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Gröbner and diagonal bases in Orlik-Solomon type algebras

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ABSTRACT

The Orlik-Solomon algebra of a matroid $\mathcal{M}$ is the quotient of the exterior algebra on the points by the ideal $\Im(\mathcal{M})$ generated by the boundaries of the circuits of the matroid. There is an isomorphism between the Orlik-Solomon algebra of a complex matroid and the cohomology of the complement of a complex arrangement of hyperplanes. In this article a generalization of the Orlik-Solomon algebras, called χ-algebras, are considered. These new algebras include, apart from the Orlik-Solomon algebras, the Orlik-Solomon-Terao algebra of a set of vectors and the Cordovil algebra of an oriented matroid. To encode an important property of the "no broken circuit bases" of the Orlik-Solomon-Terao algebras, András Szenes has introduced a particular type of bases, the so called "diagonal bases". This notion extends naturally to the χ-algebras. We give a survey of the results obtained by the authors concerning the construction of Gröbner bases of $\Im_\chi(\mathcal{M})$ and diagonal bases of Orlik-Solomon type algebras and we present the combinatorial analogue of an "iterative residue formula" introduced by Szenes.

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RESUMEN

El álgebra de Orlik-Solomon de una matroide $\mathcal{M}$ es el cuociente del álgebra exterior en los puntos por el ideal $\Im(\mathcal{M})$ generado por los acotamientos de los circuitos de la matroide. Existe un isomorfismo entre el álgebra de Orlik-Solomon de una matroide compleja y la cohomología del complemento de un arreglo complejo de hiperplanos. En este artículo se considera una generalización de las algebras de Orlik-Solomon, llamadas χ-algebras. Estas nuevas álgebras incluyen, además de las álgebras de Orlik-Solomon, el álgebra de Orlik-Solomon-Terao de un conjunto de vectores y el álgebra de Cordovil de una matroid orientada. Para recalcar una importante propiedad de las "bases de circuitos no quebrados" de las álgebras de Orlik-Solomon-Terao, András Szenes ha introducido un particular tipo de bases, llamadas "bases diagonales". Este concepto se extiende naturalemente a la χ-algebras. Damos una mirada a los resultados obtenidos por los autores referentes a la construcción de las bases de Gröbner de $\Im_\chi(\mathcal{M})$ y bases diagonales de los tipos de algebras de Orlik-Solomon, y presentamos el análogo combinatorio de una "fórmula de residuos iterativa" introducida por Szenes.

Key words and phrases: *arrangement of hyperplanes, broken circuit, cohomology algebra, matroid, oriented matroid, Orlik-Solomon algebra, Gröbner bases.*

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1 Introduction

Let $\mathcal{M} = \mathcal{M}([n])$ be a matroid on the ground set $[n] := \{1, 2, \ldots, n\}$. The Orlik-Solomon algebra of a matroid $\mathcal{M}$ is the quotient of the exterior algebra on the points by the ideal $\Im(\mathcal{M})$ generated by the boundaries of the circuits of $\mathcal{M}$. The isomorphism between the Orlik-Solomon algebra of complex matroid and the cohomology of the complement of a complex arrangement of hyperplanes was established in [12]. The Orlik-Solomon algebras have been then intensively studied. A general reference on hyperplane arrangements and Orlik-Solomon algebras is [14]. Descriptions of developments from the early 1980's to the end of 1999, together with the contributions of many authors, can be found in [9, 21].

In this article a generalization of the Orlik-Solomon algebras, called χ-algebra, is considered. These new algebras include, apart from the Orlik-Solomon algebras, the Orlik-Solomon-Terao algebra of a set of vectors [15] and the Cordovil algebra of an oriented matroid [7]. We will survey recent results concerning this family of Orlik-Solomon type algebras (see [8, 10, 11]). In this introduction, we will recall the origin of the Orlik-Solomon algebra and we will develop the different notions used in the next sections like matroids and oriented matroids, the Orlik-Solomon algebra and its generalizations, its diagonal bases and the Gröbner bases of the defining ideal.

Let V be a vector space of dimension d over some field $\mathbb{K}$. A (central) arrangement (of hyperplanes) in V, $\mathcal{A}_\mathbb{K} = \{H_1, \ldots, H_n\}$, is a finite listed set of codimension one vector subspaces. Given an arrangement $\mathcal{A}_{\mathbb{K}}$ we always suppose fixed a family of linear forms $\{\theta_{H_i} \in V^* : H_i \in \mathcal{A}_{\mathbb{K}}, \mathrm{Ker}(\theta_{H_i}) = H_i\}$, where V^* denotes the dual space of V. Let $L(\mathcal{A}_{\mathbb{K}})$ be the *intersection lattice of* $\mathcal{A}_{\mathbb{K}}$: i.e., the set of intersections of hyperplanes in $\mathcal{A}_{\mathbb{K}}$, partially ordered by reverse inclusion. There is a matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ on the ground set $[n]$ determined by $\mathcal{A}_{\mathbb{K}}$: a subset $D \subseteq [n]$ is a *dependent set* of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ iff there are scalars $\zeta_i \in \mathbb{K}$, $i \in D$, not all nulls, such that $\sum_{i \in D} \zeta_i \theta_{H_i} = 0$. A *circuit* is a minimal dependent set with respect to inclusion. If $\mathbb{K}$ is an ordered field an additional structure is obtained: to every circuit C, $\sum_{i \in C} \zeta_i \theta_{H_i} = 0$, we associate a partition (determined up to a factor ± 1) $C^+ := \{i \in C : \zeta_i > 0\}, C^- := \{i \in C : \zeta_i < 0\}$. With this new structure $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ is said a *(realizable) oriented matroid* and denoted by $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{K}})$. Oriented matroids on a ground set $[n]$, denoted $\boldsymbol{\mathcal{M}}([n])$, are a very natural mathematical concept and can be seen as the theory of generalized hyperplane arrangements, see [3].

Set $\mathfrak{M}(\mathcal{A}_{\mathbb{K}}) := V \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{K}}} H$. The manifold $\mathfrak{M}(\mathcal{A}_{\mathbb{C}})$ plays an important role in the Aomoto-Gelfand theory of multidimensional hypergeometric functions (see [16] for a recent introduction from the point of view of arrangement theory). Let K be a commutative ring. In [12, 13, 14] the determination of the cohomology K-algebra $H^*\big(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); K\big)$ from the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$ is accomplished by first defining the Orlik-Solomon K-algebra $\mathrm{OS}(\mathcal{A}_{\mathbb{C}})$ in terms of generators and relators which depends only on the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$, and then by showing that this algebra is isomorphic to $H^*\big(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); K\big)$. Aomoto suggested the study of the (graded) $\mathbb{K}$-vector space $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$, generated by the basis $\{Q(\mathcal{B}_{\mathrm{I}})^{-1}\}$, where I is an independent set of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, $\mathcal{B}_{\mathrm{I}} := \{H_i \in \mathcal{A}_{\mathbb{K}} : i \in I\}$, and $Q(\mathcal{B}_{\mathrm{I}}) = \prod_{i \in I} \theta_{H_i}$ denotes the corresponding defining polynomial. To answer a conjecture of Aomoto, Orlik and Terao have introduced in [15] a commutative $\mathbb{K}$-algebra, $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$, called the Orlik-Solomon-Terao algebra. The algebra $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$ is isomorphic to $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$ as a graded $\mathbb{K}$-vector space in terms of the equations $\{\theta_H : H \in \mathcal{A}_{\mathbb{K}}\}$. A "combinatorial analogue" of the algebra of Orlik-Solomon-Terao was introduced in [7]: to every oriented matroid $\boldsymbol{\mathcal{M}}$ was associated a commutative $\mathbb{Z}$-algebra, denoted by $\mathbb{A}(\boldsymbol{\mathcal{M}})$ and called the Cordovil algebra. The χ-algebras generalizes the three just mentioned algebras: Orlik-Solomon, Orlik-Solomon-Terao and the Cordovil algebras, see [11] or Example 2.4 below.

In section two we will give the definition of a χ-algebra and recall the principal examples. In general a χ-algebra, denoted $\mathbb{A}_\chi(\mathcal{M})$, is defined as the quotient of some kind of a finite $\mathbb{K}$-algebra $\mathfrak{A}$ by an ideal $\Im_\chi(\mathcal{M})$ of $\mathfrak{A}$ whose generators are defined from the circuits of $\mathcal{M}$ and are depending of the map χ, see Definition 2.2. In particular the first important result is that like for the original Orlik-Solomon algebra we get ***nbc***-bases of the χ-algebra (as a module) from the "no broken circuit" sets of the matroid and corresponding basis for the ideal $\Im_\chi(\mathcal{M})$.

In section three, we construct the reduced Gröbner basis of the ideal $\Im_\chi(\mathcal{M})$ for any term order $\prec$ on the set of the monomials $\mathbb{T}(\mathfrak{A})$ of the algebra $\mathfrak{A}$. This result gives as a corollary a universal Gröbner basis (a Gröbner basis who works for every term order) which is shown to be minimal. Finally we remark that the ***nbc***-bases are in some sense the bases corresponding to the Gröbner bases for the different term orders.

In section four, following Szenes [17], we define a particular type of basis of $\mathbb{A}_\chi$, the so called "diagonal basis", see Definition 4.7. The ***nbc***-bases are an important examples of diagonal bases. We construct the dual bases of these bases, see Theorem 4.8. Our definitions make also use of an "iterative residue formula" based on the matroidal operation of contraction, see Equation (4.6). This formula can be seen as the combinatorial analogue of an "iterative residue formula" introduced by Szenes, [17]. As applications we deduce nice formulas to express a pure element in a diagonal basis. We prove also that the χ-algebras verify a splitting short exact sequence, see Theorem 4.4. This theorem generalizes for the χ-algebras previous similar theorems of [7, 14].

We use [19, 20] as a general reference in matroid theory. We refer to [3] and [14] for good sources of the theory of oriented matroids and arrangements of hyperplanes, respectively.

2 χ- algebras

Let $\mathrm{IND}_\ell(\mathcal{M}) \subseteq \binom{[n]}{\ell}$ [resp. $\mathrm{DEP}_\ell(\mathcal{M}) \subseteq \binom{[n]}{\ell}$] be the family of independent [resp. dependents] sets of cardinality ℓ of the matroid $\mathcal{M}$ and set

$$\mathrm{IND}(\mathcal{M}) := \bigcup_{\ell \in \mathbb{N}} \mathrm{IND}_\ell(\mathcal{M}),$$

$$\mathrm{DEP}(\mathcal{M}) := \bigcup_{\ell \in \mathbb{N}} \mathrm{DEP}_\ell(\mathcal{M}).$$

We denote by $\mathfrak{C} = \mathfrak{C}(\mathcal{M})$ the set of circuits of $\mathcal{M}$. For shortening of the notation the singleton set $\{x\}$ is denoted by x. When the smallest element α of a circuit C, $|C| > 1$, is deleted, the remaining set, $C \setminus \alpha$, is said to be a *broken circuit*. (Note that our definition is slightly different from the standard one. In the standard definition $C \setminus \alpha$ can be empty.) A *no broken circuit* set of a matroid $\mathcal{M}$ is an independent subset of $[n]$ which does not contain any broken circuit. Let $\mathrm{NBC}_\ell(\mathcal{M}) \subseteq \binom{[n]}{\ell}$ be the set of the no broken circuit sets of cardinal ℓ of $\mathcal{M}$ and set

$$\mathrm{NBC}(\mathcal{M}) := \bigcup_{\ell \in \mathbb{N}} \mathrm{NBC}_\ell(\mathcal{M}).$$

Let $L(\mathcal{M})$ be the lattice of flats of $\mathcal{M}$. $\big($We remark that the lattice map $\phi : L(\mathcal{A}_\mathbb{K}) \to L(\mathcal{M}(\mathcal{A}_\mathbb{K}))$, determined by the one-to-one correspondence $\phi' : H_i \longleftrightarrow \{i\}$, $i = 1, \dots, n$, is a lattice isomorphism.$\big)$ For an independent set $I \in \mathrm{IND}(\mathcal{M})$, let $c\ell(I)$ be the closure of I in $\mathcal{M}$.

For every permutation $\sigma \in \mathfrak{S}_m$, let X^σ be the ordered set

$$X^\sigma := i_{\sigma(1)} \prec \cdots \prec i_{\sigma(m)} = (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(m)}).$$

When necessary we also see the set $X = \{i_1, \dots, i_m\}$, as the ordered set

$$X^{\mathrm{id}} = (i_1, \dots, i_m).$$

Set $X^\sigma \setminus x := (i_{\sigma(1)}, \ldots, \widehat{x}, \ldots, i_{\sigma(m)})$. If $Y^\beta = (j_{\beta(1)}, \ldots, j_{\beta(m')})$ and $X \cap Y = \emptyset$, let $X^\sigma \circ Y^\beta$ be the concatenation of X^σ and Y^β, i.e., the ordered set

$$X^\sigma \circ Y^\beta := (i_{\sigma(1)}, \ldots, i_{\sigma(m)}, j_{\beta(1)}, \ldots, j_{\beta(m')}).$$

Definition 2.1 Let χ be a mapping $\chi : 2^{[n]} \to \mathbb{K}$. Let us also define χ for ordered sets by $\chi(X^\sigma) = \text{sgn }(\sigma)\chi(X)$, where sgn (σ) denotes the sign of the permutation σ. Fix a set $E = \{e_1, \ldots, e_n\}$. Let $\mathfrak{A} := \mathbb{K} \oplus \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ be the graded algebra over the field $\mathbb{K}$ generated by the elements $1, e_1, \ldots, e_n$ and satisfying the relations:

- $1e_i = e_i 1 = e_i$, for all $e_i \in E$,
- $e_i^2 = 0$, for all $e_i \in E$,
- $e_j \cdot e_i = \beta_{i,j} e_i \cdot e_j$ with $\beta_{i,j} \in \mathbb{K}^*$ for all $i < j$.

By definition the χ-*boundary* of an element $e_X \in \mathfrak{A}$, $X \neq \emptyset$, is given by the formula

$$\partial e_X := \sum_{p=1}^{p=m} (-1)^p \chi(X \setminus i_p) e_{X \setminus i_p}.$$

We set $\partial e_i = 1$, for all $e_i \in E$. We extend ∂ to the $\mathbb{K}$-algebra $\mathfrak{A}$ by linearity.

Let $X = (i_1, i_2, \ldots, i_m)$. In the sequel we will denote by e_X the (pure) element of the $\mathbb{K}$-algebra $\mathfrak{A}$,

$$e_X := e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_m}.$$

By convention we set $e_\emptyset := 1$. Both the exterior $\mathbb{K}$-algebra, $\bigwedge E$, (take $\beta_{i,j} = -1$) and the polynomial algebra $\mathbb{K}[e_1, \ldots, e_n]/\langle e_i^2 \rangle$ with squares zero (take $\beta_{i,j} = 1$) considered in [7, 15], are such $\mathbb{K}$-algebras $\mathfrak{A}$ and will be the only ones to be used in the examples. It is clear that for any $x \notin X$,

$$\pm \partial e_{X \cup x} = (-1)^{m+1} \chi(X) e_X + \sum_{p=1}^{p=m} (-1)^p \chi(X \setminus i_p \circ x) e_{X \setminus i_p \cup x}.$$

From the equality $\chi(X^\sigma) = \text{sgn }(\sigma)\chi(X)$, it is easy to see that for $\sigma \in \mathfrak{S}_{|X|}$ we have

$$\partial e_X = \text{sgn }(\sigma) \sum_{p=1}^{p=m} (-1)^p \chi(X^\sigma \setminus i_{\sigma(p)}) e_{X \setminus i_{\sigma(p)}}.$$

Given an independent set I, an element $a \in c\ell(I) \setminus I$ is said *active* in I if a is the minimal element of the unique circuit contained in $I \cup a$. We say that a subset $U \subseteq [n]$ is a *unidependent* set of $\mathcal{M}([n])$ if it contains a unique circuit, denoted $C(U)$. Note that U is unidependent iff $\text{rk}(U) = |U| - 1$. We say that a unidependent set U is an *inactive unidependent* if $\min(C(U))$ is the the minimal active element of $U \setminus \min(C(U))$. We will denote by $\text{UNI}_\ell(\mathcal{M})$ for the sets of inactive unidependent sets of size ℓ and set

$$\text{UNI}(\mathcal{M}) := \bigcup_{\ell \in \mathbb{N}} \text{UNI}_\ell(\mathcal{M}).$$

Let us remark that U is a unidependent set of $\mathcal{M}$ iff for some (or every) $x \in U$, $\mathrm{rk}(x) \neq 0$, $U \setminus x$ is a unidependent set of $\mathcal{M}/x$.

Definition 2.2 ([11]) Let χ be a mapping $\chi : 2^{[n]} \to \mathbb{K}$. Let $\Im_\chi(\mathcal{M}([n]))$ be the (right) ideal of $\mathfrak{A}$ generated by the χ-boundaries $\{\partial e_C : C \in \mathfrak{C}(\mathcal{M}), |C| > 1\}$ and the set of the loops of $\mathcal{M}$, $\{e_i : \{i\} \in \mathfrak{C}(\mathcal{M})\}$. We say that $\mathbb{A}_\chi(\mathcal{M}) := \mathfrak{A}/\Im_\chi(\mathcal{M})$ is a χ-*algebra* if χ satisfies the following two properties:

(2.2.1) $\chi(I) \neq 0$ if and only if I is independent.

(2.2.2) For any two unidependents U and U' of $\mathcal{M}$ with $U' \subseteq U$ there is a scalar $\varepsilon_{U,U'} \in \mathbb{K}^*$, such that $\partial e_U = \varepsilon_{U,U'}(\partial e_{U'})e_{U\setminus U'}$.

Note that
$$\{e_C : C \in \mathfrak{C}(\mathcal{M})\} \subseteq \Im_\chi(\mathcal{M}([n])).$$
For every $X \subseteq [n]$, we denote by $[X]_{\mathbb{A}}$ or shortly by e_X when no confusion will result, the residue class in $\mathbb{A}_\chi(\mathcal{M})$ determined by the element e_X. Since $\Im_\chi(\mathcal{M})$ is a homogeneous ideal, $\mathbb{A}_\chi(\mathcal{M})$ inherits a grading from $\mathfrak{A}$. More precisely we have $\mathbb{A}_\chi(\mathcal{M}) = \mathbb{K} \oplus \mathbb{A}_1 \oplus \cdots \oplus \mathbb{A}_r$, where $\mathbb{A}_\ell = \mathfrak{A}_\ell/\mathfrak{A}_\ell \cap \Im_\chi(\mathcal{M})$ denotes the subspace of $\mathbb{A}_\chi(\mathcal{M})$ generated by the elements $\{[I]_{\mathbb{A}} : I \in \mathrm{IND}_\ell(\mathcal{M})\}$. Set

$$\boldsymbol{nbc}_\ell := \{[I]_{\mathbb{A}} : I \in \mathrm{NBC}_\ell(\mathcal{M})\} \quad \text{and} \quad \boldsymbol{nbc} := \bigcup_{\ell=0} \boldsymbol{nbc}_\ell,$$

$$\boldsymbol{dep}_\ell := \{[D]_{\mathbb{A}} : D \in \mathrm{DEP}_\ell(\mathcal{M})\} \quad \text{and} \quad \boldsymbol{dep} := \bigcup_{\ell=0} \boldsymbol{dep}_\ell,$$

$$\boldsymbol{uni}_\ell := \{[U]_{\mathbb{A}} : U \in \mathrm{UNI}_\ell(\mathcal{M})\} \quad \text{and} \quad \boldsymbol{uni} := \bigcup_{\ell=0} \boldsymbol{uni}_\ell.$$

Remark 2.3 From (2.2.1) and (2.2.2) we conclude that $\Im_\chi(\mathcal{M})$ has the basis $\boldsymbol{dep} \cup \partial\boldsymbol{uni}$ and that $\boldsymbol{nbc} := \{[I]_{\mathbb{A}} : I \in \mathrm{NBC}(\mathcal{M})\}$ is a basis of the vector space $\mathbb{A} = \mathbb{A}_\chi(\mathcal{M})$. We also have that $\boldsymbol{nbc}_\ell$ is a basis of the vector space $\mathbb{A}_\ell$. This fundamental property was first discovered for the Orlik-Solomon algebras [14], and then also for the other classical χ-algebras, see [7, 15] and the following example for more details. Note also that this implies that $[X]_{\mathbb{A}} \neq 0$ iff X is an independent set of $\mathcal{M}$.

Example 2.4 Recall the three usual χ-algebras $\mathbb{A}_\chi(\mathcal{M})$.

- Let $\mathfrak{A} = \bigwedge E$ be the exterior $\mathbb{K}$-algebra (taking $\beta_{i,j} = -1$). Setting $\chi(I^\sigma) = \mathrm{sgn}(\sigma)$ for every independent set I of a matroid $\mathcal{M}$ and every permutation $\sigma \in \mathfrak{S}_{|I|}$, we obtain the Orlik-Solomon algebra, $\mathrm{OS}(\mathcal{M})$.

- Let $\mathcal{A}_{\mathbb{K}} = \{H_i : H_i = \mathrm{Ker}(\theta_i), i = 1, 2, \ldots, n\}$ be an hyperplane arrangement and $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ its associated matroid. For every flat $F := \{f_1, \ldots, f_k\} \subseteq [n]$ of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ we choose a bases B_F of the vector subspace of $(\mathbb{K}^d)^*$ generated by $\{\theta_{f_1}, \ldots, \theta_{f_k}\}$. By taking $\mathfrak{A} = \mathbb{K}[e_1, \ldots, e_n]/\langle e_i^2 \rangle$ the polynomial algebra with squares null (taking $\beta_{i,j} = 1$) and taking for any $\{i_1, \ldots, i_\ell\} = I \in \mathrm{IND}_\ell$, $\chi(I) = \det(\theta_{i_1}, \ldots, \theta_{i_\ell})$, where the vectors are expressed in the basis $B_{c\ell(I)}$, we obtain the Orlik-Solomon-Terao algebra $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$, defined in [15].

- Let $\boldsymbol{\mathcal{M}}([n])$ be an oriented matroid. For every flat F of $\boldsymbol{\mathcal{M}}([n])$, we choose (determined up to a factor ± 1) a bases signature in the restriction of $\boldsymbol{\mathcal{M}}([n])$ to F. We define a *signature of the independents of an oriented matroid* $\boldsymbol{\mathcal{M}}([n])$ as a mapping, sgn $: \mathrm{IND}(\boldsymbol{\mathcal{M}}) \to \{\pm 1\}$, where sgn (I) is equal to the basis signature of I in the restriction of $\boldsymbol{\mathcal{M}}([n])$ to $c\ell(I)$. By taking $\mathfrak{A} = \mathbb{Q}[e_1, \dots, e_n]/\langle e_i^2 \rangle$ the polynomial algebra over the rational field $\mathbb{Q}$ with squares zero (take $\beta_{i,j} = 1$) and taking $\chi(I) = \text{sgn }(I)$ (resp. $\chi(X) = 0$) for every independent (resp. dependent) set of the matroid, we obtain the algebra $\mathbb{A}(\boldsymbol{\mathcal{M}}) \oplus_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{A}(\boldsymbol{\mathcal{M}})$ denotes the Cordovil $\mathbb{Z}$-algebra defined in [7].

3 Gröbner bases of χ-ideals

For general details on Gröbner bases of an ideal, see [1, 2]. We begin by adapting some definitions to our context. Consider the $\mathbb{K}$-algebra $\mathfrak{A}$ introduced in Definition 2.1. Note that there are monomials $e_Y, e_Z \in \mathfrak{A}$, such that $e_Y \cdot e_Z = 0$. In the standard case where $\mathfrak{A}$ is replaced by the polynomial ring $\mathbb{K}[e_1, \dots, e_n]$, this is not possible. So the the following definitions are slightly different from the standard corresponding ones given in [1, 2]. Let $\mathcal{M} = \mathcal{M}([n])$ be a matroid, $\Im_\chi(\mathcal{M})$ and $\mathbb{A}_\chi(\mathcal{M})$ the χ-ideal and χ-algebra as defined in the previous section. We will denote for shortness $\mathbb{A}(\mathcal{M})$ for $\mathbb{A}_\chi(\mathcal{M})$.

Definition 3.1 Let $\mathbb{T} = \mathbb{T}(\mathfrak{A})$ be the *set of the monomials of the* $\mathbb{K}$*-algebra* $\mathfrak{A}$, i.e., $\mathbb{T}(\mathfrak{A}) := \{e_X : X = (e_{i_1}, \dots, e_{i_m})\}$. A total ordering $\prec$ on the monomials $\mathbb{T}$ is said a *term order* on $\mathbb{T}$ if $e_\emptyset = 1$ is the minimal element and $\prec$ is compatible with the multiplication in $\mathfrak{A}$, i.e.,

$$\forall e_X, e_Y, e_Z \in \mathbb{T},\ \ (e_X \prec e_Y)\&(e_X \cdot e_Z \neq 0)\&(e_Y \cdot e_Z \neq 0) \implies e_X \cdot e_Z \prec e_Y \cdot e_Z.$$

Given a term order $\prec$ on $\mathbb{T}$ and a non-null polynomial $f \in \mathfrak{A}$, we may write

$$f = a_1 e_{X_1} + a_2 e_{X_2} + \cdots + a_m e_{X_m},$$

where $a_i \in \mathbb{K}^*$ and $e_{X_m} \prec \cdots \prec e_{X_1}$. We say that the $a_i e_{X_i}$ [resp. e_{X_i}] are the *terms* [resp. *monomials*] of f. We say that $\mathrm{lp}_\prec(f) := e_{X_1}$ [resp. $\mathrm{lt}_\prec(f) := a_1 e_{X_1}$] is the *leading monomial* [resp. *leading term*] of f (with respect to $\prec$). We also define $\mathrm{lp}_\prec(0) = \mathrm{lt}_\prec(0) = 0$. Note that in general we have $\mathrm{lp}_\prec(hg) \neq \mathrm{lp}_\prec(h)\mathrm{lp}_\prec(g)$, contrarily to the cases considered in [1, 2].

Example 3.2 A permutation $\pi \in \Sigma_n$ defines a linear reordering of the set $[n]$: $\pi^{-1}(1) <_\pi \pi^{-1}(2) <_\pi \cdots <_\pi \pi^{-1}(n)$. Consider the ordering of the set E

$$e_{\pi^{-1}(1)} \prec_\pi e_{\pi^{-1}(2)} \prec_\pi \cdots \prec_\pi e_{\pi^{-1}(n)}.$$

The corresponding degree lexicographic ordering on the monomials $\mathbb{T}$, also denoted $\prec_\pi$, is a term order on $\mathbb{T}$.

For a subset S, $S \subseteq \mathfrak{A}$ and a term order $\prec$ on $\mathbb{T}(\mathfrak{A})$, we define the *leading term ideal* of S, denoted $\mathrm{Lt}_{\prec}(S)$, as the ideal generated by the leading monomials of the polynomial in S, i.e.,

$$\mathrm{Lt}_{\prec}(S) := \langle \mathrm{lp}_{\prec}(f) : f \in S \rangle.$$

In the remaining of this section we suppose that $\mathcal{M}([n])$ is a loop free matroid.

Definition 3.3 Let $\mathcal{M}$ be a matroid. Let $\prec$ be a term order on $\mathbb{T}(\mathfrak{A})$. Consider the ideal $\Im_{\chi}(\mathcal{M})$ of $\mathfrak{A}$ A family $\mathcal{G}$ of non-null polynomials of the ideal $\Im_{\chi}(\mathcal{M})$ is called a *Gröbner basis* of the ideal $\Im_{\chi}(\mathcal{M})$ with respect to $\prec$ iff

$$\mathrm{Lt}_{\prec}(\mathcal{G}) = \mathrm{Lt}_{\prec}(\Im_{\chi}(\mathcal{M})).$$

The Gröbner basis $\mathcal{G}$ is called *reduced* if, for every element $g \in \mathcal{G}$ we have $\mathrm{lt}_{\prec}(g) = \mathrm{lp}_{\prec}(g)$, and for every two distinct elements $g, g' \in \mathcal{G}$, no term of g' is divisible by $\mathrm{lp}_{\prec}(g)$. The Gröbner basis $\mathcal{G}$ is called a *universal Gröbner basis* if it is a Gröbner basis with respect to all term orders on $\mathbb{T}(\mathfrak{A})$ simultaneously. If $\mathcal{U}$ is a universal Gröbner basis, minimal for inclusion with this property, we say that $\mathcal{U}$ is a *minimal universal Gröbner basis*.

From Definition 3.3 we conclude:

Proposition 3.4 *Let $\mathcal{G}_{\prec}$ be a Gröbner basis of the ideal $\Im_{\chi}(\mathcal{M})$ with respect to the term order $\prec$ on $\mathbb{T}(\mathfrak{A})$. Then*

$$\mathcal{B}_{\mathcal{G}_{\prec}} := \big\{ e_X + \Im_{\chi}(\mathcal{M}) : e_X \notin \mathrm{Lt}_{\prec}(\mathcal{G}) \big\}$$

is a basis of the module $\mathbb{A}_{\chi}(\mathcal{M})$. ∎

We say that the well determined basis $\mathcal{B}_{\mathcal{G}_{\prec}}$ is the *canonical basis of the χ-algebra $\mathbb{A}_{\chi}(\mathcal{M})$ for the Gröbner basis $\mathcal{G}$ of the ideal $\Im_{\chi}(\mathcal{M})$, with respect to the term order $\prec$ on $\mathbb{T}(\mathfrak{A})$.*

Consider the partition $\mathbb{T}(\mathfrak{A}) = \mathbb{T}_i(\mathfrak{A}) \biguplus \mathbb{T}_d(\mathfrak{A})$ where:

$$\mathbb{T}_i(\mathfrak{A}) := \big\{ e_I : I \in \mathrm{IND}(\mathcal{M}) \big\} \quad \text{and} \quad \mathbb{T}_d(\mathfrak{A}) := \big\{ e_D : D \in \mathrm{DEP}(\mathcal{M}) \big\}.$$

Let $\mathbb{K}[\mathbb{T}_i]$ and $\mathbb{K}[\mathbb{T}_d]$ be the $\mathbb{K}$-vector subspaces of $\mathfrak{A}$ generated by the basis $\mathbb{T}_i$ and $\mathbb{T}_d$, respectively. So $\mathfrak{A} = \mathbb{K}[\mathbb{T}_i] \oplus \mathbb{K}[\mathbb{T}_d]$. We also see the set $\mathbb{K}[\mathbb{T}_d] \subseteq \Im_{\chi}(\mathcal{M})$ as the ideal of $\mathfrak{A}$ generated by the set of monomials $\{e_C : C \in \mathfrak{C}(\mathcal{M})\}$. Let $\boldsymbol{p_i} : \mathfrak{A} \to \mathbb{K}[\mathbb{T}_i]$ be the first projection. We define the term orders on the set of monomials $\mathbb{T}_i$ in a similar way to the corresponding definition on $\mathbb{T}$. It is clear that the restriction of every term order of $\mathbb{T}$ to the subset $\mathbb{T}_i$ is also a term order on $\mathbb{T}_i$. We can also add to $\mathbb{K}[\mathbb{T}_i]$ a structure of $\mathbb{K}$-algebra with the product $\star : \mathbb{K}[\mathbb{T}_i] \times \mathbb{K}[\mathbb{T}_i] \to \mathbb{K}[\mathbb{T}_i]$, determined by the equalities

$$e_X \star e_{X'} = \boldsymbol{p_i}(e_X e_{X'}) \quad \text{for all} \quad X, X' \in \Im_{\chi}(\mathcal{M}).$$

Note that if $e_X \star e_{X'} \neq 0$, then $e_X \star e_{X'} = e_X e_{X'}$. We remember that $e_X e_{X'} \neq 0$ iff $X \cap X' = \emptyset$ and $X \cup X' \in \mathrm{IND}(\mathcal{M})$. So $\Im_{\chi_i}(\mathcal{M}) := \boldsymbol{p_i}\big(\Im_{\chi}(\mathcal{M})\big)$ is an ideal of $\mathbb{K}[\mathbb{T}_i]$.

Proposition 3.5 *Let $\prec$ be a term order on $\mathbb{T}(\mathfrak{A})$. Then the leading term ideals of $\mathfrak{A}$, $\mathrm{Lt}_{\prec}(\boldsymbol{p_i}(\Im_{\chi}(\mathcal{M})))$ and $\mathrm{Lt}_{\prec}(\Im_{\chi}(\mathcal{M}))$ are equal. In particular a Gröbner basis of the ideal $\Im_{\chi_i}(\mathcal{M})$ of $\mathbb{K}[\mathbb{T}_i]$ with respect to term order $\prec$ on $\mathbb{T}_i$ is also a Gröbner basis of the ideal $\Im_{\chi}(\mathcal{M})$ of $\mathfrak{A}$ with respect to the term order $\prec$ on $\mathbb{T}$.*

Proof. Note first that if we see $\Im_{\chi}(\mathcal{M})$ as a $\mathbb{K}$-vector space it is clear that $\Im_{\chi}(\mathcal{M}) = \Im_{\chi_i}(\mathcal{M}) \oplus \mathbb{K}[\mathbb{T}_d]$. Pick a non-null polynomial $f \in \Im_{\chi}(\mathcal{M})$ and let $e_{X_1} := \mathrm{lp}_{\prec}(f)$. So $e_{X_1} \in \Im_i(\mathcal{M})$ if $X_1 \in \mathrm{IND}(\mathcal{M})$, or $e_{X_1} \in \mathbb{K}[\mathbb{T}_d] \setminus 0$ if X_1 is a dependent set of $\mathcal{M}$. If $X_1 \in \mathrm{IND}(\mathcal{M})$ then $e_{X_1} \in \mathrm{Lt}_{\prec}(\Im_{\chi}(\mathcal{M}))$. Suppose now that X_1 is a dependent set of $\mathcal{M}$. Then there is a circuit $C \subseteq X_1$. From Definition 2.2 we know that $\partial e_C \in \Im_{\chi}(\mathcal{M})$. It is clear that $e_C \in \mathrm{Lt}_{\prec}(\boldsymbol{p_i}(\Im_{\chi}(\mathcal{M})))$ and so we have also $e_{X_1} \in \mathrm{Lt}_{\prec}(\boldsymbol{p_i}(\Im_{\chi}(\mathcal{M})))$. ∎

Remark 3.6 It is well known that a term order $\prec$ of $\mathbb{T}(\mathfrak{A})$ determines also a unique reduced Gröbner basis of $\Im_{\chi}(\mathcal{M})$ denoted $(\mathcal{G}_r)_{\prec}$. From the definitions we can deduce also that, for every pair of term orders $\prec$ and $\prec'$ on $\mathbb{T}(\mathfrak{A})$,

$$\mathcal{B}_{\mathcal{G}_{\prec}} = \mathcal{B}_{\mathcal{G}_{\prec'}} \Leftrightarrow (\mathcal{G}_r)_{\prec} = (\mathcal{G}_r)_{\prec'} \Leftrightarrow \mathrm{Lt}_{\prec}\big(\Im_{\chi}(\mathcal{M})\big) = \mathrm{Lt}_{\prec'}\big(\Im_{\chi}(\mathcal{M})\big).$$

Definition 3.7 For a term order $\prec$ on $\mathbb{T}(\mathfrak{A})$ we say that $\pi_{\prec} \in \mathfrak{S}_n$, is the *permutation compatible* with $\prec$ if, for every pair $i, j \in [n]$, we have

$$e_i \prec e_j \quad \text{iff} \quad i <_{\pi_{\prec}} j \ \big(\Leftrightarrow \ {\pi_{\prec}}^{-1}(i) < {\pi_{\prec}}^{-1}(j)\,\big).$$

Let $\mathfrak{C}_{\pi_{\prec}}$ be the subset of circuits of $\mathcal{M}$ such that:

- ○ $C \in \mathfrak{C}_{\pi_{\prec}}$ iff $\inf_{<_{\pi_{\prec}}}(C) = \alpha_{\pi}(C) \ \big(= \inf_{<_{\pi_{\prec}}}(\mathrm{cl}(C) \setminus C)\big)$ and $C \setminus \alpha_{\pi}(C)$ is inclusion minimal with this property.

In the following we replace "$\pi_{\prec}$" by "π" if no mistake can results. We recall that given a unidependent set U of the matroid $\mathcal{M}([n])$, $C(U)$ denotes the unique circuit of $\mathcal{M}$ contained in U.

Theorem 3.8 *Let $\prec$ be a term order on $\mathbb{T}(\mathfrak{A})$ compatible with the permutation $\pi \in \mathfrak{S}_n$. Then the family $\mathcal{G}_r := \big\{\partial e_{C(U)} : U \in \mathfrak{C}_{\pi_{\prec}}(\mathcal{M})\big\}$ form a reduced Gröbner basis of $\Im_{\chi_i}(\mathcal{M})$ with respect to the term order $\prec$.*

Proof. From Proposition 3.5 it is enough to prove that $(\mathcal{G}_r)_{\prec}$ is a reduced Gröbner of $\Im_{\chi_i}(\mathcal{M})$. Let f be any element of $\Im_{\chi_i}(\mathcal{M})$, we have from Theorem 2.3 that

$$f = \sum_{U \in \mathfrak{U}_{\pi}} \xi_U \partial e_U, \ \xi_U \in \mathbb{K}^{\star}.$$

Let now remark that $\mathrm{lp}_{\prec}\big(\partial e_U\big) = e_{U \setminus \alpha_{\pi}(U)}$ and that these terms are all different. We have then clearly that

$$\mathrm{lp}_{\prec}(f) = \sup_{\prec}\big\{\mathrm{lp}_{\prec}(\partial e_U) : \ U \in \mathfrak{U}_{\pi}\big\}.$$

Given an arbitrary $U' \in \mathfrak{U}_\pi(\mathcal{M})$ it is clear that $\alpha_\pi(C(U')) = \alpha_\pi(U')$. So,

$$C(U') \setminus \alpha_\pi(C(U')) \subseteq U' \setminus \alpha_\pi(U').$$

Let C' be a circuit of $\mathfrak{C}_\pi$ such that $C' \setminus \alpha_\pi(C') \subseteq C(U) \setminus \alpha_\pi(C(U))$. So we have that $\mathrm{lp}_\prec(\partial e_{C'})$ divides $\mathrm{lp}_\prec(\partial e_U)$, and $(\mathcal{G}_r)_\prec$ is a Gröbner basis.

Suppose for a contradiction that $(\mathcal{G}_r)_\prec$ is not a reduced Gröbner basis: i.e., there exists two circuits C and C' in $\mathfrak{C}_\pi$ and an element $c \in C$ such that $e_{C' \setminus \alpha_\pi(C')}$ divides $e_{C \setminus c}$ $\big(\Leftrightarrow C' \setminus \alpha_\pi(C') \subseteq C \setminus c\big)$. First we can say that $c \neq \alpha_\pi(C)$ because the sets $C' \setminus \alpha_\pi(C')$ and $C \setminus \alpha_\pi(C)$ are incomparable. This in particular implies that $\alpha_\pi(C) \in C' \setminus \alpha_\pi(C')$, and $\alpha_\pi(C') \prec \alpha_\pi(C)$. On the other hand we have $\alpha_\pi(C') \in \mathrm{cl}\big(C' \setminus \alpha_\pi(C')\big) \subseteq \mathrm{cl}(C \setminus c) = \mathrm{cl}(C \setminus \alpha_\pi(C))$, so $\alpha_\pi(C) \prec \alpha_\pi(C')$, a contradiction. ■

Corollary 3.9 *The set* $\mathcal{G}_u := \big\{\partial e_C : C \in \mathfrak{C}(\mathcal{M})\big\}$ *is a minimal universal Gröbner basis of the ideal* $\Im_\chi(\mathcal{M})$.

Proof. From Theorem 3.8, the reduced Gröbner bases constructed for the different orders $\prec$ are all contained in $\mathcal{G}_u$. We prove the minimality by contradiction. Let $C_0 = \{i_1, \dots, i_m\}$ be a circuit of $\mathcal{M}$ and let $\pi \in \mathfrak{S}_n$ be a permutation such that $\pi^{-1}(i_j) = j$, $j = 1, \dots, m$. Then $\mathcal{G}'_u := \{\partial e_C : C \in \mathfrak{C} \setminus C_0\}$ it is not a Gröbner basis because $\mathrm{lp}_{\prec_\pi}(\partial e_{C_0}) = e_{C_0 \setminus i_1}$ is not in $\mathrm{Lt}_{\prec_\pi}(\mathcal{G}'_u)$. ■

To finish this section we give an important characterization of the no broken circuit bases of the χ-algebras in terms of the Gröbner bases of their ideals.

Definition 3.10 Consider a permutation $\pi \in \mathfrak{S}_n$ and the associated re-ordering $<_\pi$ of $[n]$. When the $<_\pi$-smallest element $\inf_{<_\pi}(C)$ of a circuit $C \in \mathfrak{C}(\mathcal{M})$, $|C| > 1$, is deleted, the remaining set, $C \setminus \inf_{<_\pi}(C)$, is called a π-*broken circuit* of $\mathcal{M}$. We say that

$$\pi\text{-}\boldsymbol{nbc}(\mathcal{M}) := \{e_X : X \subseteq [n] \text{ contains no } \pi\text{-broken circuit of } \mathcal{M}\}$$

is the π-*no broken circuit bases* of $\mathbb{A}_\chi(\mathcal{M})$. As the algebra $\mathbb{A}_\chi(\mathcal{M})$ does not depend of the ordering of the elements of $\mathcal{M}$ it is clear that π-$\boldsymbol{nbc}(\mathcal{M})$ is a no broken circuit bases of $\mathbb{A}_\chi(\mathcal{M})$.

Corollary 3.11 *Let* $\mathcal{B}$ *be a basis of the module* $\mathcal{A}_\chi(\mathcal{M})$. *Then are equivalent:*

(3.11.1) $\mathcal{B}$ *is the canonical basis* $\mathcal{B}_\prec$, *for some term order* $\prec$ *on* $\mathbb{T}(\mathfrak{A})$.

(3.11.2) $\mathcal{B}$ *is the* π-*no broken circuit bases* π-$\boldsymbol{nbc}(\mathcal{M})$, *for some permutation* $\pi \in \mathfrak{S}_n$.

(3.11.3) $\mathcal{B}$ *is the canonical basis* $\mathcal{B}_{\mathcal{G}_r}$, *for some reduced Gröbner basis* $\mathcal{G}_r$ *of the ideal* $\Im_\chi(\mathcal{M})$.

Proof. (3.11.1) ⇒ (3.11.2) Let $\prec$ be a term order of $\mathbb{T}(\mathfrak{A})$. Since $\mathcal{G}_u$ is a universal Gröbner basis of $\Im_\chi(\mathcal{M})$ (see Corollary 3.9) it is trivially a Gröbner basis relatively to $\prec$. We have already remarked that the leading term of ∂e_C is $e_{C \setminus c}$ where $c = \inf_{<_{\pi_\prec}}(C)$. From Proposition 3.4 we conclude that $\mathcal{B}_\prec = \pi_\prec$-$\boldsymbol{nbc}(\mathcal{M})$.

(3.11.2) $\Rightarrow$ (3.11.3) Suppose that $\mathcal{B} = \pi$-$\boldsymbol{nbc}(\mathcal{M})$. Let $\prec_\pi$ be the degree lexicographic order of $\mathbb{T}$ determined by the permutation $\pi \in \mathfrak{S}_n$. Note that $\pi_{\prec_\pi} = \pi$. ¿From Theorem 3.8 we know that $(\mathcal{G}_r)_{\prec_\pi} = \{\partial e_C : C \in \mathfrak{C}_{\prec_\pi}\}$ is the reduced Gröbner basis of $\Im_\chi(\mathcal{M})$ with respect to the term order $\prec_\pi$. Then $\mathcal{B}$ is the canonical basis for the reduced Gröbner basis $(\mathcal{G}_r)_{\prec_\pi}$.
(3.11.3) $\Rightarrow$ (3.11.1) It is a consequence of Proposition 3.4 and Remark 3.6. ∎

4 Diagonal bases of χ-algebras

Proposition 4.1 *Let $\mathbb{A}_\chi(\mathcal{M})$ be a χ-algebra with the associated map $\chi : 2^{[n]} \to \mathbb{K}$. For any non loop element x of $\mathcal{M}([n])$, we define the two maps:*

$$\chi_{\mathcal{M}\setminus x} : 2^{[n]\setminus x} \to \mathbb{K} \quad \textit{by} \quad \chi_{\mathcal{M}\setminus x}(X) = \chi(X) \quad \textit{and} \tag{4.1}$$

$$\chi_{\mathcal{M}/x} : 2^{[n]\setminus x} \to \mathbb{K} \quad \textit{by} \quad \chi_{\mathcal{M}/x}(X) = \chi(X \circ x). \tag{4.2}$$

There are two χ-algebras, $\mathbb{A}_{\chi_{\mathcal{M}/x}}(\mathcal{M}/x)$ and $\mathbb{A}_{\chi_{\mathcal{M}\setminus x}}(\mathcal{M}\setminus x)$, associated to the maps $\chi_{\mathcal{M}\setminus x}$ and $\chi_{\mathcal{M}/x}$, respectively.

Proof. From (2.2.1) we know that $\chi(X) \neq \emptyset$ iff $X \in \mathrm{IND}(\mathcal{M})$. The deletion case being trivial, we will just prove the contraction case. We have to show that $\chi_{\mathcal{M}/x}$ verifies properties (2.2.1) and (2.2.2). The first property is verified since a set I is independent in $\mathcal{M}/x$ iff $I \cup x$ is independent in $\mathcal{M}$. To see that the second property is also verified, let U and U' be two unidependents sets of $\mathcal{M}/x$. I.e., iff $U \cup x$ and $U' \cup x$ are two unidependents sets of $\mathcal{M}$. From (2.2.1) we know that

$$\partial e_{U\cup x} = \varepsilon_{U\cup x, U'\cup x}(\partial e_{U'\cup x}) e_{U\setminus U'} \quad \text{where} \quad \varepsilon_{U\cup x, U'\cup x} \in \mathbb{K}^*.$$

Let ∂' be the $\chi_{\mathcal{M}/x}$-boundary, i.e., the linear mapping $\partial' : \mathfrak{A}/\langle e_x\rangle \to \mathfrak{A}/\langle e_x\rangle$ such that for ever $e_i \in E \setminus x$ we have $\partial' e_i = 1$, $\partial' e_\emptyset = 1$ and for every monomial e_X, $x \notin X$ and $X \neq \emptyset$,

$$\partial' e_X = \sum_{p=1}^{p=m} (-1)^p \chi_{\mathcal{M}/x}(X \setminus i_p) e_{X\setminus i_p} = \sum_{p=1}^{p=m} (-1)^p \chi(X \setminus i_p \circ x) e_{X\setminus i_p}.$$

To finish the proof we will show that there is a scalar $\tilde{\varepsilon}_{U,U'} \in \mathbb{K}^*$ such that

$$\partial' e_U = \tilde{\varepsilon}_{U,U'}(\partial' e_{U'}) e_{U\setminus U'}.$$

Let $X, X' \subseteq [n]$ be two disjoint subsets. From Definition 2.1 we known that

$$e_X \cdot e_{X'} = \beta_{X,X'} e_{X\cup X'}, \text{ where } \beta_{X,X'} = \prod \beta_{i,j}, \ (e_i \in X, e_j \in X' \text{ and } i > j).$$

So we have with $U = (i_1, \ldots, i_m)$ and $U' = (j_1, \ldots, j_k)$, $U \cap U' = \emptyset$, $x \notin U \cup U'$:

$$\pm\partial e_{U\cup x} = \sum_{p=1}^{p=m} (-1)^p \chi(U \setminus i_p \circ x) e_{U\cup x\setminus i_p} + (-1)^{m+1}\chi(U) e_U,$$

$$\partial' e_U = \sum_{p=1}^{p=m} (-1)^p \chi(U \setminus i_p \circ x) e_{U \setminus i_p},$$

$$\pm(\partial e_{U' \cup x}) e_{U \setminus U'} = \sum_{p=1}^{p=k} (-1)^p \chi(U' \setminus j_p \circ x) \cdot \beta \cdot e_{U \cup x \setminus j_p} + (-1)^{k+1} \chi(U') \cdot \beta' \cdot e_U,$$

where $\beta = \beta_{U' \cup x \setminus j_p, U \setminus U'}$ and $\beta' = \beta_{U', U \setminus U'}$.

$$(\partial' e_{U'}) e_{U \setminus U'} = \sum_{p=1}^{p=k} (-1)^p \chi(U' \setminus j_p \circ x) \cdot \beta_{U' \setminus j_p, U \setminus U'} \cdot e_{U \setminus j_p}.$$

After remarking that

$$\beta_{U' \cup x \setminus j_p, U \setminus U'} \beta^{-1}_{U' \setminus j_p, U \setminus U'} = \beta_{x, U \setminus U'}$$

does not depend on j_p, we can deduce that

$$\partial' e_U = \tilde{\varepsilon}_{U,U'} (\partial' e_{U'}) e_{U \setminus U'} \quad \text{with} \quad \tilde{\varepsilon}_{U,U'} = \pm \varepsilon_{U \cup x, U' \cup x} \cdot \beta^{-1}_{x, U \setminus U'}.$$

∎

Proposition 4.2 *For every non loop element x of $\mathcal{M}([n])$, there is a unique monomorphism of vector spaces, $\mathfrak{i}_x : \mathbb{A}(\mathcal{M} \setminus x) \to \mathbb{A}(\mathcal{M})$, such that such that for every $I \in \mathrm{IND}(\mathcal{M} \setminus x)$, we have $\mathfrak{i}_x(e_I) = e_I$.*

Proof. By a reordering of the elements of the matroid $\mathcal{M}$ we can suppose that $x = n$. It is clear that

$$\mathrm{NBC}(\mathcal{M} \setminus x) = \big\{X : X \subseteq [n-1] \text{ and } X \in \mathrm{NBC}(\mathcal{M})\big\},$$

so the proposition is a consequence of Equation (4.1). ∎

Proposition 4.3 *For every non loop element x of $\mathcal{M}([n])$, there is a unique epimorphism of vector spaces, $\mathfrak{p}_x : \mathbb{A}(\mathcal{M}) \to \mathbb{A}(\mathcal{M}/x)$, such that, for every e_I, $I \in \mathrm{IND}(\mathcal{M})$, we have*

$$\mathfrak{p}_x(e_I) := \begin{cases} e_{I \setminus x} & \text{if} \quad x \in I, \\ \frac{\chi(I \setminus y, x)}{\chi(I \setminus y, y)} e_{I \setminus y} & \text{if there is } y \in I \text{ parallel to } x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. From Remark 2.3, it is enough to prove that $\mathfrak{p}_x(\partial e_U) = 0$, for all unidependent $U = (i_1, \dots, i_m)$. We recall that if $x \in U$ then $U \setminus x$ is a unidependent set of $\mathcal{M}/x$. There are only the following four cases:

- If U contains x but no y parallel to x then:

$$\pm \mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x((-1)^m \chi(U \setminus x) e_{U \setminus x} + \sum_{i_p \in U \setminus x} (-1)^p \chi(U \setminus \{i_p, x\} \circ x) e_{U \setminus i_p}))$$

$$= \sum_{i_p \in U \setminus x} (-1)^p \chi(U \setminus \{i_p, x\} \circ x) e_{U \setminus \{i_p, x\}} = 0$$

from Proposition 4.1.

∘ If U does not contain x but contains a y parallel to x then:

$$\begin{aligned}\pm\mathfrak{p}_x(\partial e_U) &= \mathfrak{p}_x\big((-1)^m\chi(U\setminus y)e_{U\setminus y} + \sum_{i_p\in U\setminus y}(-1)^p\chi(U\setminus\{i_p,y\}\circ y)e_{U\setminus i_p}\big)\\ &= \sum_{i_p\in U\setminus y}(-1)^p\chi(U\setminus\{i_p,y\}\circ y)\frac{\chi(U\setminus\{i_p,x\}\circ x)}{\chi(U\setminus\{i_p,y\}\circ y)}e_{U\setminus\{i_p,y\}} = 0\end{aligned}$$

like previously since $U\setminus y$ is again a unidependent of $\mathcal{M}/x$.

∘ If U contains x and a y parallel to x then:

$$\begin{aligned}\pm\mathfrak{p}_x(\partial e_U) &= \mathfrak{p}_x(\chi(U\setminus\{x,y\}\circ y)e_{U\setminus x} - \chi(U\setminus\{x,y\}\circ x)e_{U\setminus y})\\ = \chi(U\setminus\{x,y\}\circ y)&\frac{\chi(U\setminus\{x,y\}\circ x)}{\chi(U\setminus\{x,y\}\circ y)}e_{U\setminus\{x,y\}} - \chi(U\setminus\{x,y\}\circ x)e_{U\setminus\{x,y\}} = 0.\end{aligned}$$

∘ If U does not contain x nor a y parallel to x then:

$$\mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x\big(\sum_{i_p\in U}(-1)^p\chi(U\setminus i_p)e_{U\setminus i_p}\big) = 0. \qquad \blacksquare$$

Theorem 4.4 *For every element x of a simple $\mathcal{M}([n])$, there is a splitting short exact sequence of vector spaces*

$$0\to\mathbb{A}(\mathcal{M}\setminus x)\xrightarrow{\mathfrak{i}_x}\mathbb{A}(\mathcal{M})\xrightarrow{\mathfrak{p}_x}\mathbb{A}(\mathcal{M}/x)\to 0. \tag{4.4}$$

Proof. From the definitions we know that the composite map $\mathfrak{p}_x\circ\mathfrak{i}_x$, is the null map so $\mathrm{Im}(\mathfrak{i}_x)\subseteq\mathrm{Ker}(\mathfrak{p}_x)$. We will prove the equality $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. By a reordering of the elements of $[n]$ we can suppose that $x = n$. The minimal broken circuits of $\mathcal{M}/n$ are the minimal sets X such that either X or $X\cup\{n\}$ is a broken circuit of $\mathcal{M}$ (see the Proposition 3.2.e of [5]). Then

$$\mathrm{NBC}(\mathcal{M}/n) = \big\{X : X\subseteq[n-1] \ \text{ and } \ X\cup\{n\}\in\mathrm{NBC}(\mathcal{M})\big\} \quad \text{and}$$

$$\mathrm{NBC}(\mathcal{M}) = \mathrm{NBC}(\mathcal{M}\setminus n)\biguplus\big\{I\cup n : I\in\mathrm{NBC}(\mathcal{M}/n)\big\}. \tag{4.5}$$

So $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. There is a morphism of modules

$$\mathfrak{p}_n^{-1}:\mathbb{A}(\mathcal{M}/n)\to\mathbb{A}, \quad \text{where} \quad \mathfrak{p}_n^{-1}([I]_{\mathbb{A}(\mathcal{M}/n)}) := [I\,\cup\,n]_{\mathbb{A}},\ \forall I\in\mathrm{NBC}(\mathcal{M}/n).$$

It is clear that the composite map $\mathfrak{p}_n\circ\mathfrak{p}_n^{-1}$ is the identity map. From Equation (4.5) we conclude that the exact sequence (4.4) splits. ∎

Similarly to [17] (see also [4]), we now construct, making use of iterated contractions, the dual bases $\boldsymbol{nbc}^*_\ell = (b^*_i)$ of the bases $\boldsymbol{nbc}_\ell := (b_j)$ of the vector space $\mathbb{A}_\ell$. More precisely $\boldsymbol{nbc}^*_\ell$ is the basis of $\mathbb{A}^*_\ell$ the vector space of the linear forms such that $\langle b^*_i, b_j\rangle = \delta_{ij}$ (the Kronecker delta).

We associate to the ordered independent set $I^\sigma := (i_{\sigma(1)}, \ldots, i_{\sigma(p)})$ of $\mathcal{M}$ the linear form on $\mathbb{A}_\ell$, $\mathfrak{p}_{I^\sigma} : \mathbb{A}_\ell \to \mathbb{K}$, defined as the composite of the maps $\mathfrak{p}_{e_{i_{\sigma(p)}}}$, $\mathfrak{p}_{e_{i_{\sigma(p-1)}}}, \ldots, \mathfrak{p}_{e_{i_{\sigma(1)}}}$, i.e.,

$$\mathfrak{p}_{I^\sigma} := \mathfrak{p}_{e_{i_{\sigma(1)}}} \circ \mathfrak{p}_{e_{i_{\sigma(2)}}} \circ \cdots \circ \mathfrak{p}_{e_{i_{\sigma(p)}}}. \tag{4.6}$$

We call $\mathfrak{p}_{I^\sigma}$ the *iterated residue* with respect to the ordered independent set I^σ. We remark that the map $\mathfrak{p}_{I^\sigma}$ depends on the order chosen on I^σ and not only on the underlying set I. We associate to I^σ the flag of flats of $\mathcal{M}$,

$$\mathbf{Flag}(I^\sigma) := c\ell\big(\{i_{\sigma(p)}\}\big) \subsetneq c\ell\big(\{i_{\sigma(p)}, i_{\sigma(p-1)}\}\big) \subsetneq \cdots \subsetneq c\ell\big(\{i_{\sigma(p)}, \ldots, i_{\sigma(1)}\}\big).$$

Proposition 4.5 *Let $J \in \mathrm{IND}_\ell(\mathcal{M})$ then we have $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ iff there is a unique permutation $\tau \in \mathfrak{S}_\ell$ such that $\boldsymbol{Flag}(J^\tau) = \boldsymbol{Flag}(I^\sigma)$. And in this case we have $\mathfrak{p}_{I^\sigma}(e_J) = \chi(I^\sigma)/\chi(J^\tau)$. In particular we have $\mathfrak{p}_{I^\sigma}(e_I) = 1$ for any independent set I and any permutation σ.*

Proof. The first equivalence is easy to prove in both direction. To obtain the expression of $\mathfrak{p}_{I^\sigma}(e_J)$ we just need to iterate ℓ times the residue. This gives:

$$\mathfrak{p}_{I^\sigma}(e_J) = \frac{\chi(J \setminus j_{\tau(\ell)} \circ i_{\sigma(\ell)})}{\chi(J \setminus j_{\tau(\ell)} \circ j_{\tau(\ell)})} \times \frac{\chi(J \setminus \{j_{\tau(\ell)}, j_{\tau(\ell-1)}\} \circ i_{\sigma(\ell-1)} \circ i_{\sigma(\ell)})}{\chi(J \setminus \{j_{\tau(\ell)}, j_{\tau(\ell-1)}\} \circ j_{\tau(\ell-1)} \circ i_{\sigma(\ell)})} \times \cdots$$
$$\cdots \times \frac{\chi(I^\sigma)}{\chi(j_{\tau(1)} \circ I^\sigma \setminus i_{\sigma(1)})}.$$

After simplification we obtain the announced formula. The last result is clear. ∎

Remark 4.6 The fact that $\mathfrak{p}_{I^\sigma}(e_J)$ is null depends on the permutation σ. For example, for any simple matroid of rank 2 we have $\mathfrak{p}_{13}(e_{12}) = 0$ and $\mathfrak{p}_{31}(e_{12}) \neq 0$. But if $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ then its value does not depend on σ. We mean by this that if there are two permutations σ and σ' such that $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ and $\mathfrak{p}_{I^{\sigma'}}(e_J) \neq 0$ then $\mathfrak{p}_{I^\sigma}(e_J) = \mathfrak{p}_{I^{\sigma'}}(e_J)$.

Definition 4.7 ([17]) We say that the subset $\mathbb{I}_\ell \subseteq \big\{[I]_{\mathbb{A}} : I \in \mathrm{IND}_\ell(\mathcal{M})\big\}$ is a *diagonal basis* of $\mathbb{A}_\ell$ if and only if the following three conditions hold:

(4.7.1) For every $[I]_{\mathbb{A}} \in \mathbb{I}_\ell$ there is a fixed permutation of the set I denoted $\sigma_I \in \mathfrak{S}_\ell$;

(4.7.2) $\big|\mathbb{I}_\ell\big| \geq \dim(\mathbb{A}_\ell)$;

(4.7.3) For every $[I]_{\mathbb{A}}, [J]_{\mathbb{A}} \in \mathbb{I}_\ell$ and every permutation $\tau \in \mathfrak{S}_\ell$, the equality $\mathbf{Flag}\,(J^\tau) = \mathbf{Flag}\,(I^{\sigma_I})$ implies $J = I$.

Theorem 4.8 *Suppose that $\mathbb{I}_\ell$ is a diagonal basis of $\mathbb{A}_\ell$. Then $\mathbb{I}_\ell$ is a basis of $\mathbb{A}_\ell$ and $\mathbb{I}_\ell^* := \{\mathfrak{p}_{I^{\sigma_I}} : [I]_{\mathbb{A}} \in \mathbb{I}_\ell\}$ is the dual basis of $\mathbb{I}_\ell$.*

Proof. Pick two elements $[I]_{\mathbb{A}}, [J]_{\mathbb{A}} \in \mathbb{I}_\ell$. Note that $\mathfrak{p}_{I^{\sigma_I}}(e_J) = \delta_{IJ}$ (the Kronecker delta), from Condition (4.7.2) and Proposition 4.5. The elements of $\mathbb{I}_\ell$ are linearly independent: suppose that $[J] = \sum \zeta_j [I_j]$, $\zeta_j \in \mathbb{K} \setminus 0$; then $1 = \mathfrak{p}_{J^{\sigma_J}}([J]) = \mathfrak{p}_{J^{\sigma_J}}\left(\sum \zeta_j [I_j]\right) = 0$, a contradiction. It is clear also that $\mathbb{I}_\ell^*$ is the dual basis of $\mathbb{I}_\ell$. ∎

The following result gives an interesting explanation of results of [6, 7].

Corollary 4.9 *$\boldsymbol{nbc}_\ell(\mathcal{M})$ is a diagonal basis of $\mathbb{A}_\ell$ where σ_I is the identity for every $[I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$. For a given $[J]_{\mathbb{A}} \in \mathbb{A}_\ell$, suppose that*

$$[J]_{\mathbb{A}} = \sum \xi(I,J)[I]_{\mathbb{A}}, \text{ where } [I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M}) \text{ and } \xi(I,J) \in \mathbb{K}. \tag{4.9.2}$$

Then are equivalent:

- $\xi(I,J) \neq 0$,
- $\boldsymbol{Flag}(I) = \boldsymbol{Flag}(J^\tau)$ *for some permutation* τ.

If $\xi(I,J) \neq 0$ we have $\xi(I,J) = \frac{\chi(I)}{\chi(J^\tau)}$. In particular if $\mathbb{A}$ is the Orlik-Solomon algebra then $\xi(I,J) = sgn\ (\tau)$.

Proof. By hypothesis (4.7.1) and (4.7.2) are true. We claim that $\boldsymbol{nbc}_\ell(\mathcal{M})$ verifies (4.7.3). Suppose for a contradiction that $J \neq I$, $[J]_{\mathbb{A}}, [I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$ and there is $\tau \in \mathfrak{S}_\ell$, such that $\mathbf{Flag}(J^\tau) = \mathbf{Flag}(I)$. Set $I = (i_1, \ldots, i_\ell)$ and $J = (j_{\tau(1)}, \ldots, j_{\tau(\ell)})$, and suppose that $j_{\tau(m+1)} = i_{m+1}, \ldots, j_{\tau(\ell)} = i_\ell$ and $i_m \neq j_{\tau(m)}$. Then there is a circuit C of $\mathcal{M}$ such that

$$i_m, j_{\tau(m)} \in C \subseteq \{i_m, j_{\tau(m)}, i_{m+1}, i_{m+2}, \ldots, i_\ell\}.$$

If $j_{\tau(m)} < i_m$ [resp. $i_m < j_{\tau(m)}$] we conclude that

$$I \notin \mathrm{NBC}_\ell(\mathcal{M}) \quad [\text{resp.} \quad J \notin \mathrm{NBC}_\ell(\mathcal{M})]$$

a contradiction. So $\boldsymbol{nbc}_\ell(\mathcal{M})$ is a diagonal basis of $\mathbb{A}_\ell$.

From Theorem 4.8 we conclude that $\boldsymbol{nbc}_\ell^* := \left\{\mathfrak{p}_I : [I]_{\mathbb{A}} \in \boldsymbol{nbc}\right\}$ is the dual basis of $\boldsymbol{nbc}$. Suppose now that $[J]_{\mathbb{A}} = \sum \xi_I [I]_{\mathbb{A}}$, where $[I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$ and $\xi_I \in k$. Then $\xi_I = \mathfrak{p}_I(e_J)$ and the remaining follows from Proposition 4.5. ∎

Making full use of the matroidal notion of iterated residue, see Equation (4.6), we are able to prove the following result very close to Proposition 2.1 of [18].

Proposition 4.10 *Consider the set of vectors $\mathcal{V} := \{v_1, \ldots, v_k\}$ in the plane $x_d = 1$ of $\mathbb{K}^d$. Set $\mathcal{A}_{\mathbb{K}} := \{H_i : H_i = \mathrm{Ker}(v_i) \subseteq (\mathbb{K}^d)^*, i = 1, \ldots, k\}$ and let $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$ be its Orlik-Solomon-Terao corresponding algebra. Fix a diagonal basis $\mathbb{I}_\ell \subseteq \{[I]_{\mathbb{A}} : I \in \mathrm{IND}_\ell(\mathcal{M})\}$ of $\mathbb{A}_\ell$ and let $\mathbb{I}_\ell^* = \{\mathfrak{p}_{I^{\sigma_I}} : [I]_{\mathbb{A}} \in \mathbb{I}_\ell\}$ be the corresponding dual basis. Then, for any $e_J \in \mathbb{A}_\ell \setminus 0$, we have*

$$\sum_{I \in \mathbb{I}_\ell} \mathfrak{p}_{I^{\sigma_I}}(e_J) = \sum_{I \in \mathbb{I}_\ell} \left\langle \mathfrak{p}_{I^{\sigma_I}}, e_J \right\rangle = 1.$$

Proof. We have for any $\ell+1$-subset of $\mathcal{V}$, $\sum_{p=1}^{p=\ell+1}(-1)^p\chi(U\setminus i_p)=0$. (This is the development of a determinant with two lines of 1.) For any rank ℓ unidependent $U=\{i_1,\dots,i_{\ell+1}\}$ of the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, we have

$$\partial e_U=\sum_{p=1}^{p=\ell+1}(-1)^p\chi(U\setminus i_p)e_{U\setminus i_p}.$$

Since the sum of the coefficients in these relations is 0 and that these relations are generating, see Remark 2.3, we can deduce that the sum of the coefficients in any relation in $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$ is also equal to 0 which concludes the proof. ∎

5 Examples

In this section we will show on a small example the different results of the three previous sections.

Consider the the set of 6 points $\{p_1,\dots,p_6\}$ in the affine plane $z=1$ of three dimensional real vector space $\mathbb{R}^3$, whose coordinates are indicated in Figure 1. Set $v_i:=\overrightarrow{(0,p_i)}$, $i=1,\dots,6$. And let $\mathcal{A}$ be the corresponding hyperplane arrangement of $(\mathbb{R}^3)^*$, $\mathcal{A}:=\{H_i=\mathrm{Ker}(v_i),\ i=1,\dots,6\}$. Let $\mathcal{M}(\mathcal{A})$ [resp. $\boldsymbol{\mathcal{M}}(\mathcal{A})$] be the corresponding rank three [resp. oriented] matroid. So like in Example 2.4, the arrangement $\mathcal{A}$ defines the three classical Orlik-Solomon type algebras: the original Orlik-Solomon algebra $\mathrm{OS}(\mathcal{M}(\mathcal{A}))$ through $\mathcal{M}(\mathcal{A})$, the Orlik-Solomon-Terao algebra $\mathrm{OT}(\mathcal{A})$ directly from the v_i and the Cordovil algebra $\mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A}))$ from $\boldsymbol{\mathcal{M}}(\mathcal{A})$.

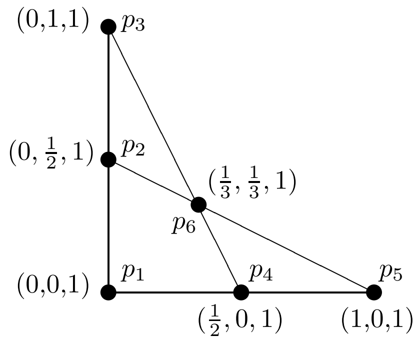

Figure 1: The rank 3 matroid on the set $\{p_1,\dots,p_6\}$.

Let $\mathbb{A}_\chi$ be a χ-algebra on $\mathcal{M}(\mathcal{A})$. We know that

$$\boldsymbol{nbc}_3=\{e_{124},e_{125},e_{126},e_{134},e_{135},e_{136}\}$$

together with $\sigma_{124}=\sigma_{125}=\sigma_{134}=\sigma_{135}=\sigma_{136}=\sigma_{156}=\mathrm{id}$ is a diagonal basis of $\mathbb{A}_3$, from Corollary 4.9. Directly from the Definition 4.7 we see that $\mathfrak{B}_3=\{e_{124},e_{125},e_{134},e_{135},e_{136},e_{156}\}$ with $\sigma_{124}=\sigma_{134}=\sigma_{135}=\sigma_{136}=\sigma_{156}=\mathrm{id}$ and

$\sigma_{125} = (132)$ is also a diagonal basis of $\mathbb{A}_3$. We will look at expressions on the basis $\boldsymbol{nbc}_3$ (resp. $\mathfrak{B}_3$) of the vector space $\mathbb{A}_3$, of some elements of the type e_B, B basis of $\mathcal{M}(\mathcal{A})$, for the three χ-algebras of Example 2.4. Especially, we will verify as stated in Remark 4.6 that $\mathfrak{p}_{125^{\mathrm{id}}}(e_{235}) = \mathfrak{p}_{125^{(132)}}(e_{235})$. Let also point out that for the Orlik-Solomon-Terao algebra, we will have have $\sum_{I\in\mathfrak{B}} \mathfrak{p}_{I^\sigma}(e_J) = 1$ as proved in Proposition 4.10. Finally recall that $\mathbb{T}$ is set of the monomials of $\mathfrak{A}$ and set $\mathbb{T}_\ell := \{e_X \in \mathbb{T} : |X| = \ell\}$.

(a) Let us first take the Orlik-Solomon algebra $\mathrm{OS}(\mathcal{M}(\mathcal{A}))$:

From Remark 2.3 , the basis of $\mathrm{OS}(\mathcal{M}(\mathcal{A}))$ is simply the $\boldsymbol{nbc}$-bases:

$$\boldsymbol{nbc}(\mathcal{M}) = \mathbb{T}_0 \cup \mathbb{T}_1 \cup \boldsymbol{nbc}_2 \cup \boldsymbol{nbc}_3,$$

with $\boldsymbol{nbc}_2 = \{e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{24}, e_{25}, e_{26}, e_{34}, e_{35}, e_{36}\}$, and

$$\boldsymbol{nbc}_3 = \{e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}\}.$$

The basis of $\Im_\chi(\mathcal{M}(\mathcal{A}))$ is the union of the dependents and of the boundaries of the inactive unidependents:

$$\partial\boldsymbol{uni}_3 \cup \boldsymbol{dep}_3 \cup \partial\boldsymbol{uni}_4 \cup \mathbb{T}_4 \cup \mathbb{T}_5 \cup \mathbb{T}_6$$

where $\partial\boldsymbol{uni}_3 = \{\partial e_{123}, \partial e_{145}, \partial e_{256}, \partial e_{346}\}$, $\boldsymbol{dep}_3 = \{e_{123}, e_{145}, e_{256}, e_{346}\}$ and $\partial\boldsymbol{uni}_4$ is the set

$$\{\partial e_{1234}, \partial e_{1235}, \partial e_{1236}, \partial e_{1245}, \partial e_{1246}, \partial e_{1256}, \partial e_{1345}, \partial e_{1346}, \partial e_{1356}, \partial e_{1456}\}.$$

Note that we have

$$|\boldsymbol{nbc}_2| + |\partial\boldsymbol{uni}_3| = 11 + 4 = 15 = \dim(\mathfrak{A}_2)$$

and

$$|\boldsymbol{nbc}_3| + |\partial\boldsymbol{uni}_4| + |\boldsymbol{dep}_3| = 6 + 10 + 4 = 20 = \dim(\mathfrak{A}_3).$$

Take first on $[n]$ the natural order. We have then for the leading term ideal

$$\mathrm{Lt}_<(\mathcal{G}) = \langle e_{BC} : BC \text{ broken circuit}\rangle.$$

We obtain explicitly:

$$\mathrm{Lt}_<(\mathcal{G}) = \langle e_{23}, e_{45}, e_{56}, e_{46}, e_{246}, e_{345}, e_{356}\rangle.$$

Always for the natural order, from Theorem 3.8, we obtain for the reduced Gröbner basis:

$$\mathcal{G}_r = \big\{\partial e_{123}, \partial e_{145}, \partial e_{256}, \partial e_{346}\big\}.$$

If we take now the term order $\prec_\pi$ on $\mathbb{T}(\mathfrak{A})$, defined by the permutation $\pi := (234561)$, we get now:

$$\mathrm{Lt}_\prec(\mathcal{G}) = \langle e_{13}, e_{15}, e_{56}, e_{46}, e_{146}, e_{345}, e_{165}\rangle,$$

and then for the corresponding reduced Gröbner basis:

$$\mathcal{G}_r = \{\partial e_{123}, \partial e_{145}, \partial e_{256}, \partial e_{346}, \partial e_{2345}\}.$$

Finally from Corollary 3.9, we get the minimal universal Gröbner basis

$$\mathcal{G}_u = \{\partial e_C : C \in \mathfrak{C}(\mathcal{M})\}.$$

We obtain explicitly:

$$\mathcal{G}_u = \{\partial e_{123}, \partial e_{145}, \partial e_{256}, \partial e_{346}, \partial e_{1246}, \partial e_{1356}, \partial e_{2345}\}.$$

Now we will use the results of Section 4 to express pure elements in different diagonal bases. Consider the diagonal basis $\boldsymbol{nbc}_3$ of the $\mathbb{K}$-vector space $\mathrm{OS}(\mathcal{M}(\mathcal{A}))_3$. So we have:

$$e_{156} = \mathrm{sgn}(165)e_{125} + \mathrm{sgn}(156)e_{126} = -e_{125} + e_{126}$$

and

$$e_{235} = \mathrm{sgn}\ (325)e_{125} + \mathrm{sgn}\ (235)e_{135} = -e_{125} + e_{135}.$$

For the diagonal basis $\mathfrak{B}_3$ of the $\mathbb{K}$-vector space $\mathrm{OS}(\mathcal{M}(\mathcal{A}))_3$, we have:

$$e_{126} = \mathrm{sgn}(162)\mathrm{sgn}(152)e_{125} + \mathrm{sgn}(126)e_{156} = e_{125} + e_{156}$$

and

$$e_{235} = \mathrm{sgn}(152)\mathrm{sgn}(352)e_{125} + \mathrm{sgn}(235)e_{135} = -e_{125} + e_{135}.$$

(*b*) Let us take the Orlik-Solomon-Terao algebra $\mathrm{OT}(\mathcal{A})$:

For the different bases and Gröbner bases we obtain formally the same results. There is in fact differences which are hidden by the operator ∂ (indeed ∂ is function of χ).

For the diagonal basis $\boldsymbol{nbc}_3$ of the $\mathbb{K}$-vector space $\mathrm{OT}(\mathcal{A})_3$ we have:

$$e_{156} = \frac{\det(125)}{\det(165)}e_{125} + \frac{\det(126)}{\det(156)}e_{126} = \frac{3}{2}e_{125} - \frac{1}{2}e_{126}$$

and

$$e_{235} = \mathrm{sgn}\ (325)e_{125} + \mathrm{sgn}\ (235)e_{135} = -e_{125} + e_{135}$$

For the diagonal basis $\mathfrak{B}_3$ of the $\mathbb{K}$-vector space $\mathrm{OT}(\mathcal{A})_3$ we have:

$$e_{126} = \frac{\det(152)}{\det(162)}e_{125} + \frac{\det(156)}{\det(126)}e_{156} = 3e_{125} - 2e_{156}.$$

and

$$e_{235} = \frac{\det(152)}{\det(352)}e_{125} + \frac{\det(135)}{\det(235)}e_{135} = -e_{125} + 2e_{135}.$$

(c) Let us take the Cordovil $\mathbb{Z}$-algebra $\mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A}))$:

For the diagonal basis $\boldsymbol{nbc}_3$ of the $\mathbb{K}$-vector space $\mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A}))_3$ we have:

$$e_{156} = \chi(125)\chi(165)e_{125} + \chi(126)\chi(156)e_{126} = e_{125} - e_{126}$$

and

$$e_{235} = \text{sgn }(325)e_{125} + \text{sgn }(235)e_{135} = -e_{125} + e_{135}.$$

For the diagonal basis $\mathfrak{B}_3$ of the $\mathbb{K}$-vector space $\mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A}))_3$ we have:

$$e_{126} = \chi(152)\chi(162)e_{125} + \chi(156)\chi(126)e_{156} = e_{125} - e_{156}$$

and

$$e_{235} = \text{sgn}(152)\text{sgn}(352)e_{125} + \text{sgn}(235)e_{135} = -e_{125} + e_{135}.$$

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