometrical graphs; Spectral characteristics, Inverse problems, Method of spectral mappings.*

AMS Classification: *34A55, 34L05, 47E05.*

1 Introduction

We study the inverse spectral problem of recovering arbitrary order differential operators on compact star-type graphs from a system of spectra. We prove a corresponding uniqueness theorem and provide a constructive procedure for the solution of this inverse problem. For studying this inverse problem we develop the ideas of the method of spectral mappings [1]. The obtained results are natural generalizations of the well-known results on inverse problems for the differential operators on an interval ([1]-[4]). We note that boundary value problems on graphs (networks, trees) often appear in natural sciences and engineering (see [5] and the references therein).

Consider a compact star-type graph T in $\mathbf{R^m}$ with the set of vertices $V = \{v_0, \dots, v_p\}$ and the set of edges $\mathcal{E} = \{e_1, \dots, e_p\}$, where $v_0, \dots, v_{p-1}$ are the boundary vertices, v_p is the internal vertex, and $e_p = [v_0, v_p]$, $e_j = [v_p, v_j]$, $j = \overline{1, p-1}$, $e_1 \cap \ldots \cap e_p = \{v_p\}$. For simplicity we suppose that the length of each edge is equal to 1 (it follows from the proofs that the results remain true for arbitrary lengths of the edges). Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x \in [0, 1]$. It is convenient for us to choose the following orientation: $x = 0$ corresponds to the boundary vertices $v_0, \dots, v_{p-1}$, and $x = 1$ corresponds to the internal vertex v_p. An integrable function Y on T may be represented as $Y(x) = \{y_j(x)\}_{j=\overline{1,p}}$, $x \in [0, 1]$, where the function $y_j(x)$ is defined on the edge e_j.

Fix $n \geq 2$. Let $q_\nu(x) = \{q_{\nu j}(x)\}_{j=\overline{1,p}}$, $\nu = \overline{0, n-2}$ be integrable complex-valued functions on T. Consider the following n-th order differential equation on T:

$$y_j^{(n)}(x) + \sum_{\nu=0}^{n-2} q_{\nu j}(x) y_j^{(\nu)}(x) = \lambda y_j(x), \quad j = \overline{1, p}, \tag{1}$$

where λ is the spectral parameter, $q_{\nu j}(x)$ are complex-valued integrable functions, and $y_j^{(\nu)}(x) \in AC[0, 1]$, $j = \overline{1, p}$, $\nu = \overline{0, n-1}$. Denote by $q = \{q_\nu\}_{\nu=\overline{0,n-2}}$ the set of the coefficients of equation (1); q is called the potential. Consider the linear forms

$$U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(1), \quad j = \overline{1, p-1}, \ \nu = \overline{0, n-1},$$

where $\gamma_{j\nu\mu}$ are complex numbers, and $\gamma_{j\nu} := \gamma_{j\nu\nu} \neq 0$. The linear forms $U_{j\nu}$ will be used in matching conditions in the internal vertex v_p for for special solutions of equation (1).

Fix $s = \overline{1, p-1},\ k = \overline{1, n-1},\ \mu = \overline{k, n}$. Let $\Lambda_{sk\mu} := \{\lambda_{lsk\mu}\}_{l\geq 1}$ be the set of the eigenvalues of the boundary value problem $L_{sk\mu}$ for equation (1) with the boundary conditions

$$y_s^{(\nu-1)}(0) = 0, \quad \nu = \overline{1, k-1}, \mu$$

$$y_j^{(\xi-1)}(0) = 0, \quad \xi = \overline{1, n-k},\ j = \overline{1, p} \setminus s,$$

and with the matching conditions

$$\left.\begin{aligned} &U_{j\nu}(y_j) + y_p^{(\nu)}(1) = 0, \quad j = \overline{1, p-1},\ \nu = \overline{0, k-1}, \\ &\sum_{j=1}^{p-1} U_{j\nu}(y_j) + y_p^{(\nu)}(1) = 0, \quad \nu = \overline{k, n-1}. \end{aligned}\right\} \tag{2}$$

The inverse problem of recovering the potential from the system of spectra is formulated as follows.

Inverse Problem 1. Given the spectra $\Lambda := \{\Lambda_{sk\mu}\}$, $s = \overline{1, p-1}$, $1 \leq k \leq \mu \leq n$, construct the potential q.

This inverse problem is a generalization of the well-known inverse problems for differential operators on an interval from a system of spectra (see [1-4]). For example, if $n = p = 2$, then Inverse Problem 1 is the classical Borg's inverse problem of recovering Sturm-Liouville operators from two spectra.

2 Auxiliary propositions

Let $\Psi_{sk}(x, \lambda) = \{\psi_{skj}(x, \lambda)\}_{j=\overline{1,p}}$, $s = \overline{1, p-1}$, $k = \overline{1, n}$, be solutions of equation (1) satisfying the boundary conditions

$$\left.\begin{aligned} &y_s^{(\nu-1)}(0) = \delta_{k\nu}, \quad \nu = \overline{1, k}, \\ &y_j^{(\xi-1)}(0) = 0, \quad \xi = \overline{1, n-k},\ j = \overline{1, p} \setminus s, \end{aligned}\right\} \tag{3}$$

and the matching conditions (2). Here and in the sequel, $\delta_{k\nu}$ is the Kronecker symbol. The function Ψ_{sk} is called the Weyl-type solution of order k with respect to the boundary vertex v_s. We introduce the matrices $M_s(\lambda) = [M_{sk\nu}(\lambda)]_{k,\nu=\overline{1,n}}$, $s = \overline{1, p-1}$, where $M_{sk\nu}(\lambda) := \psi_{sks}^{(\nu-1)}(0, \lambda)$. It follows from the definition of ψ_{skj} that $M_{sk\nu}(\lambda) = \delta_{k\nu}$ for $k \geq \nu$, and $\det M_s(\lambda) \equiv 1$. The matrix $M_s(\lambda)$ is called the Weyl-type matrix with respect to the boundary vertex v_s. Denote by $M = \{M_s\}_{s=\overline{1,p-1}}$ the set of the Weyl-type matrices.

Let $\lambda = \rho^n$. The ρ - plane can be partitioned into sectors S of angle $\frac{\pi}{n}$ $\Big(\arg\rho \in \Big(\frac{\nu\pi}{n}, \frac{(\nu+1)\pi}{n}\Big),\ \nu = \overline{0, 2n-1}\Big)$ in which the roots $R_1, R_2, \dots, R_n$ of the equation $R^n - 1 = 0$ can be numbered in such a way that

$$Re(\rho R_1) < Re(\rho R_2) < \ldots < Re(\rho R_n), \quad \rho \in S. \tag{4}$$

We assume that the regularity condition for matching from [6] is fulfilled. The following assertion was proved in [6].

Lemma 1. *Fix a sector S with the property (4). For $x \in (0,1),\ \nu = \overline{0, n-1},\ s = \overline{1, p-1},\ k = \overline{1, n}$, the following asymptotical formula holds*

$$\psi_{sks}^{(\nu)}(x, \lambda) = \frac{\omega_k}{\rho^{k-1}} (\rho R_k)^\nu \exp(\rho R_k x)[1], \quad \rho \in S,\ |\rho| \to \infty,$$

where

$$\omega_k := \frac{\Omega_{k-1}}{\Omega_k}, \quad k = \overline{1, n}, \quad \Omega_k := \det[R_\xi^{\nu-1}]_{\xi,\nu=\overline{1,k}}, \quad \Omega_0 := 1.$$

For $s = \overline{1, p-1},\ k = \overline{1, n-1},\ \mu = \overline{k+1, n}$,

$$M_{sk\mu}(\lambda) = m_{k\mu}\rho^{\mu-k}[1], \quad \rho \in S,\ |\rho| \to \infty, \tag{5}$$

where $m_{k\mu}$ are constants which do not depend on the potential.

Let $\{C_{kj}(x, \lambda)\}_{k=\overline{1,n}},\ j = \overline{1, p}$ be the fundamental system of solutions of equation (1) on the edge e_j under the initial conditions $C_{kj}^{(\nu-1)}(0, \lambda) = \delta_{k\nu},\ k, \nu = \overline{1, n}$. For each fixed $x \in [0, 1]$, the functions $C_{kj}^{(\nu-1)}(x, \lambda),\ k, \nu = \overline{1, n},\ j = \overline{1, p}$, are entire in λ of order $1/n$. Moreover,

$$\det[C_{kj}^{(\nu-1)}(x, \lambda)]_{k,\nu=\overline{1,n}} \equiv 1. \tag{6}$$

Using the fundamental system of solutions $\{C_{kj}(x, \lambda)\}_{k=\overline{1,n}}$, one can write

$$\psi_{skj}(x, \lambda) = \sum_{\mu=1}^{n} M_{skj\mu}(\lambda) C_{\mu j}(x, \lambda), \quad j = \overline{1, p},\ s = \overline{1, p-1},\ k = \overline{1, n}, \tag{7}$$

where the coefficients $M_{skj\mu}(\lambda)$ do not depend on x. In particular, $M_{sks\mu}(\lambda) = M_{sk\mu}(\lambda)$, and

$$\psi_{sks}(x, \lambda) = C_{ks}(x, \lambda) + \sum_{\mu=k+1}^{n} M_{sk\mu}(\lambda) C_{\mu s}(x, \lambda). \tag{8}$$

It follows from (6) and (8) that $\det[\psi_{sks}^{(\nu-1)}(x, \lambda)]_{k,\nu=\overline{1,n}} \equiv 1$.

Fix $k = \overline{1,n}$, $s = \overline{1,p-1}$. According to (2) and (3),

$$\left.\begin{aligned} &U_{j\nu}(\psi_{skj}(x,\lambda)) + \psi_{skp}^{(\nu)}(1,\lambda) = 0, \quad j = \overline{1,p-1},\ \nu = \overline{0,k-1}, \\ &\sum_{j=1}^{p-1} U_{j\nu}(\psi_{skj}(x,\lambda)) + \psi_{skp}^{(\nu)}(1,\lambda) = 0, \quad \nu = \overline{k,n-1}, \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} &\psi_{sks}^{(\nu-1)}(0,\lambda) = \delta_{k\nu}, \quad \nu = \overline{1,k}, \\ &\psi_{skj}^{(\xi-1)}(0,\lambda) = 0, \quad \xi = \overline{1,n-k},\ j = \overline{1,p} \setminus s. \end{aligned}\right\} \tag{10}$$

Substituting the representation (7) into (9) and (10) we obtain a linear algebraic system with respect to $M_{skj\mu}(\lambda)$. Solving this system by Cramer's rule one gets

$$M_{skj\mu}(\lambda) = \frac{\Delta_{skj\mu}(\lambda)}{\Delta_{sk}(\lambda)},$$

where the functions $\Delta_{skj\mu}(\lambda)$ and $\Delta_{sk}(\lambda)$ are entire in λ of order $1/n$. Thus, the functions $M_{skj\mu}(\lambda)$ are meromorphic in λ, and consequently, the Weyl-type solutions and the Weyl-type matrices are meromorphic in λ. In particular,

$$M_{sk\mu}(\lambda) = \frac{\Delta_{sk\mu}(\lambda)}{\Delta_{sk}(\lambda)}, \quad s = \overline{1,p-1},\ k = \overline{1,n-1},\ \mu = \overline{k+1,n}, \tag{11}$$

where $\Delta_{sk\mu}(\lambda) := \Delta_{sks\mu}(\lambda)$, $\Delta_{sk}(\lambda) := \Delta_{skk}(\lambda)$. The function $\Delta_{sk\mu}(\lambda)$ is the characteristic function of the boundary value problem $L_{sk\mu}$, and its zeros coincide with the eigenvalues $\Lambda_{sk\mu} := \{\lambda_{lsk\mu}\}_{l\geq 1}$ of $L_{sk\mu}$.

The functions $\Delta_{sk\mu}(\lambda)$ are entire in λ of order $1/n$. By Hadamard's factorization theorem, the functions $\Delta_{sk\mu}(\lambda)$ are uniquely determined up to multiplicative constants $c_{sk\mu}$ by their zeros:

$$\Delta_{sk\mu}(\lambda) = c_{sk\mu} \prod_{l=1}^{\infty} \Big(1 - \frac{\lambda}{\lambda_{lsk\mu}}\Big)$$

(the case when $\Delta_{sk\mu}(0) = 0$ requires evident modifications). Then, by virtue of (11),

$$M_{sk\mu}(\lambda) = M_{sk\mu}^0 \prod_{l=1}^{\infty} \Big(1 - \frac{\lambda}{\lambda_{lsk\mu}}\Big)\Big(1 - \frac{\lambda}{\lambda_{lskk}}\Big)^{-1}, \quad s = \overline{1,p-1},\ k = \overline{1,n-1},\ \mu = \overline{k+1,n}. \tag{12}$$

Using (5) we obtain

$$M_{sk\mu}^0 = \lim_{|\rho|\to\infty} m_{mk}\rho^{\mu-k} \prod_{l=1}^{\infty} \Big(1 - \frac{\lambda}{\lambda_{lskk}}\Big)\Big(1 - \frac{\lambda}{\lambda_{lsk\mu}}\Big)^{-1}. \tag{13}$$

Thus, using the given spectra Λ, one can construct uniquely the Weyl-type matrices M by (12) and (13). In other words, the following assertion holds.

Theorem 1. *The specification of the system of spectra* $\Lambda := \{\Lambda_{sk\mu}\}$, $s = \overline{1, p-1}$, $1 \le k \le \mu \le n$, *uniquely determines the Weyl-type matrices* $M = \{M_s\}_{s=\overline{1,p-1}}$ *by (12)-(13).*

Fix $s = \overline{1, p-1}$, and consider the following inverse problem on the edge e_s.

Inverse Problem 2. Given the Weyl-type matrix M_s, construct the functions $q_{\nu s}$, $\nu = \overline{0, n-2}$ on the edge e_s.

It was proved in [6] that this inverse problem has a unique solution, i.e. the specification of the Weyl-type matrix M_s uniquely determines the potential on the edge e_s. Moreover, using the method of spectral mappings one can get a constructive procedure for the solution of Inverse Problem 2. It can be obtained by the same arguments as for n-th order differential operators on a finite interval (see [1, Ch.2] for details).

Now we define an auxiliary Weyl-type matrix with respect to the internal vertex v_p. Let $\psi_{pk}(x, \lambda)$, $k = \overline{1, n}$, be solutions of equation (1) on the edge e_p under the conditions

$$\psi_{pk}^{(\nu-1)}(1, \lambda) = \delta_{k\nu},\ \nu = \overline{1, k}, \quad \psi_{pk}^{(\xi-1)}(0, \lambda) = 0,\ \xi = \overline{1, n-k}. \tag{14}$$

We introduce the matrix $M_p(\lambda) = [M_{pk\nu}(\lambda)]_{k,\nu=\overline{1,n}}$, where $M_{pk\nu}(\lambda) := \psi_{pk}^{(\nu-1)}(1, \lambda)$. Clearly, $M_{pk\nu}(\lambda) = \delta_{k\nu}$ for $k \ge \nu$, and $\det M_p(\lambda) \equiv 1$. The matrix $M_p(\lambda)$ is called the Weyl-type matrix with respect to the internal vertex v_p. Consider the following inverse problem on the edge e_p.

Inverse Problem 3. Given the Weyl-type matrix M_p, construct the functions $q_{\nu p}$, $\nu = \overline{0, n-2}$ on the edge e_p.

This inverse problem is the classical one, since it is the inverse problem of recovering n-th order differential equation on a finite interval from its Weyl-type matrix. This inverse problem has been solved in [1]. In particular, it is proved that the specification of the Weyl-type matrix M_p uniquely determines the potential on the edge e_p. Moreover, in [1] an algorithm for the solution of Inverse Problem 3 is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

3 Solution of the inverse problem from spectra

In this section we obtain a constructive procedure for the solution of Inverse Problem 1. Our plan is the following.

Step 1. Using (12)-(13) construct the Weyl-type matrices $M = \{M_s\}_{s=\overline{1,p-1}}$.

Step 2. Solving Inverse Problem 2 for each fixed $s = \overline{1,p-1}$, we find the functions $q_{\nu s}$, $\nu = \overline{0,n-2}$, $s = \overline{1,p-1}$, i.e. we find the potential q on the edges $e_1, \ldots, e_{p-1}$.

Step 3. Using the knowledge of the potential on the edges $e_1, \ldots, e_{p-1}$, we construct the Weyl-type matrix M_p.

Step 4. Solving Inverse Problem 3 we find the functions $q_{\nu p}$, $\nu = \overline{0,n-2}$, i.e. we find the potential on the edge e_p.

Steps 1, 2 and 4 have been already studied in Section 2. It remains to fulfil Step 3.

Suppose that Steps 1-2 are already made, and we found the functions $q_{\nu s}$, $\nu = \overline{0,n-2}$, $s = \overline{1,p-1}$, i.e. we found the potential q on the edges $e_1, \ldots, e_{p-1}$. Fix $s = \overline{1,p-1}$. All calculations below will be made for this fixed s. Using the knowledge of the potential on the edge e_s, we calculate the functions $C_{ks}(x,\lambda)$, $k = \overline{1,n}$, and the functions $\psi_{sks}(x,\lambda)$, $k = \overline{1,n}$, by (8).

Now we are going to construct the Weyl-type matrix M_p using $\psi_{sks}(x,\lambda)$, $k = \overline{1,n}$. Fix $s = \overline{1,p-1}$. Denote

$$z_{p1}(x,\lambda) := \frac{\psi_{s1p}(x,\lambda)}{\psi_{s1p}(1,\lambda)}.$$

The function $z_{p1}(x,\lambda)$ is a solution of equation (1) on the edge e_p, and $z_{p1}(1,\lambda) = 1$. Moreover, by virtue of (10), one has $z_{p1}^{(\xi-1)}(0,\lambda) = 0$, $\xi = \overline{1,n-k}$. Taking (14) into account we conclude that the solutions $z_{p1}(x,\lambda)$ and $\psi_{p1}(x,\lambda)$ satisfy the same boundary conditions, and consequently, $z_{p1}(x,\lambda) \equiv \psi_{p1}(x,\lambda)$. Thus,

$$\psi_{p1}(x,\lambda) = \frac{\psi_{s1p}(x,\lambda)}{\psi_{s1p}(1,\lambda)}. \tag{15}$$

Similarly, we calculate

$$\psi_{pk}(x,\lambda) = \frac{\det[\psi_{s\mu p}(1,\lambda), \ldots, \psi_{s\mu p}^{(k-2)}(1,\lambda), \psi_{s\mu p}(x,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu p}^{(\xi-1)}(1,\lambda)]_{\xi,\mu=\overline{1,k}}}, \ k = \overline{2,n-1}. \tag{16}$$

Since $M_{pk\nu}(\lambda) = \psi_{pk}^{(\nu-1)}(1,\lambda)$, it follows from (15)-(16) that

$$M_{p1\nu}(\lambda) = \frac{\psi_{s1p}^{(\nu-1)}(1,\lambda)}{\psi_{s1p}(1,\lambda)}, \quad \nu = \overline{2,n}, \tag{17}$$

$$M_{pk\nu}(\lambda) = \frac{\det[\psi_{s\mu p}(1,\lambda), \ldots, \psi_{s\mu p}^{(k-2)}(1,\lambda), \psi_{s\mu p}^{(\nu-1)}(1,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu p}^{(\xi-1)}(1,\lambda)]_{\xi,\mu=\overline{1,k}}}, \tag{18}$$

$$k = \overline{2,n-1}, \quad \nu = \overline{k+1,n}.$$

Using the matching conditions (9) we get

$$U_{j\nu}(\psi_{skj}) = U_{s\nu}(\psi_{sks}), \quad 0 \le \nu < k \le n-1. \tag{19}$$

Since the functions ψ_{sks} were already calculated, the right-hand sides in (19) are known. For each fixed $k = \overline{1, n-1}$, we successively use (19) for $\nu = 0, 1, \dots, k-1$, and calculate recurrently the functions

$$\psi_{skj}^{(\nu)}(1,\lambda), \quad k = \overline{1, n-1},\ \nu = \overline{0, k-1},\ j = \overline{1, p-1} \setminus s. \tag{20}$$

Furthermore, it follows from (7) and (10) that $M_{skj\mu}(\lambda) = 0$ for $\mu = \overline{1, n-k}$, $j = \overline{1, p-1} \setminus s$, and consequently,

$$\psi_{skj}(x,\lambda) = \sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda) C_{\mu j}(x,\lambda), \quad k = \overline{1, n-1},\ j = \overline{1, p-1} \setminus s.$$

This yields

$$\psi_{skj}^{(\nu)}(1,\lambda) = \sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda) C_{\mu j}^{(\nu)}(1,\lambda), \quad \nu = \overline{0, n-1},\ k = \overline{1, n-1},\ j = \overline{1, p-1} \setminus s. \tag{21}$$

Fix $k = \overline{1, n-1}$, $j = \overline{1, p-1} \setminus s$, and consider a part of the relations (21), namely, for $\nu = \overline{0, k-1}$. They form a linear algebraic system with respect to the functions $M_{skj\mu}(\lambda)$, $\mu = \overline{n-k+1, n}$. Solving this system by Cramer's rule we find these functions. Substituting them into (21) for $\nu \ge k$, we calculate the functions

$$\psi_{skj}^{(\nu)}(1,\lambda), \quad k = \overline{1, n-1},\ \nu = \overline{k, n-1},\ j = \overline{1, p-1} \setminus s. \tag{22}$$

Substituting now the functions (20) and (22) into (9) we find

$$\psi_{skp}^{(\nu)}(1,\lambda), \quad k = \overline{1, n-1},\ \nu = \overline{0, n-1}. \tag{23}$$

Since the functions (23) are known, one can calculate the Weyl-type matrix M_p via (17)-(18).

Thus, we have obtained the solution of Inverse Problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 2. *The specification of the spectra Λ uniquely determines the potential q on T. The solution of Inverse Problem 1 can be obtained by the following algorithm.*

Algorithm 1. *Given the spectra* Λ.

1) *Construct the Weyl-type matrices* $M = \{M_s\}_{s=\overline{1,p-1}}$ *via (12)-(13).*
2) *Find the functions* $q_{\nu s}$, $\nu = \overline{0, n-2}$, $s = \overline{1, p-1}$, *by solving Inverse Problem 2 for each* $s = \overline{1, p-1}$.
3) *Fix* $s = \overline{1, p-1}$, *and calculate* $C_{ks}^{(\nu)}(1, \lambda)$ *for* $k = \overline{1, n}$, $\nu = \overline{0, n-1}$.
4) *Construct the functions* $\psi_{sks}^{(\nu)}(1, \lambda)$, $k = \overline{1, n-1}$, $\nu = \overline{0, n-1}$ *by the formula*

$$\psi_{sks}^{(\nu)}(1, \lambda) = C_{ks}^{(\nu)}(1, \lambda) + \sum_{\mu=k+1}^{n} M_{sk\mu}(\lambda) C_{\mu s}^{(\nu)}(1, \lambda).$$

5) *Find the functions* $\psi_{skj}^{(\nu)}(1, \lambda)$, $k = \overline{1, n-1}$, $\nu = \overline{0, k-1}$, $j = \overline{1, p-1} \setminus s$, *by using the recurrent formulae (19).*
6) *Calculate* $M_{skj\mu}(\lambda)$, $k = \overline{1, n-1}$, $\mu = \overline{n-k+1, n}$, $j = \overline{1, p-1} \setminus s$, *by solving the linear algebraic systems*

$$\sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda) C_{\mu j}^{(\nu)}(1, \lambda) = \psi_{skj}^{(\nu)}(1, \lambda), \quad \nu = \overline{0, k-1},$$

for each fixed $k = \overline{1, n-1}$, $j = \overline{1, p-1} \setminus s$.

7) *Construct the functions* $\psi_{skj}^{(\nu)}(1, \lambda)$, $k = \overline{1, n-1}$, $\nu = \overline{k, n-1}$, $j = \overline{1, p-1} \setminus s$, *by the formula*

$$\psi_{skj}^{(\nu)}(1, \lambda) = \sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda) C_{\mu j}^{(\nu)}(1, \lambda), \quad \nu \geq k.$$

8) *Find the functions* $\psi_{skp}^{(\nu)}(1, \lambda)$, $k = \overline{1, n-1}$, $\nu = \overline{0, n-1}$, *by (9).*
9) *Calculate the Weyl-type matrix* M_p *via (17)-(18).*
10) *Construct the functions* $q_{\nu p}$, $\nu = \overline{0, n-2}$, *by solving Inverse Problem 3.*

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