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On the classical 2−orthogonal polynomials sequences of Sheffer-Meixner type

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ABSTRACT

The polynomial sequences of Sheffer-Meixner type designed by $\{S_n\}_{n\geq 0}$, are defined by the generating function

$$G(x,t) = A(t)e^{xH(t)} = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!}$$

We are interested, in this work, in studying the sequences when they are 2−orthogonal. We will give the general properties of these sequences, and we study in details those which are classical.

RESUMEN

Las sucesiones polinomiales del tipo Sheffer-Meixner denotadas por $\{S_n\}_{n\geq 0}$ son definidas por la función generatriz

$$G(x,t) = A(t)e^{xH(t)} = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!}$$

En este trabajo estamos interesados en estudiar aquellas sucesiones que son 2− ortogonales. Mostraremos sus propiedades generales y estudiaremos en detalle aquellas que son clásicas.

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1 Introduction.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its algebraic dual. Let us given d scalar linear forms $\Gamma^1, \Gamma^2, \cdots, \Gamma^d$ defined from $\mathcal{P}$ into $\mathbb{C}$.

A monic sequence $\{P_n\}_{n\geq 0}$ (i.e. $P_n(x) = x^n + \cdots$, $n \geq 0$) is said $d-$orthogonal with respect to $\Gamma = (\ \Gamma^1, \Gamma^2, \cdots, \Gamma^d\ )^T$ when it satisfies **[7, 10, 12, 18, 19, 23]**

$$\begin{cases} \langle \Gamma^\alpha, x^m P_n(x)\rangle = 0, & n \geq md+\alpha, \quad m \geq 0 \\ \langle \Gamma^\alpha, x^m P_{md+\alpha-1}(x)\rangle \neq 0, & m \geq 0, \end{cases} \tag{1.1}$$

for every $1 \leq \alpha \leq d$, and where $\langle\ ,\ \rangle$ is the dual bracket between $\mathcal{P}$ and $\mathcal{P}'$.

Among the $d-$orthogonal sequences, we will be interested here by a particular class, but nevertheless important. Indeed, theses sequences have many applications and have extensively investigated.

This class consists of sequences of polynomials $\{S_n\}_{n\geq 0}$ defined by the generating function

$$G(x,t) = A(t)e^{xH(t)} = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!} \tag{1.2}$$

where

$$A(t) = \sum_{n\geq 0} a_n t^n \qquad \text{and} \qquad H(t) = \sum_{n\geq 1} h_n t^n$$

with

$$A(0) = 1,\ H(0) = 0 \text{ and } H'(0) = 1$$

These sequences are said to be of Sheffer-Meixner type.

The case $d = 1$ has been first studied by Meixner **[20]** and Sheffer **[22]** and then, completed by other authors **[2, 13, 21].**

Meixner has shown that this class consists of 5 sequences, namely, Hermite polynomials, Laguerre polynomials, Charlier polynomials, Meixner polynomials and Meixner-Pollaczek polynomials.

In the case $d = 2$ **[5]**, we have shown that the functions H and A satisfy, respectively the equations

$$\begin{cases} H'(t) = \dfrac{1}{(1-\alpha t)(1-\beta t)(1-\gamma t)}, & \alpha,\beta,\gamma \in \mathbb{C} \\[2ex] \dfrac{A'(t)}{A(t)} = \dfrac{\sigma_0+\sigma_1 t+\sigma_2 t^2}{(1-\alpha t)(1-\beta t)(1-\gamma t)}, & \sigma_0,\sigma_1,\sigma_2 \in \mathbb{C}; \quad \sigma_2 \neq 0. \end{cases} \tag{1.3}$$

If we note by J the inverse function of H (i.e. $J(H(t) = t$), and by $D = \dfrac{d}{dx}$, then we have [**1, 20**]

$$J(D)S_{n+1}(x) = (n+1)S_n(x), \quad n \geq 0. \tag{1.4}$$

Moreover, the polynomials S_n $(n \geq 0)$ are characterized by the four-term relation [**5**]

$$\begin{array}{rl} S_{n+3}(x) = & \left[(x-\sigma_0) + (n+2)(\alpha+\beta+\gamma)\right] S_{n+2}(x) \\ & -(n+2)\left[\sigma_1 + (n+1)\left(\alpha\beta+\alpha\gamma+\beta\gamma\right)\right] S_{n+1}(x) \\ & -\left(n+1\right)\left(n+2\right)\left(\sigma_2 - n\alpha\beta\gamma\right) S_n(x), \quad n \geq 0 \end{array} \tag{1.5}$$

We also have proved that this class is composed of 9 sequences, namely

$$\left\{ \begin{array}{lll} (\mathbf{a}) & \alpha=\beta=\gamma=0, & \\ (\mathbf{b}) & \alpha=\beta\neq 0 \text{ and } \gamma=0, & \\ (\mathbf{c}) & \alpha\neq 0 \text{ and } \beta=\gamma=0, & \\ (\mathbf{d1}) & \alpha\neq\beta\neq 0 \text{ and } \gamma=0, & \alpha\ ,\beta\in\mathbb{R}, \\ (\mathbf{d2}) & \alpha\neq\beta\neq 0 \text{ and } \gamma=0, & \alpha\ ,\beta\in\mathbb{C}, \\ (\mathbf{e}) & \alpha=\beta=\gamma\neq 0, & \\ (\mathbf{f}) & \alpha=\beta\neq\gamma\neq 0, & \\ (\mathbf{g1}) & \alpha\neq\beta\neq\gamma\neq 0 & \alpha,\beta\ ,\gamma\in\mathbb{R}, \\ (\mathbf{g2}) & \alpha\neq\beta\neq\gamma\neq 0 & \alpha,\beta\in\mathbb{R} \text{ and } \gamma\in\mathbb{C}. \end{array} \right. \tag{1.6}$$

We recall in paragraph 2, the principal properties of the $d-$orthogonal sequences [**6, 11, 18, 19**]. The paragraph 3 is denoted to the characterization of those which are in addition classical [**3, 4, 12**].

In paragraph 4, we show that the sequences **(a), (b), (c)** and **(d)** are classical sequences and we give certain of their properties. Whereas in paragraph 5, we exhibit an integral representation of the forms with respect to which these sequences are 2−orthogonal in cases (**a**) and (**b**).

2 Properties of $d-$orthogonal sequences.

Definition 2.1 *Let $\{P_n\}_{n\geq 0}$ be a sequence monic polynomials. We call dual sequence of the sequence $\{P_n\}_{n\geq 0}$, the sequence of linear forms $\{\mathcal{F}_n\}_{n\geq 0}$ defined by*

$$\langle \mathcal{F}_n, P_n(x)\rangle = \delta_{n,m}, \quad n,m \geq 0 \tag{2.1}$$

Proposition 2.2 *[11] If we denote by D the operator of derivation i.e. $D = \dfrac{d}{dx}$ and by $\{\tilde{\mathcal{F}}_n\}_{n\geq 0}$ the dual sequence associated to the monic sequence $\{Q_n\}_{n\geq 0}$ of the derivatives of $\{P_n\}_{n\geq 0}$, and defined by*

$$Q_n(x) = \frac{DP_{n+1}(x)}{n+1}, \qquad n \geq 0,$$

then

$$D\tilde{\mathcal{F}}_n = -(n+1)\mathcal{F}_n, \tag{2.2}$$

with
$$\left\langle D\tilde{\mathcal{F}}_n, r(x)\right\rangle = -\left\langle \tilde{\mathcal{F}}_n, Dr(x)\right\rangle, \quad \forall r \in \mathcal{P}.$$

Proposition 2.3 *[18, 19] Let $\mathcal{L} \in \mathcal{P}'$ be and q an integer, in order that $\mathcal{L}$ satisfies*

$$\begin{array}{ll} \langle \mathcal{L}, P_n(x)\rangle = 0 & n \geq q \\ \langle \mathcal{L}, P_{q-1}(x)\rangle \neq 0 & \end{array} \tag{2.3}$$

it is necessary and sufficient that there exists $\lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq q-1$, $\lambda_{q-1} \neq 0$, such that

$$\mathcal{L} = \sum_{\nu=0}^{q-1} \lambda_\nu \mathcal{F}_\nu. \tag{2.4}$$

Corollary 2.4 *According to the preceding lemma, we have*

$$\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha \mathcal{F}_\nu, \quad \lambda_{\alpha-1}^\alpha \neq 0, \quad 1 \leq \alpha \leq d,$$

and in a equivalent manner

$$\mathcal{F}_\nu = \sum_{\alpha=0}^{\nu+1} \tau_\alpha^\nu \Gamma^\alpha; \quad \tau_{\nu+1}^\nu \neq 0, \quad 0 \leq \nu \leq d-1.$$

Consequently, every $d-$orthogonal sequence $\{P_n\}_{n\geq 0}$ with respect to $\Gamma = (\Gamma^1, \Gamma^2, \cdots, \Gamma^d)^T$ is also $d-$orthogonal with respect to $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_{d-1})^T$.

Theorem 2.5 *[18, 23] With the same notations as previously we have the following equivalences*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is $d-$orthogonal with respect to $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_{d-1})^T$.*

(b) *The sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence of order $d+1$($d \geq 1$)*

$$P_{m+d+1}(x) = (x - \beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-\nu-1}(x), \qquad m \geq 0 \tag{2.5}$$

with the initial data

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad \text{and if } d \geq 2 \\ P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x), \qquad 2 \leq n \leq d \end{cases} \tag{2.6}$$

where $\gamma_{m+1}^0 \neq 0$, $m \geq 0$. (Regularity conditions).

(c) *For every (n, v), $n \geq 0$, $0 \leq \nu \leq d-1$, there exists d polynomials $V^\mu(n,\nu)$, $(0 \leq \mu \leq d-1)$ such that*

$$\mathcal{F}_{nd+\nu} = \sum_{\mu=0}^{d-1} V^\mu(n,\nu)\mathcal{F}_\mu, \quad n \geq 0, \quad 0 \leq \nu \leq d-1, \tag{2.7}$$

where

$$\begin{cases} \deg V^{\mu}(n,\mu)=n, & 0\le\mu\le d-1, & \\ \deg V^{\mu}(n,\nu)\le n, & 0\le\mu\le\nu-1, & \text{if } 1\le\nu\le d-1, \\ \deg V^{\mu}(n,\nu)\le n-1, & \nu+1\le\mu\le d-1, & \text{if } 0\le\nu\le d-2. \end{cases} \tag{2.8}$$

Theorem 2.6 *[18] For every sequence $\{P_n\}_{n\ge0}$ $d-$orthogonal with respect to $\mathcal{F}=(\mathcal{F}_0,\mathcal{F}_1,\cdots,\mathcal{F}_{d-1})^T$, the following statements are equivalent*

(a) *It exists $\mathcal{L}\in\mathcal{P}'$ and an integer $s\ge1$ such that*

$$\begin{cases} \langle\mathcal{L},\ P_n(x)\rangle=0, & n\ge s, \\ \langle\mathcal{L},\ P_{s-1}(x)\rangle\neq0. \end{cases} \tag{2.9}$$

(b) *It exists $\mathcal{L}\in\mathcal{P}'$ and d polynomials ϕ^{α}, $0\le\alpha\le d-1$ such that*

$$\mathcal{L}=\sum_{\alpha=0}^{d-1}\phi^{\alpha}\mathcal{F}_{\alpha},$$

with the following properties

if $s-1=qd+r$, $0\le r\le d-1$, we have

$$\begin{cases} \deg\phi^{r}=q, & 0\le r\le d-1, & if\ \ d\ge2, \\ \deg\phi^{\alpha}\le q, & 0\le\alpha\le r-1, & if\ \ 1\le r\le d-1, \\ \deg\phi^{\alpha}\le q-1, & r+1\le\alpha\le d-1, & if\ \ 0\le r\le d-2. \end{cases} \tag{2.10}$$

3 The $d-$orthogonal sequences and the finite differences operators $\Delta_\omega$and ∇_ω.

Let us consider the progressive finite differences operators Δ_ω (Hahn's operator) and regressive operator ∇_ω, defined respectively by

$$\begin{aligned} \Delta_\omega f(x) &= \frac{f(x+\omega)-f(x)}{\omega}, \quad \text{and} \\ \nabla_\omega f(x) &= \frac{f(x)-f(x-\omega)}{\omega}=\Delta_{-\omega}f(x) \end{aligned}$$

These operators enjoy the following properties

Proposition 3.1 *Let $\mathcal{F}\in\mathcal{P}'$ then we have*

$$\langle\mathcal{F},\ \Delta_\omega f(x)\rangle=-\langle\nabla_\omega\mathcal{F},\ f(x)\rangle,\quad \forall f\in C^{\infty}. \tag{3.1}$$

Proof. We know that

$$\Delta_\omega f(x)=\frac{e^{\omega D}-1}{\omega}f(x),$$

and that by definition we have

$$\langle D\mathcal{F},\ f(x)\rangle=-\langle\mathcal{F},Df(x)\rangle,$$

therefore

$$\begin{aligned}\langle \mathcal{F}, \Delta_\omega f(x)\rangle &= \left\langle \mathcal{F}, \sum_{k\geq 0} \frac{\omega^k}{(k+1)!} D^{k+1} f(x) \right\rangle = \left\langle \sum_{k\geq 0} \frac{(-1)^{k+1}\omega^k}{(k+1)!} D^{k+1}\mathcal{F},\ f(x)\right\rangle \\ &= \left\langle \frac{e^{-\omega D}-1}{\omega}\mathcal{F},\ f(x)\right\rangle = -\left\langle \nabla_\omega \mathcal{F},\ f(x)\right\rangle.\end{aligned}$$

∎

Proposition 3.2 *Let $\{Q_n^\omega\}_{n\geq 0}$ be the sequence of the monic polynomials defined by*

$$Q_n^\omega(x) = \frac{\Delta_\omega P_{n+1}(x)}{n+1} = \frac{P_{n+1}(x+\omega) - P_{n+1}(x)}{(n+1)\omega}, \quad n \geq 0 \tag{3.2}$$

and $\{\tilde{\mathcal{F}}_n\}_{n\geq 0}$ the dual sequence associated to the sequence $\{Q_n^\omega\}_{n\geq 0}$, then we have

$$\nabla_\omega \tilde{\mathcal{F}}_n = \Delta_{-\omega}\tilde{\mathcal{F}}_n = -(n+1)\mathcal{F}_{n+1}; \quad n \geq 0. \tag{3.3}$$

Proof. Indeed, we have

$$\delta_{n,m} = \left\langle \tilde{\mathcal{F}}_n, Q_m(x)\right\rangle = \frac{1}{m+1}\left\langle \tilde{\mathcal{F}}_n, \Delta_\omega P_{m+1}(x)\right\rangle = -\frac{1}{m+1}\left\langle \Delta_{-\omega}\tilde{\mathcal{F}}_n, P_{m+1}(x)\right\rangle,$$

i.e.

$$-\left\langle \Delta_{-\omega}\tilde{\mathcal{F}}_n, P_{n+1}(x)\right\rangle = (m+1)\delta_{n,m}$$

but from the lemma (2.1), $\exists\ \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq n+1$, such that

$$\Delta_{-\omega}\tilde{\mathcal{F}}_n = \sum_{\nu=0}^{n+1} \lambda_\nu^n \mathcal{F}_\nu,$$

with $\lambda_\nu^n = 0,\ 0 \leq \nu \leq n$ and $\lambda_{n+1}^n = n+1$. ∎

Lemma 3.3 *We have the following properties*

$$\Delta_\omega\left[(x-\omega)^m P_n(x)\right] = x\Delta_\omega\left[(x-\omega)^{m-1}P_n(x)\right] + (x-\omega)^{m-1}P_n(x), \quad m \geq 0 \tag{3.4}$$

and

$$x^m \Delta_\omega P_n(x) = \Delta_\omega\left[(x-\omega)^m P_n(x)\right] - \left[mx^{m-1} - \frac{m(m-1)}{2}\omega x^{m-2} + R_{m-3}^\omega(x)\right]P_n(x), \tag{3.5}$$

$m \geq 0$, *where $R_{m-3}^\omega(x)$ is a polynomial of degree $(m-3)$ in x.*

Proof. Clearly

$$\begin{aligned}\Delta_\omega\left[(x-\omega)^m P_n(x)\right] &= \frac{x^m P_n(x+\omega) - (x-\omega)^m P_n(x)}{\omega} \\ &= \frac{x[x^{m-1}P_n(x+\omega) - (x-\omega)^{m-1}P_n(x)] + \omega(x-\omega)^{m-1}P_n(x)}{\omega} \\ &= x\Delta_\omega\left[(x-\omega)^{m-1}P_n(x)\right] + (x-\omega)^{m-1}P_n(x), \quad m \geq 0.\end{aligned}$$

Repeating m times the expression (3.4) we get

$$\Delta_\omega\left[(x-\omega)^m P_n(x)\right] = x^m \Delta_\omega P_n(x) + \left[\sum_{k=0}^{m-1} x^{m-k}(x-\omega)^k\right] P_n(x),$$

as

$$\begin{aligned}\sum_{k=0}^{m-1} x^{m-k}(x-\omega)^k &= \sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\omega^j x^{m-1-j}\\ &= \sum_{j=0}^{m-1}\sum_{k=j}^{m-1}\binom{k}{j}(-1)^j\omega^j x^{m-1-j}\\ &= mx^{m-1} - \frac{m(m-1)}{2}\omega x^{m-2} + R_{m-3}^{\omega}(x).\end{aligned}$$

from which we obtain (3.5). ∎

Definition 3.4 *[4, 11, 14, 15] A sequence of polynomials $\{P_n\}_{n\geq 0}$ $d-$orthogonal $(d\geq 1)$ with respect to $\mathcal{F}=(\mathcal{F}_0,\mathcal{F}_1,\cdots,\mathcal{F}_{d-1})^T$,those the monic sequence of finite differences $\{Q_n^\omega\}_{n\geq 0}$ defined by*

$$Q_n^\omega(x) = \frac{\Delta_\omega P_{n+1}(x)}{n+1}\ ,\quad n\geq 0$$

is also $d-$orthogonal $(d\geq 1)$ with respect to $\tilde{\mathcal{F}} = \left(\tilde{\mathcal{F}}_0,\tilde{\mathcal{F}}_1,\cdots,\tilde{\mathcal{F}}_{d-1}\right)^T$ is said to be classical.

Remark 3.5 *In the case $\omega=0$, the operator Δ_ω becomes $D=\dfrac{d}{dx}$.*

Theorem 3.6 *With the above hypothesis we have the following equivalence*
(a) The sequence $\{P_n\}_{n\geq 0}$ is classical $d-$orthogonal.
(b) The functional $\mathcal{F}$ satisfies the vectorial functional equation

$$\nabla_\omega(\Phi\mathcal{F}) + \Psi\mathcal{F} = 0, \tag{3.6}$$

where Ψ and Φ are 2 matrices $d\times d$ of polynomials

$$\Psi(x) = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0\\ 0 & 0 & 2 & . & . & . & 0\\ . & . & . & & & & .\\ . & . & . & & & & .\\ . & . & . & & & & .\\ 0 & 0 & 0 & . & . & . & d-1\\ \psi(x) & \xi_1 & \xi_2 & . & . & . & \xi_{d-1}\end{pmatrix} \tag{3.7}$$

and ψ is a polynomial of degree 1 and ξ_μ, $1\leq\mu\leq d-1$ are constants,

$$\Phi(x) = \begin{pmatrix} \phi_0^0(x) & \phi_0^1(x) & . & . & . & \phi_0^{d-1}(x)\\ . & . & & & & .\\ . & . & & & & .\\ . & . & & & & .\\ \phi_{d-2}^0(x) & \phi_{d-2}^1(x) & . & . & . & \phi_{d-2}^{d-1}(x)\\ \phi_{d-1}^0(x) & \phi_{d-1}^1(x) & . & . & . & \phi_{d-1}^{d-1}(x)\end{pmatrix} \tag{3.8}$$

where ϕ_α^ν, $0 \leq \alpha, \nu \leq d-1$ *are polynomials such that*

$$\begin{cases} \deg \phi_\alpha^\nu \leq 1, \quad 0 \leq \nu \leq \alpha+1 & \text{if} \quad 0 \leq \alpha \leq d-2 \\ \deg \phi_\alpha^\nu = 0, \quad \alpha+2 \leq \nu \leq d-1 & \text{if} \quad 0 \leq \alpha \leq d-3 \\ \deg \phi_{d-1}^0 \leq 2 \quad \text{and} \quad \deg \phi_{d-1}^\nu \leq 1, & 1 \leq \nu \leq d-1 \end{cases} \tag{3.9}$$

In addition, if we write

$$\begin{cases} \psi(x) = e_1 x + e_0, \qquad \phi_{d-1}^0(x) = c_2 x^2 + c_1 x + c_0 \\ \\ \phi_\alpha^{\alpha+1}(x) = k_\alpha x + l_\alpha, \quad 0 \leq \alpha \leq d-2, \end{cases}$$

then

$$\begin{cases} c_2 \neq \dfrac{e_1}{m+1}, \quad m \geq 0, \qquad e_1 \neq 0, \\ \\ k_\alpha \neq \dfrac{\alpha+1}{m+1}, \quad m \geq 0, \ \text{for } 0 \leq \alpha \leq d-2. \end{cases} \tag{3.10}$$

Remark 3.7 *a) It is easy to show that :*

$$\begin{cases} \tilde{\mathcal{F}} = \Phi \mathcal{F} \\ \nabla_\omega \tilde{\mathcal{F}} = -\Psi \mathcal{F} \end{cases} \tag{3.11}$$

(b) When $\omega = 0$ *the functional equation (3.6) may be written* ***[11]***

$$\Psi \mathcal{F} + D(\Phi \mathcal{F}) = 0 \tag{3.12}$$

and the conditions (3.7), (3.8), (3.9) and (3.10) remain unchanged.

c) The proof of this theorem is the same as in the case $\omega = 0$ *[**11**], if we take into account the relation (3.4).*

4 Classification of the sequences $2-$orthogonal of Sheffer-Meixner type.

Let us consider now the sequences of polynomials $\{S_n\}_{n\geq 0}$, Sheffer-Meixner type defined by the relation (1.5).

We noted by $\{m_n^\omega\}_{n\geq 0}$ and $\{M_n^\omega\}_{n\geq 0}$ the sequences of monic polynomials defined respectively by

$$m_n(x) = \frac{DS_{n+1}(x)}{n+1}; \quad n \geq 0, \tag{4.1}$$

and

$$M_n^\omega(x) = \frac{\Delta_\omega S_{n+1}(x)}{n+1}; \quad n \geq 0. \tag{4.2}$$

Then we have

Lemma 4.1 *In the case **(b)** (the case **(a)** if $\alpha = 0$), the sequence of derivatives of monic polynomials defined by the relation (4.1) satisfies the following recurrence*

$$\begin{cases} m_{n+3}(x) = \left[(x-\sigma_0) + (2n+5)\,\alpha\right] m_{n+2}(x) \\ \qquad\quad -(n+2)\left[\sigma_1 + (n+2)\alpha^2\right] m_{n+1}(x) - (n+1)(n+2)\sigma_2 m_n(x); \quad n \geq 0 \\ m_0(x) = 1;\ m_1(x) = x - \sigma_0 + \alpha;\ \ m_2(x) = (x - \sigma_0 + 3\alpha)\, m_1(x) - \left(\sigma_1 + \alpha^2\right) \end{cases} \tag{4.3}$$

Proof. Indeed, in the case (**b**) J is such that [**5**]

$$J(D) = \frac{D}{1+\alpha D},$$

then by the relation (1.4) we have

$$DS_{n+1}(x) = (n+1)\left[S_n(x) + \alpha DS_n(x)\right]$$

consequently

$$S_{n+1}(x) = m_{n+1}(x) - (n+1)\alpha m_n(x); \quad n \geq 0.$$

Differentiating the recurrence(1.5) and replacing S_{n+1} by $\{m_\nu\}_{\nu=n-1}^{n+1}$,we obtain the relation (4.3). ∎

Lemma 4.2 *In the case **(d)** (the case **(c)** if $\beta = 0$), the sequence of finite differences of monic polynomials defined by the relation (4.2) satisfies the following recurrence*

$$\begin{cases} M_{n+3}^{\alpha-\beta}(x) = \left[(x+\alpha-\sigma_0) + (n+2)\,(\alpha+\beta)\right] M_{n+2}^{\alpha-\beta}(x) \\ \qquad\quad -(n+2)\left[\sigma_1 + (n+2)\alpha\beta\right] M_{n+1}^{\alpha-\beta}(x) - (n+1)(n+2)\sigma_2 M_n^{\alpha-\beta}(x);\ n \geq 0 \\ M_0^{\alpha-\beta}(x) = 1;\ M_1^{\alpha-\beta}(x) = x - \sigma_0 + \alpha;\ M_2^{\alpha-\beta}(x) = (x - \sigma_0 + 2\alpha + \beta)\, M_1^{\alpha-\beta}(x) \\ \qquad\quad -\sigma_1 - \alpha\beta \end{cases} \tag{4.4}$$

Proof. Indeed, in the case (**d**) the function J is such that [**5**]

$$J(D) = \frac{\Delta_{\alpha-\beta}}{1+\alpha\Delta_{\alpha-\beta}},$$

i.e. by the relation (1.4).

$$\Delta_{\alpha-\beta}S_{n+1}(x) = (n+1)\left[\alpha\Delta_{\alpha-\beta}S_n(x) + S_n(x)\right],$$

consequently

$$S_{n+1}(x) = M_{n+1}^{\alpha-\beta}(x) - (n+1)\alpha M_n^{\alpha-\beta}(x); \quad n \geq 0.$$

By acting the operator $\Delta_{\alpha-\beta}$ on the recurrence (1.5) and replacing S_{n+1} by $\{M_{n+1}^{\alpha-\beta}\}_{\nu=n-1}^{n+1}$, we obtain the relation (4.4). ∎

Thus, we have the following classification.

Theorem 4.3 *The sequences **(a)**, **(b)**, **(c)** and **(d)** are classical sequences and the 2− orthogonal polynomials sequences $\{m_n\}_{n\geq 0}$ and $\left\{M_n^{\alpha-\beta}\right\}_{n\geq 0}$ are "2−Kernel" polynomial **[8, 9, 17]** for the 2− orthogonal polynomials sequences $\{S_n\}_{n\geq 0}$.*

5 Integral representation of the functional $\mathcal{F}_0$ and $\mathcal{F}_1$.

In this paragraph, we will be interested by the integral representation problem of the linear functional $\mathcal{F}_0$ and $\mathcal{F}_1$ in the cases **(a)** and **(b)**.

5.1 Properties of the functional $\mathcal{F}_0$ and $\mathcal{F}_1$.

Lemma 5.1 *In the case **(d)** (a fortiori the cases **(a)**, (**b**) and (**c**)) we have*

$$\Phi(x) = \begin{bmatrix} 1 & -\alpha \\ -\dfrac{\alpha}{\sigma_2}(x-\sigma_0) & 1+\alpha\dfrac{\sigma_1}{\sigma_2} \end{bmatrix} \quad \textit{and} \quad \Psi(x) = \begin{bmatrix} 0 & 1 \\ \dfrac{1}{\sigma_2}(x-\sigma_0) & -\dfrac{\sigma_1}{\sigma_2} \end{bmatrix}$$

Proof. With the same notations as in theorem (3.1), we have

$$\deg \phi_0^0(x) \leq 1,\ \deg \phi_0^1(x) \leq 1,\ \ \deg \phi_1^0(x) \leq 2,\ \ \deg \phi_1^1(x) \leq 1,\ \ \text{and}\ \deg \psi(x) \leq 1.$$

Putting

$$\begin{cases} \phi_0^0(x) &=& d_0 + d_1 x \\ \phi_0^1(x) &=& e_0 + e_1 x \\ \phi_1^0(x) &=& a_0 + a_1 x + a_2 x^2 \\ \phi_1^1(x) &=& b_0 + b_1 x \\ \psi(x) &=& c_0 + c_1 x \end{cases}$$

the relations (3.11) may be written respectively

$$\begin{cases} \tilde{\mathcal{F}}_0 &=& (d_0 + d_1 x)\,\mathcal{F}_0 + (e_0 + e_1 x)\,\mathcal{F}_1 \\ \tilde{\mathcal{F}}_1 &=& \left(a_0 + a_1 x + a_2 x^2\right)\mathcal{F}_0 + (b_0 + b_1 x)\,\mathcal{F}_1 \end{cases} \tag{R5.0}$$

and

$$\begin{cases} \nabla_{\alpha-\beta}\tilde{\mathcal{F}}_0 &=& -\mathcal{F}_1 \\ \nabla_{\alpha-\beta}\tilde{\mathcal{F}}_1 &=& -(c_0 + c_1 x)\,\mathcal{F}_0 - \xi_1 \mathcal{F}_1 \end{cases} \tag{R5.1}$$

By letting, firstly, the functional $\tilde{\mathcal{F}}_0$ and $\tilde{\mathcal{F}}_1$ act successively on $S_0(x), S_1x), S_2(x)$ $S_3(x)$, and $S_0(x), S_1(x), \cdots, S_4(x)$, respectively we determine the coefficients of the polynomials $\Phi_i^j(x)$, $(i,j=0,1)$, secondly, we let $\nabla_{\alpha-\beta}\tilde{\mathcal{F}}_1$ act on $S_0(x), S_1x)$ and $S_2(x)$ to determine the coefficients c_0, c_1 and ξ_1. ∎

Proposition 5.2 *For $\alpha \neq 0$, the functional $\mathcal{F}_0$ is solution of the equation*

$$\begin{aligned} &\nabla_{\alpha-\beta}\left\{\nabla_{\alpha-\beta}[\left(\alpha^2 x - \sigma_2 - \alpha\sigma_1 - \alpha^2\sigma_0\right)\mathcal{F}_0] - (2\alpha x - \sigma_1 - 2\alpha\sigma_0)\,\mathcal{F}_0\right\} \\ &+ (x-\sigma_0)\,\mathcal{F}_0 = 0, \end{aligned} \tag{5.1}$$

Proof. From the relation (3.6) we see that

$$\begin{cases} \nabla_{\alpha-\beta}\mathcal{F}_0 - \alpha\nabla_{\alpha-\beta}\mathcal{F}_1 = -\mathcal{F}_1 \\ -\dfrac{\alpha}{\sigma_2}\nabla_{\alpha-\beta}\left[(x-\sigma_0)\mathcal{F}_0\right] + \left(1+\alpha\dfrac{\sigma_1}{\sigma_2}\right)\nabla_{\alpha-\beta}\mathcal{F}_1 = -\dfrac{1}{\sigma_2}(x-\sigma_0)\mathcal{F}_0 + \dfrac{\sigma_1}{\sigma_2}\mathcal{F}_1 \end{cases}$$

and by substitution we obtain the relation

$$\mathcal{F}_1 = \nabla_{\alpha-\beta}\left\{\left[\frac{\alpha^2}{\sigma_2}(x-\sigma_0) - \left(1+\alpha\frac{\sigma_1}{\sigma_2}\right)\right]\mathcal{F}_0\right\} - \frac{\alpha}{\sigma_2}(x-\sigma_0)\mathcal{F}_0 \tag{5.2}$$

Therefore, letting $\nabla_{\alpha-\beta}$ act on this last one and replacing $\nabla_{\alpha-\beta}\mathcal{F}_1$ and $\mathcal{F}_1$ by there respective values with respect to $\nabla^2_{\alpha-\beta}\mathcal{F}_0$, $\nabla_{\alpha-\beta}\mathcal{F}_0$ and $\mathcal{F}_0$ in the first relation, we find the expected result. ∎

Remark 5.3 *In the case (**b**) (the case (**a**) if $\alpha = 0$), the relations (5.1) and (5.2) may be written respectively*

$$\begin{aligned} &D\left\{D[\left(\alpha^2 x - \sigma_2 - \alpha\sigma_1 - \alpha^2\sigma_0\right)\mathcal{F}_0] - \left(2\alpha x - \sigma_1 - 2\alpha\sigma_0\right)\mathcal{F}_0\right\} \\ &+ (x-\sigma_0)\,\mathcal{F}_0 = 0 \end{aligned} \tag{5.3}$$

and

$$\mathcal{F}_1 = \frac{\alpha}{\sigma_2}D\left\{\left[\alpha(x-\sigma_0) - \left(\frac{\sigma_2}{\alpha}+\sigma_1\right)\right]\mathcal{F}_0\right\} - \frac{\alpha}{\sigma_2}(x-\sigma_0)\mathcal{F}_0 \tag{5.4}$$

5.2 Determination of weight functions in the cases (a) and (b).

The problem consists now in representing the functional $\mathcal{F}_0$ and $\mathcal{F}_1$ as an integral by putting

$$\begin{cases} \langle \mathcal{F}_0, p(x)\rangle &= \int_C F_0(x)p(x)dx, \ \text{ and} \\ \langle \mathcal{F}_1, p(x)\rangle &= \int_C F_1(x)p(x)dx, \quad \forall p \in \mathcal{P} \end{cases} \tag{5.5}$$

where the weight functions $F_0(x)$ and $F_1(x)$ are supposed "booth regular " and C is a contour to be determined.

Proposition 5.4 *If F_0 is a weight function representing the functional $\mathcal{F}_0$ and C the contour of this representation, then F_0 and C must satisfy, respectively in the case (b) (the case (**a**) if $\alpha = 0$)*

$$\Theta(x)\frac{d^2F_0(x,\alpha)}{dx^2} + \left[\Omega(x)+2\alpha^2\right]\frac{dF_0(x,\alpha)}{dx} + \left[\Pi(x) - 2\alpha\right]F_0(x,\alpha) = 0 \tag{5.6}$$

and

$$\left[\Theta(x)F_0(x,\alpha)p'(x) - \left\{(\Theta(x)F_0(x,\alpha))' + \Omega(x)F_0(x,\alpha)\right\}p(x)\right]_C = 0, \quad \forall p \in \mathcal{P} \tag{5.7}$$

where

$$\begin{cases} \Theta(x) = \alpha^2 x - \left(\sigma_2 + \alpha\sigma_1 + \alpha^2\sigma_0\right) \\ \Omega(x) = -2\alpha x + \left(2\alpha^2 + \sigma_1 + 2\alpha\sigma_0\right) \\ \Pi(x) = x - \sigma_0. \end{cases}$$

Proof. A solution of the equation (5.3) must satisfy

$$\langle D\left\{D[\Theta(x)\mathcal{F}_0] + \Omega(x)\mathcal{F}_0\right\} + \Pi(x)\mathcal{F}, p(x)\rangle = 0, \qquad \forall p \in \mathcal{P}\ ,$$

i.e.

$$\langle \mathcal{F}_0, \Theta(x)p''(x)\rangle - \langle \mathcal{F}_0, \Omega(x)p'(x)) + \langle \mathcal{F}, \Pi(x)p(x)\rangle = 0,$$

as

$$\int_C \Theta(x)F_0(x,\alpha)p''(x)dx - \int_C \Omega(x)F_0(x,\alpha)p'(x)dx + \int_C \Pi(x)F_0(x,\alpha)p(x)dx = 0,$$

by an integration by parts we obtain

$$\begin{array}{l} \left[\Theta(x)F_0(x,\alpha)p'(x) - \{(\Theta(x)F_0(x,\alpha))' + \Omega(x)F_0(x,\alpha)\}\, p(x)\right]_C \\ + \int_C \left\{[\Theta(x)F_0(x,\alpha)]'' + [\Omega(x)F_0(x,\alpha)]' + \Pi(x)F_0(x,\alpha)\right\} p(x)dx = 0, \end{array}$$

in particular if we take

$$[\Theta(x)F_0(x,\alpha)]'' + [\Omega(x)F_0(x,\alpha)]' + \Pi(x)F_0(x,\alpha) = 0$$

and

$$\left[\Theta(x)F_0(x,\alpha)p'(x) - \{(\Theta(x)F_0(x,\alpha))' + \Omega(x)F_0(x,\alpha)\}\, p(x)\right]_C = 0, \ \forall p \in \mathcal{P}.$$

∎

Remark 5.5 *In the case* ***(a)****, the weight function F_0 and the contour C must satisfy respectively*

$$-\sigma_2 \frac{d^2F_0(x)}{dx^2} + \sigma_1 \frac{dF_0(x)}{dx} + (x - \sigma_0)F_0(x) = 0 \tag{5.8}$$

and

$$\left[-\sigma_2 \{F_0(x)p(x)\}' + \sigma_1 F_0(x)p(x)\right]_C = 0, \qquad \forall p \in \mathcal{P} \tag{5.9}$$

Theorem 5.6 *When $\sigma_2 < 0$, the differential equation (5.6) has a general solution*

$$F_0(x,\alpha) = (x+k)^{\frac{\lambda}{2}} e^{\frac{x}{\alpha}} \left\{ c_1 J_\lambda \left[q(x+k)^{\frac{1}{2}} \right] + c_2 Y_\lambda \left[q(x+k)^{\frac{1}{2}} \right] \right\} \tag{5.10}$$

where

$$\lambda = \left| \frac{\alpha^3 - \alpha\sigma_1 - 2\sigma_2}{\alpha^2} \right|, \quad k = -\frac{\alpha^2\sigma_0 + \alpha\sigma_1 + \sigma_2}{\alpha^2} \ \text{ and } \ q = \frac{\sqrt{-\sigma_2}}{\alpha^2}$$

and J_λ and Y_λ are the Bessel functions of first and second kind respectively.

Proof. The equation (5.6) can be written

$$\begin{array}{l} (x+k)\, \dfrac{d^2F_0(x,\alpha)}{dx^2} - 2\left[\dfrac{x}{\alpha} - \left(\dfrac{\sigma_0}{\alpha} + \dfrac{\sigma_1}{\alpha^2} + 2\right)\right] \dfrac{dF_0(x,\alpha)}{dx} \\ + \dfrac{1}{\alpha^2}\,(x - \sigma_0 - 2\alpha)\, F_0(x,\alpha) = 0 \end{array} \tag{5.11}$$

Let us denote by

$$r(x) = \frac{\frac{x}{\alpha} - \left(\frac{\sigma_0}{\alpha} + \frac{\sigma_1}{\alpha^2} + 2\right)}{x+k}$$

and put

$$F_0(x) = W(x) \exp\left[\int r(x)dx\right],$$

then equation (5.11) may be written

$$(x+k)^2 \frac{d^2W(x)}{dx^2} - \frac{\sigma_2}{\alpha^2}\left[x - \alpha - \sigma_0 - \frac{\sigma_1}{2\sigma_2}\left(\alpha^2 - \frac{\sigma_1}{2}\right)\right] W(x) = 0.$$

This last equation admits as a general solution

$$W(x) = (x+k)^{\frac{1}{2}} \left\{ c_1 J_\lambda \left[2q(x+k)^{\frac{1}{2}}\right] + c_2 Y_\lambda \left[2q(x+k)^{\frac{1}{2}}\right]\right\}.$$

as

$$\int r(x)dx = (x+k)^{\frac{\lambda-1}{2}} \exp(\frac{x}{\alpha}),$$

we find (5.10). ∎

Theorem 5.7 *In the case (**b**), choosing, as a contour, the interval* $C =]-k, \infty[$ *, then the function*

$$F_0^b(x,\alpha) = Const.(x+k)^{\frac{\lambda}{2}} e^{\frac{x}{\alpha}} J_\lambda \left[2\frac{\sqrt{-\sigma_2}}{\alpha^2}(x+k)^{\frac{1}{2}}\right], \alpha < 0 \quad \text{and } \sigma_2 < 0 \qquad (5.12)$$

is an integral representation of the functional $\mathcal{F}_0$*, i.e.*

$$\langle \mathcal{F}_0, p(x)\rangle = \int_C F_0^b(x,\alpha)p(x)dx, \quad \forall p \in \mathcal{P}.$$

Proof. We have

$$\begin{cases} \lim\limits_{x\to -k^+} F_0^b(x,\alpha) = \lim\limits_{x\to\infty} F_0^b(x,\alpha) = 0 \\ \lim\limits_{x\to -k^+} (x+k)F_0^b(x,\alpha) = \lim\limits_{x\to -k^+} (x+k)\frac{dF_0^b(x,\alpha)}{dx} = 0 \\ \lim\limits_{x\to\infty} (x+k)F_0^b(x,\alpha) = \lim\limits_{x\to\infty} \frac{dF_0^b(x,\alpha)}{dx} = 0 \end{cases}$$

consequently the condition (5.7) is satisfied.

As $F_0^b(x,\alpha)$ is a solution of the equation (5.6), with the choice of the interval $C =]-k, \infty[$ as a contour, $F_0^b(x,\alpha)$ may be an integral representation of $\mathcal{F}_0$. ∎

Corollary 5.8 *In the case (**b**), the function $F_1^b(x,\alpha)$ defined by*

$$F_1^b(x,\alpha) = \frac{\alpha}{\sigma_2}\left\{\left[\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right]\frac{dF_0^b(x,\alpha)}{dx} - (x-\sigma_0-\alpha)F_0^b(x,\alpha)\right\} \quad (5.13)$$

is an integral representation of $\mathcal{F}_1$.

Proof. From the relation (5.4) we have

$$\begin{aligned}\langle \mathcal{F}_1, p(x)\rangle &= \frac{\alpha}{\sigma_2}\langle D\left[\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right]\mathcal{F}_0 - (x-\sigma_0)\mathcal{F}_0, p(x)\rangle \\ &= \frac{\alpha}{\sigma_2}\langle D\left[\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right]\mathcal{F}_0, p(x)\rangle - \frac{\alpha}{\sigma_2}\langle (x-\sigma_0)\mathcal{F}_0, p(x)\rangle \\ &= -\frac{\alpha}{\sigma_2}\int_C \left[\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right]F_0^b(x,\alpha)p'(x)dx \\ &\quad -\frac{\alpha}{\sigma_2}\int_C (x-\sigma_0)\,F_0^b(x,\alpha)p(x)dx, \qquad \forall p\in\mathcal{P}.\end{aligned}$$

Integrating by parts the first term in the right hand side we find

$$\begin{aligned}\int_C F_1^b(x,\alpha)p(x)dx = &\ \frac{\alpha}{\sigma_2}\int_C\left[\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right]\frac{dF_0^b(x,\alpha)}{dx}p(x)dx \\ &-\frac{\alpha}{\sigma_2}\int_C (x-\sigma_0-\alpha)F_0^b(x,\alpha)p(x)dx \\ &-\frac{\alpha}{\sigma_2}\left[\left\{\alpha\,(x-\sigma_0) - \sigma_1 - \frac{\sigma_2}{\alpha}\right\}F_0^b(x,\alpha)p(x)\right]_C\end{aligned}$$

As the last term is zero we obtain the relation (5.13). ∎

Theorem 5.9 *When $\alpha = 0$ (the case (**a**)), the equation (5.8) admits as general solution*

$$\begin{aligned}F_0(x) &= \left(x-\sigma_0+\frac{\sigma_1^2}{4\sigma_2}\right)^{\frac{1}{2}}\exp(\frac{\sigma_1}{2\sigma_2}x)\left\{k_1 J_{\frac{1}{3}}\left[\frac{2}{3\sqrt{-\sigma_2}}\left(x-\sigma_0+\frac{\sigma_1^2}{4\sigma_2}\right)^{\frac{3}{2}}\right]\right. \\ &\left. + k_2 Y_{\frac{1}{3}}\left[\frac{2}{3\sqrt{-\sigma_2}}\left(x-\sigma_0+\frac{\sigma_1^2}{4\sigma_2}\right)^{\frac{3}{2}}\right]\right\}\end{aligned} \quad (5.14)$$

Proof. The equation (5.8) may also be written as

$$\frac{d^2F_0(x)}{dx^2} - \frac{\sigma_1}{\sigma_2}\frac{dF_0(x)}{dx} - \frac{1}{\sigma_2}(x-\sigma_0)F_0(x) = 0. \quad (5.15)$$

Let us put

$$F_0(x) = V(x)\exp(\frac{\sigma_1}{2\sigma_2}),$$

by substitution, V must then satisfy

$$\frac{d^2V(x)}{dx^2} - \frac{1}{\sigma_2}(x - \sigma_0 + \frac{\sigma_1^2}{4\sigma_2})V(x) = 0.$$

This equation is of the type $\frac{d^2V(X)}{dX^2} - \frac{1}{\sigma_2}XV(X)$, where $X = x - \sigma_0 + \frac{\sigma_1^2}{4\sigma_2}$, the general solution of which is

$$V(X) = X^{\frac{1}{2}}\left\{k_1 J_{\frac{1}{3}}\left(\frac{2}{3\sqrt{-\sigma_2}}X^{\frac{3}{2}}\right) + k_2 Y_{\frac{1}{3}}\left(\frac{2}{3\sqrt{-\sigma_2}}X^{\frac{3}{2}}\right)\right\}.$$

Going back to the initial variable x and the function F_0, we find (5.13). ∎

Theorem 5.10 *Choosing as a contour the interval* $C =]\sigma_0 - \frac{\sigma_1^2}{4\sigma_2}, \infty[$, *the function*

$$F_0^a(x) = Const.\left(x - \sigma_0 + \frac{(\sigma_1)^2}{4\sigma_2}\right)^{\frac{1}{2}} exp\ (\frac{\sigma_1}{2\sigma_2}x)J_{\frac{1}{3}}\left[\frac{2}{3\sqrt{-\sigma_2}}\left(x - \sigma_0 + \frac{(\sigma_1)^2}{4\sigma_2}\right)^{\frac{3}{2}}\right],$$

$\sigma_2 < 0$ *is an an integral representation of functional* $\mathcal{F}_0$ *in the case* ***(a)***.

Proof. As F_0^a is a solution of (5.8) and

$$\begin{cases} \lim\limits_{x\to a^+} F_0^a(x) = \lim\limits_{x\to\infty} F_0^a(x) = 0, \text{ and} \\ \lim\limits_{x\to a^+} \frac{dF_0^b(x)}{dx} = 0 = \lim\limits_{x\to\infty} \frac{dF_0^a(x)}{dx} = 0, \text{where } a = \frac{\sigma_1^2}{4\sigma_2} - \sigma_0, \end{cases}$$

the conditions of the proposition (5.2) are then satisfied and F_0^a is an integral representation of the functional $\mathcal{F}_0$. ∎

Corollary 5.11 *In the case* ***(a)****, the function* $F_1^a(x)$ *defined by*

$$F_1^a(x) = -\frac{dF_0^a(x)}{dx} \tag{5.16}$$

is an integral representation of the functional $\mathcal{F}_1$.

Proof. It suffices to note, according to (5.4), that $\mathcal{F}_1 = -D\mathcal{F}_0$. ∎

Remark 5.12 *We just proved that the class of* $2-$*orthogonal polynomials of Sheffer-Meixner type consists of* 9 *sequences.* 5 *of which are classical and* 2 *of them have continuous weight functions. The investigation of the last* 3 *sequences* $(c), (d1)$ *and* $(d2)$ *will be the subject of another talk.*

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