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Optimization of differential inclusions of Bolza type with state constraints and duality

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ABSTRACT

Sufficient conditions for optimization are obtained and duality theorems are also derived for the problems under consideration on the basis of the apparatus of locally conjugate mappings and the subdifferential calculus.

RESUMEN

Se obtienen condiciones suficientes para optimización y se derivan teoremas de dualidad para los problemas, bajo consideraciones en las bases de los aparatos de funciones conjugadas localmente y del cálculo subdiferencial.

Key words and phrases: *Multi-valued, conjugate mappings, subdifferential, sufficient conditions, conjugate function, duality*

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1 Introduction

The present article is devoted to an investigation of the problem of Bolza type differential inclusions with state constraints:

$$I(x(.),t_1)=\varphi(x(t_1),t_1)+\int_{t_0}^{t_1} g(x(t),t)dt \to inf, \tag{1}$$

$$\dot{x}(t)\in a(x(t),t), t\in[t_o,t_1], \tag{2}$$

$$x(t_0)=x_0, x(t_1)\in M, \tag{3}$$

$$x(t)\in F(t), t\in[t_0,t_1], \tag{4}$$

where a is a bounded non-autonomous convex multi-valued mapping [1], $a(.,t):\mathbb{R}^n\to 2^{\mathbb{R}^n}$, the target set $M\subset\mathbb{R}^n$ is a convex set of final states, g is a convex function, $g,\varphi:\mathbb{R}^{n+1}\to\mathbb{R}^1$ and $F:[t_0,t_1]\to 2^{\mathbb{R}^n}$ is a convex-valued function. The initial moment of the time t_0 is fixed, and the last moment t_1 is generally free. An admissible solution $x(t)$ of the differential inclusion (2) with boundary conditions (3) is an absolutely continuous function ($x(t)\in F(t)$ for all $t\in[t_0,t_1]$).

In our optimization problem we use the apparatus of locally conjugate mapping and we observe that relationship between locally conjugate mapping and conjugate function is useful for detailed investigations.

In Section **2.**, using locally conjugate mapping[1], we formulate sufficient conditions of optimality. In addition we show that conjugate variable has jumps,which are typical for control systems with state constraints and among sufficient conditions there appears a condition of jumps(see [3]), where the number of jump points may be countable.

In Section **3.**, we prove the theorem of duality for convex problems, and we show that conjugate differential inclusion play the role of extremal relation for a direct and dual problem. For construction of the dual problem, the convex continuous problem is interchanged with the discrete approximation problem and results from [8] are used.

Former investigations[1,13-15,16] have made an intensive development of the theory of extremal problems described by multivalued mappings with discrete time and with lumped parameters. Many problems in economic dynamics, as well as classical problems on optimal control, differential games, and so on, can be reduced to such investigations.

The papers [11-12] are a survey of optimality conditions for optimal control problems involving differential inclusions and so-called differential-difference inclusions. The papers[11,17-18,23] establish necessary conditions for optimal control problems with state constraints, formulated in terms of differential inclusions.

Definition 1.1: *1) $h(\bar{x},x)$ is called the upper convex approximation(UCA) of a function $g(x)$ at a point $x\in domg=\{x:|g(x)|<+\infty\}$ [1] if:*

i) $h(\bar{x},x)\geq F(\bar{x},x)=\sup\limits_{\tau(.)}\limsup\limits_{\lambda\downarrow 0}\frac{1}{\lambda}(g(x+\lambda\bar{x}+\tau(\lambda))-g(x))$

$\lambda^{-1}\tau(\lambda)\to 0, \quad \lambda\downarrow 0$ *for all* $\bar{x}\neq 0$.

ii) $h(\bar{x},x)$ is a convex closed (lower semicontinuous) positive homogeneous function of $\bar{x}$.

2) The set $\partial h(0,x) = \{x^* \in \mathbb{R}^n : h(\bar{x},x) \geq < \bar{x}, x^* >, \bar{x} \in \mathbb{R}^n\}$, is called a subdifferential of the function g at the point x and is denoted by $\partial g(x)$, there symbol $< .,. >$ denotes scalar product. It is known that when $g(x)$ is convex, the given definition coincides with the usual definition of the subdifferential.(see [1])

3) The mapping $a^*(y^*;z) = \{x^* : (-x^*, y^*) \in K_a^*(z)\}$ is called a locally conjugate mapping (LCM) to the convex mapping a at the point z.

Theorem 1.1: *Let $a : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be convex-valued closed bounded continuous mapping such that the function $W_a(x,y^*) = \inf\limits_{y\in a(x)} < y, y^* >$ is continuous differentiable on x. Let us suppose that the vector $\bar{z}_1 = (\bar{x}_1, \bar{y}_1)$ satisfies the inequality*

$$< \bar{x}_1, \frac{\partial W_a(x_0,y^*)}{\partial x} > - < \bar{y}_1, y^* > < 0.$$

Then the following statements are true for a point $z_0 = (x_0, y_0), y_0 \in a(x_0, y^*)$:

i)The cone

$$K_a(z_0) = \left\{ \bar{z} :< \bar{x}, \frac{\partial W_a(x_0,y^*)}{\partial x} > - < \bar{y}, y^* > < 0 \right\}$$

is the smooth local tent, which is the cone of tangent directions to gfa(graph of a) at the point z_0.

ii) LCM a^* corresponding to the cone $K_a(z_0)$ may be given by the formula

$$a^*(y^*; z_0) = \left\{ \frac{\partial W_a(x_0,y^*)}{\partial x} \right\}$$

Proof. If $S_a(x,y^*)$, $y^* \in \mathbb{R}^n$, is the support function to $a(x)$, then by the theory of convex analysis it is known that $y \in a(x)$ if and only if $< y, y^* > \leq S_a(x,y^*)$ for all $y^* \in \mathbb{R}^n$. Since $S_a(x,y^*) = -W_a(x,-y^*)$, the preceding inequality means that $< y, y^* > \geq W_a(x,y^*)$. Thus $a(x)$ is given by

$$a(x) = \{y : W_a(x,y^*) - < y, y^* > \leq 0\}, \quad y^* \in \mathbb{R}^n. \tag{5}$$

Suppose

$$f_{y^*}(z) = W_a(x,y^*) - < y, y^* >, \tag{6}$$

then by the Lemma 3.1[1,p.225], $f_{y^*}(z)$ is continuous on y^* and is continuous differentiable on z. By the Theorem 2.2[1,p.211], UCA(upper convex approximation) $h_{y^*}(\bar{z},z)$ of the function $f_{y^*}(z)$ is

$$h_{y^*}(\bar{z},z) = < \bar{z}, \frac{\partial W_a(x,y^*)}{\partial x} \times \{-y^*\} > . \tag{7}$$

Furthermore $f_{y^*}(z_0) = 0$ and $f_{y^*}(z)$ has an UCA $h_{y^*}(\bar{z}, z_0)$, which is continuous on $\bar{z}$ and by the condition on the vector $\bar{z}_1$, we have $h_{y^*}(\bar{z}_1, z_0) < 0$. Then applying Theorem 3.3[1,p.234] by (7) we see that i) of the theorem follows. Since in this case

$$-con\partial f_{y^*}(z_0) = con\left\{-\frac{\partial W_a(x_0, y^*)}{\partial x}, y^*\right\},$$

then by the same Theorem 3.3[1,p.234] the equality

$$a^*(y^*; z_0) = \left\{\frac{\partial W_a(x_0, y^*)}{\partial x}\right\}$$

holds. This, in turn, implies that ii) is correct. The Theorem is proved. ∎

Let $O^+(gfa)$ be the recession cone[2] to a convex function a in the space $Z = X \times Y$, i.e.

$$O^+(gfa) = \{\bar{z} : z + \lambda\bar{z} \in gfa, \lambda \geq 0, \forall z \in gfa\}. \tag{8}$$

For such convex function a, let us define

$$\Omega_a(x^*, y^*) = \inf\{- < x, x^* > + < y, y^* >: (x, y) \in gfa\}. \tag{9}$$

It is evident that

$$\Omega_a(x^*, y^*) = \inf_x\{- < x, x^* > + W_a(x, y^*)\}. \tag{10}$$

Definition 1.2: The function

$$a^*(y^*) = \{x^* : (-x^*, y^*) \in (O^+gfa)^*\}$$

is called conjugate function to a convex function a. It is clear that if mapping a is superlinear[5], i.e. gfa is a cone, then this definition coincides with the definition of B.H.Pshenichnyi [1].

Conjugate function can be used in different problems connected with duality theorems.

Definition 1.3: Multivalued mapping a is called quasisuperlinear if its graph is in the form of

$$gfa = M + K,$$

where M is a convex compactum, K is a closed convex cone.

Lemma 1.1: For a convex mapping a we have

$$dom\Omega_a = \{(-x^*, y^*) : \Omega_a(x^*, y^*) > -\infty\} \subseteq (O^+gfa)^*.$$

If a is a quasisuperlinear mapping then

$$dom\Omega_a = K^*.$$

Proof. Let us assume the contrary: let $(-x_0^*, y_0^*) \in dom\Omega_a$, but $(-x_0^*, y_0^*) \notin (O^+gfa)^*$. It means that there exists a pair $(\bar{x}_0, \bar{y}_0) \in O^+gfa$, for which

$$- < x_0^*, \bar{x}_0 > + < y_0^*, \bar{y}_0 >< 0.$$

By the definition of O^+gfa, we have

$$(x, y) + \lambda(\bar{x_0}, \bar{y_0}) \in gfa, \quad (x, y) \in gfa, \quad \lambda > 0.$$

Then

$$- < x + \lambda\bar{x}_0, x_0^* > + < y + \lambda\bar{y}_0, y_0^* >= - < x_0^*, x > + < y_0^*, y > +$$

$$+\lambda\{- < \bar{x}_0, x_0^* > + < \bar{y}_0, y_0^* >\} \to -\infty \quad \text{for} \quad \lambda \to +\infty,$$

which contradicts the fact that $(-x_0^*, y_0^*) \in dom\Omega_a$. This proves the first statement of the lemma. Furthermore, when a is a quasisuperlinear mapping, applying Result 9.1.2[2] and Lemma3.6.1[1],we get

$$(O^+gfa)^* = [O^+(M + K)]^* = (O^+M)^* \cap (O^+K)^* = \mathbb{R}^n \cap K^* = K^*.$$

On the other hand

$$dom\Omega_a = dom(\Omega_M + \Omega_K) = dom\Omega_M \cap dom\Omega_K = dom\Omega_K = K^*.$$

Hence

$$dom\Omega_a = K^*.$$

Lemma is proved.

The following example shows that the inverse inclusion generally is not true. In fact, let $a : X \to 2^Y (X, Y$ one-dimensional axises) is given as:

$$a(x) = \{y : y \geq x^2\} \quad , \quad gfa = \{(x, y) : y \geq x^2\}.$$

Check that $O^+gfa = \{0\} \times Y^+$, where Y^+ is the positive y-axis. Therefore $(O^+gfa)^* = \{(-x^*, y^*) : x^* \in X, y^* \in Y^+\}$. Then it is clear that $(-x_0^*, y_0^*) \in (O^+gfa)^*$, $x_0^* = 1, y_0^* = 0$, but $(-x_0^*, y_0^*) \notin dom\Omega_a$.

Lemma 1.2: Let a be a quasisuperlinear mapping and $W_a(., y^*)$ be proper closed function. Then the relation

$$\sup_{x^* \in a^*(y^*)} \{< x, x^* > +\Omega_M(x^*, y^*)\} = \inf_{y \in a(x)} < y, y^* >$$

holds.

Proof. From Lemma 1.1, we have

$$dom\Omega_a = (O^+gfa)^* = K^*.$$

Therefore with regard to Theorem 4.1.III[1] we find the relation

$$\sup_{x^*}\{\Omega_a(x^*,y^*)+ < x,x^* >\} =$$

$$\sup_{x^*}\{< x,x^* > +\Omega_M(x^*,y^*) : x^* \in a^*(y^*)\} = W_a(x,y^*).$$

Remark 1.2.1: If $M = \{0\}$, then $\Omega_M = 0$ and so the result of the above lemma coincides with the result of the Theorem 4.5.III[1,p.129].

Lemma 1.3: Let a be a convex mapping. Then the point x_0 is a solution of the problem

$$\inf_{x}\{- < x,x^* > +W_a(x,y^*)\}, \quad x^*,y^* \in \mathbb{R}^n$$

if and only if

$$x^* \in a^*(y^*,z_0), \quad y_0 \in a(x_0,y^*).$$

Proof. By the Theorem 2.1.IV[1], x_0 is a minimum point of the convex function

$$- < x,x^* > +W_a(x,y^*)$$

if and only if

$$0 \in \partial_x[- < x_0,x^* > +W_a(x_0,y^*)],$$

i.e.

$$x^* \in \partial_x W_a(x_0,y^*).$$

And, therefore by the definition of Ω_a it is evident that $y_0 \in a(x_0,y^*)$. Then by the Theorem 2.1.III[1], we find the required result.

Theorem 1.2: Let a be a convex-valued closed bounded continuous mapping, satisfying the Lipschitz condition, and let the function $W_{a_z}(\bar{x},y^*)$ be closed, where

$$a_z(\bar{x}) = \{\bar{y} : (\bar{x},\bar{y}) \in K_a(z)\}.$$

Then for arbitrary $y \in a(x,y^*), z = (x,y) \in gfa$, the function $W_{a_z}(.,y^*)$ is an UCA for $W_a(.,y^*)$ and, besides,

$$a^*(y^*;z) = \partial_x W_a(x,y^*).$$

Proof. If $\bar{z} = (\bar{x},\bar{y}) \in K_a(z), z = (x,y), y \in a(x)$, then by the definition of the cone of tangent directions, there is a function $\tau(\lambda)$, $\lambda^{-1}\tau(\lambda) \to 0$, $\lambda \downarrow 0$ $(\tau(\lambda) \in Z = X \times Y)$ such that $z + \lambda\bar{z} + \tau(\lambda) \in gfa$ for a sufficiently small $\lambda \geq 0$. That means

$$y + \lambda\bar{y} + \tau_y(\lambda) \in a(x + \lambda\bar{x} + \tau_x(\lambda)), \tau = (\tau_x,\tau_y), \tau_x(\lambda) \in X, \tau_y(\lambda) \in Y.$$

Since a satisfies the Lipschitz condition, $W_a(x, y^*)$ also satisfies the same condition by Lemma 3.2.V[1,p.226]. For such functions we have

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x}, y^*) - W_a(x, y^*)).$$

It is easily shown that

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x} + \tau_x(\lambda), y^*) - W_a(x, y^*))$$

holds independently from the choice of $\tau(\lambda)$. From the definition of $W_a(x, y^*)$ and from the condition $y \in a(x, y^*)$ it follows that

$$\frac{1}{\lambda}(W_a(x + \lambda\bar{x} + \tau_x(\lambda), y^*) - W_a(x, y^*)) \leq \frac{1}{\lambda}(< y + \lambda\bar{y} + \tau_y(\lambda), y^* > - < y, y^* >) =$$

$$< \bar{y}, y^* > + < \frac{\tau_y(\lambda)}{\lambda}, y^* > .$$

Then we have

$$F(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}(W_a(x + \lambda\bar{x} + \tau_x(\lambda), y^*) - W_a(x, y^*))$$

$$\leq limsup_{\lambda \downarrow 0}[< \bar{y}, y^* > + < \lambda^{-1}\tau_y(\lambda), y^* >] =< \bar{y}, y^* > .$$

It means that

$$F(\bar{x}, x) \leq \inf_{\bar{y}}\{< \bar{y}, y^* >: \quad \bar{y} \in a_z(\bar{x})\}.$$

In addition, given $\bar{x} \notin doma_z$ let us put $W_{a_z}(\bar{x}, y^*) = +\infty$. Then, by applying Lemma 1.2 to a_z, we get

$$W_{a_z}(\bar{x}, y^*) = \sup_{x^*}\{< \bar{x}, x^* >: \quad x^* \in a_z^*(y^*)\}.$$

But on the other hand, by the definition, $a^*(y^*; z) = a_z^*(y^*)$. Hence

$$F(\bar{x}, x) \leq W_{a_z}(\bar{x}, y^*) = \sup_{x^*}\{< \bar{x}, x^* >: \quad x^* \in a^*(y^*; z)\},$$

where $W_{a_z}(\bar{x}, y^*)$ is positive homogenous convex closed function of $\bar{x}$, i.e. $W_{a_z}(\bar{x}, y^*)$ is an UCA function of $W_a(., y^*)$ at the point x. Now to conclude the proof, it remains only to apply Theorem 3.2.II[1], thus we find

$$\partial W_a(x, y^*) = \partial h(0, x) = a^*(y^*; z).$$

Let us investigate the relation between conjugate function and LCM(Locally Conjugate Mapping). We need the following two theorems.

Let $K_M(z)$ be the cone of tangent directions to a convex set $M \subseteq Z = X \times Y$ at a point $z \in M$, i.e.

$$K_M(z) = con(M - z) = \{\bar{z} : \bar{z} = \lambda(z_1 - z), \lambda > 0, z_1 \in M\}. \quad (11)$$

Theorem 1.3: Let O^+M be the recession cone of a convex closed set $M \subset Z$. Then we have

$$\bigcap_{z \in M} K_M(z) = O^+M.$$

Proof. Let us show that

$$M = \bigcap_{z \in M} (z + K_M(z)). \quad (12)$$

In fact, let $z_0 \in M$ be an arbitrary fixed point. It is evident that all vectors as $\bar{z} = z_0 - z$ (in definition (11) they corresponds to $\lambda = 1$) belong to the cone $K_M(z)$, i.e. $z_0 \in z + K_M(z), z \in M$, then $z_0 \in \bigcap_{z \in M} (z + K_M(z))$. Conversely, if we have the last inclusion then $z_0 \in z + K_M(z)$ or there are such $z_1 \in M$ and a number $\gamma > 0$, that $z_0 - z = \gamma(z_1 - z) \in K_M(z)$. Hence $z_0 = \gamma z_1 + (1 - \gamma)z \in M$. Formula (12) follows.

On the other hand, we easily show that

$$O^+[\bigcap_{z \in M} (z + K_M(z))] = \bigcap_{z \in M} [O^+(z + K_M(z))].$$

In fact if z is an arbitrary point of closed convex set $M = \bigcap_{z \in M} (z + K_M(z))$ then by the definition of the recession cone, it is evident that, directed ray $z + \lambda\bar{z}$, $\forall\lambda \geq 0$, is contained in any cone $z + K_M(z), z \in M$. But it means that

$$\bar{z} \in \bigcap_{z \in M} [O^+(z + K_M(z))].$$

Therefore

$$O^+M = O^+[\bigcap_{z \in M} (z + K_M(z))] = \bigcap_{z \in M} [O^+(z + K_M(z))] = \bigcap_{z \in M} K_M(z).$$

Theorem is proved.

Remark 1.3.1: In the statement of the above Theorem, the closedness of M is essential.

Proof. Actually, let $M = \{(x, y) : x > 0, y > 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$. Clearly, $O^+M = M$. The set M contains points $(x_0, y_0) + \lambda(0, y_0)$, where $x_0 > 0$, $y_0 > 0$ are fixed. But $(0, y_0) \notin O^+M$.

Theorem 1.4: Let M be a closed convex set and let $K_M^*(z)$ be the conjugate cone to the cone of tangent directions $K_M(z), z \in M$. Then

$$\overline{\bigcup_{z \in M} K_M^*(z)} = (O^+M)^*,$$

where the bar denotes closure.
Proof. It is sufficient to show that

$$\overline{\bigcup_{z\in M} K_M^*(z)} = (\bigcap_{z\in M} K_M(z))^*. \tag{13}$$

Get any fixed point $z_0^* \in \overline{\bigcup_{z_0\in M} K_M^*(z)}$. Then there exists a sequence $z_n^* \to z_0^*$, $z_n^* \in \bigcup_{z\in M} K_M^*(z)$. Let us define sequence $\{z_n\}$ by the relation $z_n^* \in K_M^*(z_n)$. Note that $z_n^* \in \bigcup_{z\in M} K_M^*(z)$ implies the existence of $z_n \in M$ such that $z_n^* \in K_M^*(z_n)$.

On the other hand, since $K_M(z_n) \supseteq \bigcap_{z\in M} K_M(z)$ it is evident that $K_M^*(z_n) \subseteq (\bigcap_{z\in M} K_M(z))^*$. So that $z_n^* \in (\bigcap_{z\in M} K_M(z))^*$, and therefore $z_0^* \in (\bigcap_{z\in M} K_M(z))^*$.

Let us prove the converse inclusion in (13). Let us $z_1^* \in (\bigcap_{z\in M} K_M(z))^*$ be arbitrary fixed point and let us assume the contrary i.e. let $z_1^* \notin \overline{\bigcup_{z\in M} K_M^*(z)}$. Then $z_1^* \notin K_M^*(z)$ for any $z \in M$. In other words, there exists a vector $\bar{z}_1 (\bar{z}_1 \neq 0)$ such that

$$< z_1^*, \bar{z}_1 >< 0, \quad \bar{z}_1 \in K_M(z), \quad \forall z \in M$$

or

$$< z_1^*, \bar{z}_1 >< 0, \quad \bar{z}_1 \in \bigcap_{z\in M} K_M(z), \quad \text{i.e.} \quad z_1^* \notin (\bigcap_{z\in M} K_M(z))^*.$$

This contradiction shows that

$$(\bigcap_{z\in M} K_M(z))^* \subseteq \bigcup_{z\in M} K_M^*(z).$$

The proof of the theorem is over now.
Theorem 1.5: Let a be a closed convex mapping. Then the conjugate function $a^*(y^*)$ and the LCM of a implies the following relation

$$a^*(y^*) = \overline{\bigcup_{z\in gfa} a^*(y^*; z)}, \quad y \in a(x, y^*).$$

Proof. Setting $M = gfa$ as in the previous theorem, we obtain

$$a^*(y^*) = \overline{\bigcup_{z\in gfa} a^*(y^*; z)}.$$

By the Theorem 2.1.III[1] $z = (x, y)$, $y \notin a(x, y^*)$, implies $a^*(y^*; z) = \emptyset$.

2 Sufficient conditions of the optimization.

According to [1], the LCM (Locally Conjugate Mapping) a^* of the multi-valued mapping a at a point $z = (x, y) \in gfa(., t), t \in [t_0, t_1]$, is defined as follows:

$$a^*(y^*, (x, y), t) = \{x^* : (-x^*, y^*) \in K_a^*(z, t)\}, y^* \in \mathbb{R}^n,$$

where $K_a^*(z, t)$ is the conjugate cone to the cone of tangent directions $K_a(z, t)$.
Let us define

$$W_a(x, y^*, t) = \begin{cases} \inf\{< y, y^* >: y \in a(x, t)\}, & a(x, t) \neq \emptyset \\ +\infty & a(x, t) = \emptyset, \end{cases}$$

$a(x, y^*, t) = \{y \in a(x, t) :< y, y^* >= W_a(x, y^*, t)\}$ and
$W_M(x^*) = \inf_{y \in M} < x^*, y >$.

Note that for a convex mapping a, the LCM coincides with the subdifferential[1] $\partial_x W_a(\tilde{x}, y^*, t)$ of the function $W_a(., y^*, t)$ at the point $\tilde{x}$.It is known that

$$a^*(y^*, (\tilde{x}, \tilde{y}), t) = \begin{cases} \partial_x W_a(\tilde{x}, y^*, t), & \tilde{y} \in a(\tilde{x}, y^*, t) \\ \emptyset, & \tilde{y} \notin a(\tilde{x}, y^*, t). \end{cases}$$

Let $\tilde{x}(t), t \in [t_0, t_1], \tilde{x}(t_0) = x_0$, be any admissible solution of the problem (1)-(4). Let us construct the conjugate differential inclusion of the conjugate variable $x^*(t)$ by

a) $-\dot{x}^*(t) \in a^*(x^*(t); (\tilde{x}(t), \dot{\tilde{x}}(t)), t) + \partial g(\tilde{x}, t), t \in [t_0, t_1], \quad a.e;$
$\dot{\tilde{x}}(t) \in a(\tilde{x}(t), x^*(t), t), t \in [t_0, t_1] \quad a.e;$
which should be fulfilled for all $x \in F(t)$. The solution $x^*(t), t \in [t_0, t_1]$, satisfies the conjugate differential inclusion a) almost everywhere and is in the form of the sum of absolutely continuous functions and jump functions. Let us denote points of jumps and values of jumps $x^*(t)$ by
$\tau_i (i = 1, 2, \ldots), t_0 < \tau_i < t_1,$
$x_i^* = x^*(\tau_i + 0) - x^*(\tau_i - 0) \quad (i = 1, 2, \ldots),$
respectively.

If the following condition

$$- < x^*(t), \tilde{x}(t) >< W_{M \cap F(t)}(-x^*(t)), \qquad t_0 \leq t < t_1$$

holds, the admissible trajectory $\tilde{x}(t)$ would be called strictly transversal on the set M. Note that this definition guarantees that point $\tilde{x}(t) \notin M$ for every $t \in [t_0, t_1)$.

If the inequality

$$I(x(.), \theta) < I(x(.), \theta')$$

holds for any $\theta, \theta' \in [t_0, t_1]$ with $\theta < \theta'$ and for any admissible trajectory of the differential inclusion (2) with initial condition $x(t_0) = x_0$, then the function $I(x(.), t)$ is called monotone increasing with respect to argument t.

Theorem 2.1: Let $\tilde{x}(t), t \in [t_0, t_1]$, be any admissible trajectory of the problem

(1)-(4) and let there exists absolutely continuous function $x^*(t)$ which satisfies the inclusion a). Furthermore assume that $I(x(.),t)$ is monotone increasing with respect to argument t for any admissible trajectory $x(t), t \in [t_0, t_1]$, of the differential inclusion (2) and the following conditions are satisfied:

1) $x^*(t_1) \in \partial\varphi(\tilde{x}(t_1), t_1), x^*(t_1) \in K^*_M(\tilde{x}(t_1))$;

2) the jumps x^*_i satisfy $<\tilde{x}(\tau_i), x^*_i> = W_{F(\tau_i)}(x^*_i)$;

3) $\tilde{x}(t)$ is strictly transversal on M.

Then trajectory $\tilde{x}(t)$ is optimal.

Proof. Let $x(t) \in F(t)$ be an arbitrary admissible trajectory, realising the transition from the interval $[t_0, \theta]$ to the set M. Let us show that

$$I(x(.), \theta) \geq I(\tilde{x}(.), t_1).$$

Using $\partial_x W_a(., x^*(t), t)$ as the representation of LCM and by the Moreau-Rockafellar Theorem[4] we can rewrite the inclusion a) as follows:

$$-\dot{x}^*(t) \in \partial_x[W_a(\tilde{x}(t), x^*(t), t) + g(\tilde{x}(t), t)],$$

i.e.

$$W_a(x(t), x^*(t), t) - W_a(\tilde{x}(t), x^*(t), t) + g(x(t), t) - g(\tilde{x}(t), t) \geq \\ < -\dot{x}^*(t), x(t) - \tilde{x}(t) >, \quad t \in [t_0, t_1], \tag{14}$$

$W_a(\tilde{x}(t), x^*(t), t) = <\dot{\tilde{x}}(t), x^*(t)>$.

Since $W_a(x(t), x^*(t), t) \leq <\dot{x}(t), x^*(t)>$, from (14) we have

$$d\psi(t)/dt \geq g(\tilde{x}(t), t) - g(x(t), t) \tag{15}$$

for almost every $t \in [t_0, t_1]$, where $\psi(t) = < x(t) - \tilde{x}(t), x^*(t) >$.

Then integrating (15) we find

$$\int_{t_0}^{t_1} \dot{\psi}(t)dt = < x(t_1) - \tilde{x}(t_1), x^*(t_1) > \geq \int_{t_0}^{t_1} [g(\tilde{x}(t), t) - g(x(t), t)]dt. \tag{16}$$

$x(t), \tilde{x}(t)$ are absolutely continuous, therefore $\psi(t)$ can be represented by the sum of absolutely continuous functions and jump functions (see [9]).

$$\psi(\theta) = \psi(t_0) + \int_{t_0}^{\theta} \dot{\psi}(t)dt + \sum_{i \in J(\theta)} [\psi(\tau_i + 0) - \psi(\tau_i - 0)], \\ J(t) = \{i : \tau_i \in [t_0, t]\}. \tag{17}$$

Let us compute the values of the jumps of the function $\psi(t)$ at points $\tau_i (i = 1, 2, \ldots)$. Using the condition 2) of the theorem, we find

$$\psi(\tau_i + 0) - \psi(\tau_i - 0) = < x(\tau_i) - \tilde{x}(\tau_i), x^*_i > = < x(\tau_i), x^*_i > - W_{F(\tau_i)}(x^*_i).$$

Then by the relation $x(\tau_i) \in F(\tau_i)$, it is evident that

$$\psi(\tau_i + 0) - \psi(\tau_i - 0) \geq 0 \quad \forall \tau_i \in [t_0, \theta],$$

i.e.

$$\sum_{i \in J(\theta)} [\psi(\tau_i + 0) - \psi(\tau_i - 0)] \geq 0.$$

By the condition 1) of the theorem and definition of dual cone the inequality $< x(t_1) - \tilde{x}(t_1), x^*(t_1) > \geq 0$ holds. Since t_1 is free the last inequality is correct for any $t_1 = \theta$.

Obviously, the inequality (16) is correct for any $t_1 = \theta$. Therefore from (17), it is evident that $\psi(\theta) \geq \psi(t_0)$ i.e.

$$< x(\theta) - \tilde{x}(\theta), x^*(\theta) > \geq < x(t_0) - \tilde{x}(t_0), x^*(t_0) > = 0.$$

From the last inequality and condition 3) of the Theorem

$$- < x(\theta), x^*(\theta) > \quad \leq - < \tilde{x}(\theta), x^*(\theta) > \quad < W_{M \cap F(\theta)}(-x^*(\theta)). \tag{18}$$

Let $\triangle I = I(x(.), \theta) - I(\tilde{x}(.), t_1)$ be the increment of the target functional I, obtained by the transition from the trajectory $\tilde{x}(t)$ to the trajectory $x(t)$. Then

$$\triangle I = \varphi(x(\theta), \theta) + \int_{t_0}^{\theta} g(x(t), t)dt - \varphi(\tilde{x}(t_1), t_1) - \int_{t_0}^{t_1} g(\tilde{x}(t), t)dt$$

$$= \varphi(x(\theta), \theta) + \int_{t_0}^{\theta} g(x(t), t)dt - \varphi(x(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), t)dt + \varphi(x(t_1), t_1) +$$

$$+ \int_{t_0}^{t_1} g(x(t), t)dt - \varphi(\tilde{x}(t_1), t_1) - \int_{t_0}^{t_1} g(\tilde{x}(t), t)dt.$$

On the other hand from the inequality (16) and by condition 1) of the Theorem we obtain:

$$\int_{t_0}^{t_1} [g(x(t), t) - g(\tilde{x}(t), t)]dt + \varphi(x(t_1), t_1) - \varphi(\tilde{x}(t_1), t_1) \geq 0.$$

Since (16) is correct for any $t \in [t_0, t_1]$, the last relation implies

$$\triangle I \geq \varphi(x(\theta), \theta) + \int_{t_0}^{\theta} g(x(t), t)dt - \varphi(x(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), t)dt. \tag{19}$$

To prove the optimality of $\tilde{x}(t)$ let us assume the contrary, i.e. let for any admissible trajectory $x(t)$, $t \in [t_0, \theta], x(t_0) = x_0, x(\theta) \in M$, $\triangle I < 0$, i.e. $I(x(.), \theta) < I(\tilde{x}(.), t_1)$. Then by the inequality (19), we have $I(x(.), \theta) < I(x(.), t_1)$. Since $I(x(.), t)$ is monotone we conclude that

$$\theta < t_1. \tag{20}$$

Thus by the inequalities (18) and (20) we have $x(\theta) \notin M \cap F(\theta)$. Hence $x(\theta) \notin M$, i.e. the trajectory $x(t)$ cannot realize the transition from the interval $[t_0, \theta]$ to the set M. It means that, $\tilde{x}(t)$ is the optimal trajectory.

Remark 2.1.1: If t_1 is fixed then $\theta = t_1$, and then $\triangle I \geq 0$ (see (19)), i.e. $\tilde{x}(t)$ is optimal. Moreover, in that case the condition of monotone increasingness of $I(x(.), t)$ on t is superfluous .

Remark 2.1.2: Condition of monotonicity of $I(x(.), t)$ on t for any admissible trajectory $x(t)$ is not very restrictive and we can verify it. For example it is fulfilled for high speed problems and for problems with quadratic criteria of quality and in case when $\varphi(x,t) \equiv 0, g(x,t) \geq 0$.

Remark 2.1.3: Suppose a is a convex-valued closed bounded continuous mapping and $W_a(x, y^*)$ is continuous differentiable on x. Theorem 1.1 and the condition a) of Theorem 2.1 imply

$$-\dot{x}^*(t) \in \frac{\partial W_a(\tilde{x}(t), x^*(t), t)}{\partial x} + \partial g(\tilde{x}(t), t).$$

3 Duality.

Let us reconsider the problem (1)-(4), given in the Introduction. This problem is called a convex problem if the functions, multivalued mapping and the set are convex and the function F is convex-valued. Now consider (1)-(4) as a convex problem. Let us denote by $\varphi^*(., t_1)$ and $g^*(., t)$ conjugate functions [1,10] to functions $\varphi(., t_1)$ and $g(., t)$, respectively. Let us recall the equality

$$\Omega_a(x^*, y^*, t) = \inf\{- < x, x^* > + < y, y^* >: (x, y) \in gfa\}.$$

Evidently $\Omega_a(x^*, y^*, t) = \inf_x\{- < x, x^* > + W_a(x, y^*, t)\}$. The following problem is called the dual problem to (1)-(4):

$$\begin{aligned} \sup_{x^*(t), \xi^*(t), u^*(t), v^*(t_1)} & \{-\varphi^*(v^*(t_1) - \xi^*(t_1), t_1) - \int_{t_0}^{t_1} g^*(u^*(t), t) dt \\ & + < x(t_0), x^*(t_0) > + \int_{t_0}^{t_1} \Omega_a(-\xi^*(t) - \dot{x}^*(t) - u^*(t), x^*(t), t) dt \\ & - \int_{t_0}^{t_1} W_{F(t)}(\xi^*(t)) dt + W_M(v^*(t_1) - x^*(t_1) - \xi^*(t_1))\}. \end{aligned} \qquad (21)$$

Here $x^*(t), \xi^*(t), u^*(t)$ and $v^*(t_1)$, are absolutely continuous functions. Let us denote the expression in curly brackets by $I_*(x^*(t), \xi^*(t), u^*(t), v^*(t_1), t_1)$.

Theorem 3.1. For any admissible solutions $x(t)$ and $\{x^*(t), \xi^*(t), u^*(t), v^*(t_1)\}$ of the direct problem (1)-(4) and the dual problem (21), respectively, the relation

$$I(x(t), t_1) \geq I_*(x^*(t), \xi^*(t), u^*(t), v^*(t_1), t_1)$$

holds.

Proof. By the definitions of the conjugate function, Ω_a, W_M and $W_{F(t)}$, we have

$$I_*(x^*(t), \xi^*(t), u^*(t), v^*(t_1), t_1) \leq -< x(t_1), v^*(t_1) - \xi^*(t_1) > +\varphi(x(t), t) -$$
$$\int_{t_0}^{t_1} [< x(t), u^*(t) > -g(x(t), t)]dt + < x(t_0), x^*(t_0) > +$$
$$+ \int_{t_0}^{t_1} [- < x(t), -\xi^*(t) - \dot{x}^*(t) - u^*(t) > + < \dot{x}(t), x^*(t) >]dt -$$
$$\int_{t_0}^{t_1} < x(t), \xi^*(t) > dt + < x(t_1), v^*(t_1) - x^*(t_1) - \xi^*(t_1) >=$$
$$I(x(t), t_1) + < x(t_0), x^*(t_0) > + \int_{t_0}^{t_1} d < x(t), x^*(t) > - < x(t_1), x^*(t_1) >=$$
$$I(x(t), t_1). \quad (22)$$

Theorem is proved.

Theorem 3.2: Let the trajectory $x(t), t \in [t_0, t_1]$, be a solution of the direct convex problem (1)-(4). Further, let $x^*(t)$, $\xi^*(t), u^*(t)$ and $v^*(t_1)$ be functions such that $x^*(t)$ satisfies the dual differential inclusion a), $u^*(t) \in \partial g(\tilde{x}(t), t)$, $v^*(t_1) - \xi^*(t_1) \in \partial\varphi(\tilde{x}(t_1), t_1)$, $\xi^*(t) \in K^*_{F(t)}(\tilde{x}(t))$ and $v^*(t_1) - x^*(t_1) - \xi^*(t_1) \in K^*_M(\tilde{x}(t_1))$.

Then $\{x^*(t), \xi^*(t), u^*(t), v^*(t_1)\}$ is a solution of the dual problem and in this case, the values of the two problems coincide.

Proof. By the definitions of locally conjugate mapping and conjugate cone we have

$$< \xi^*(t) + \dot{x}^*(t) + u^*(t), x - \tilde{x}(t) > + < x^*(t), y - \dot{\tilde{x}}(t) > \geq 0$$

at almost every $t \in [t_0, t_1]$ and all $x \in F(t), (x, y) \in gfa(., t)$. It means that $(-\xi^*(t) - \dot{x}^*(t) - u^*(t), x^*(t)) \in dom\Omega_a, \quad t \in [t_0, t_1]$.

If we consider $\partial_x g(x, t) \subset domg^*(., t)$ and $\partial_x\varphi(x, t_1) \subset dom\varphi^*(., t_1)$ then we may conclude that $\{x^*(t), \xi^*(t), u^*(t), v^*(t_1)\}$ is an admissible solution of the dual problem.

Further, by Lemma 1.3 and from the conjugate differential inclusion a) it is clear that

$$\Omega_a(\xi^*(t) - \dot{x}^*(t) - u^*(t), x^*(t), t) =$$
$$- < \tilde{x}(t), -\xi^*(t) - \dot{x}^*(t) - u^*(t) > +W_a(\tilde{x}(t), x^*(t), t), t \in [t_0, t_1]. \quad (23)$$

From conditions of the theorem and from the fact that $\dot{\tilde{x}}(t) \in a(\tilde{x}(t), x^*(t), t)$, $t \in [t_0, t_1]$, it follows that

$$g^*(u^*(t), t) =< \tilde{x}(t), u^*(t) > -g(\tilde{x}(t), t),$$
$$\varphi^*(v^*(t_1) - \xi^*(t_1), t_1) =< \tilde{x}(t_1), v^*(t_1) - \xi^*(t_1) > -\varphi(\tilde{x}(t_1), t_1),$$
$$W_{F(t)}(\xi^*(t)) =< \xi^*(t), \tilde{x}(t) >, \quad t \in [t_0, t_1], \quad (24)$$
$$W_M(v^*(t_1) - x^*(t_1) - \xi^*(t_1)) =< v^*(t_1) - x^*(t_1) - \xi^*(t_1), \tilde{x}(t_1) >,$$
$$W_a(\tilde{x}(t), x^*(t), t) =< \dot{\tilde{x}}(t), x^*(t) >, \quad t \in [t_0, t_1].$$

From relations (23), (24) and the proof of Theorem 3.1(see (22)), we get the required result.

4 Examples about the construction of the dual problem.

Let consider the following problem

$$\begin{aligned} I(x(.),t_1) = \varphi(x(t_1),t_1) \to \inf \\ \dot{x}(t) = f(x(t),u(t)), u(t) \in U \subset \mathbb{R}^n, \quad t \in [t_0,t_1], \\ x(t_0) = x_0, \quad x(t_1) \in M = \{x_1\}, \end{aligned} \tag{25}$$

where $f(x,u)$ is differentiable on x and $a(x) = f(x,U)$ is convex. Let us replace the problem (25) with the following:

$$\begin{aligned} I(x(.),t_1) \to \inf \\ \dot{x}(t) \in a(x(t)) \\ x(t_0) = x_0, \quad x(t_1) = x_1. \end{aligned} \tag{26}$$

It is obvious that

$$W_a(x,y^*) = \inf_{u \in U} < y^*, f(x,u) > . \tag{27}$$

Then,if $\tilde{u}$ is a solution of the problem (27), and $\tilde{x}$ is a solution of the problem which is formulated in Lemma 1.3, then the following relation is valid

$$x^* = f_x^{'*}(\tilde{x},\tilde{u})y^*, \tag{28}$$

where $f_x^{'*}$ is matrix conjugate to the matrix $f_x^{'}$. When $F(t) \equiv \mathbb{R}^n$ and $M = \mathbb{R}^n$, $W_{F(t)}$ and W_M in (24) show that

$$\xi^*(t) = 0, v^*(t_1) = x^*(t_1). \tag{29}$$

Since $g(x,t) \equiv 0$ in the problem (25), then

$$g^*(u^*,t) = \begin{cases} 0 & ,u^* = 0 \\ \infty & ,u^* \neq 0. \end{cases} \tag{30}$$

Considering Ω_a in various intervals (see(21)) and using (29) and (30), we obtain

$$\sup_{x^*(t)} \{-\varphi^*(x^*(t_1),t_1) + \int_{t_0}^{t_1} \Omega_a(-\dot{x}^*(t),x^*(t),t)dt\}. \tag{31}$$

From relations (28)-(30) we have

$$-\dot{x}^*(t) = f_x^{'*}(\tilde{x}(t),\tilde{u}(t))x^*(t), \quad t \in [t_0,t_1], \tag{32}$$

$$W_a(\tilde{x}(t), x^*(t)) =< x^*(t), f(\tilde{x}(t), \tilde{u}(t)) > .$$

Thus the dual problem is defined by the formulas (31) and (32).

Let us consider the problem with polyhedral mapping[1]

$$a(x) = \{y : Ax - By \leq d\},$$

where A, B are $(m \times n)$ matrices and d is an m-dimensional column-vector. We compute easily that the LCM is given by

$$a^*(y^*; (\tilde{x}, \tilde{y})) = \{A^*\lambda : y^* = B^*\lambda, \lambda \geq 0, < A\tilde{x} - B\tilde{y} - d, \lambda >= 0\}.$$

Using the last formula it is easy to show that the dual problem consists of the following:

$$I_*(x^*(.), t_1) \to \sup,$$

$$-\dot{x}^*(t) = A^*\lambda(t), \quad t \in [t_0, t_1],$$

$$x^*(t) = B^*\lambda(t), \quad t \in [t_0, t_1],$$

$$< A\tilde{x}(t) - B\dot{\tilde{x}}(t) - d, \lambda(t) >= 0,$$

$$\lambda(t) \geq 0, \quad t \in [t_0, t_1].$$

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