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Matrix Liapunov's Functions Method and Stability Analysis of Continuous Systems

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1 Introduction

This paper contains main results of qualitative analysis of motions for large scale dynamical systems described by ordinary differential equations in terms of matrix-valued Liapunov's functions.

The paper is arranged as folows.

Section 2 deals with stability problems for continuous large scale dynamical systems. The definitions and sufficient conditions for various types of motion stability of nonautonomous and nonlinear systems are presented. The main theorems of the Section are supplied with corollaries which illustrate the generality of the results obtained and indicate the sources for the assertions.

In Section 3 general theorems of Section 2 are suplied with a constructive algorithm of constructing the Liapunov functions in terms of matrix-valued auxiliary function. The conditions for various types of stability of zero solution for a wide class of large-scale systems are formulated in terms of the property of having a fixed sign of special matrices.

Section 4 sets out conditions of exponential stability with respect to a part of variables. These conditions are established in terms of matrix-valued function constructed by the method proposed in Section 3.

Liapunov functions for linear nonautonomous and autonomous systems in Section 5 are constructed by adapting general algorithm from Section 3.

Section 6 presents a discussion of the algorithm and a numerical example which demonstrate the efficiency of the proposed method of constructing the Liapunov's functions in terms of matrix-valued auxiliary function as compared with the Bellman–Bailey approach based on the vector Liapunov's functions.

In final Section 7 some unsolved problems of the method of matrix Liapunov's functions are presented.

Thus, this paper provides a development of the direct Liapunov method consisting in both the establishment of general theorems and proposition of a new method of constructing of appropriate Liapunov functions for some classes of linear and nonlinear dynamical systems.

2 The Direct Liapunov's Method via Matrix-Valued Functions

In this Section the notions of motion stability corresponding to the motion properties of nonautonomous systems are presented being necessary in subsequent presentation. Basic notions of the method of matrix-valued Liapunov functions are discussed and general theorems and some corollaries are set out.

Throughout this Section, real systems of ordinary differential equations will be considered. Notations will be used.

2.1 Stability concept in the sense of Liapunov

We consider systems which can appropriately be described by ordinary differential equations of the form

$$\frac{dy_i}{dt} = Y_i(t,\, y_1, \dots, y_n), \qquad i = 1, 2, \dots, n, \tag{2.1}$$

or in the equivalent vector form

$$\frac{dy}{dt} = Y(t, y), \tag{2.2}$$

where $x \in R^n$, $Y(t,y) = (Y_1(t,y), \dots, Y_n(t,y))^{\mathrm{T}}$, $Y\ \mathcal{T} \times R^n \to R^n$. In the present Section we will assume that the right-hand part of (2.2) satisfies the solution existence and uniqueness conditions of the Cauchy problem

$$\frac{dy}{dt} = Y(t, y), \qquad y(t_0) = y_0, \tag{2.3}$$

for any $(t_0, y_0) \in \mathcal{T} \times \Omega$, where $\Omega \subset R^n$, $0 \in \Omega$ and Ω is an open connected subset of $R^{\cdot}$.

It is clear that the solution of problem (2.3) may not exist on R (on R_+), even if the right-hand part $Y(t, y)$ of system (2.3) is definite and continuous for all $(t, y) \in \mathcal{T} \times R^n$.

For example, the Cauchy problem for the equation

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0, \tag{2.4}$$

where y is a scalar, has a unique solution $y(t) = \operatorname{tg} t$, existing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, only, while the right-hand part of equation (2.4) is definite on the whole plane (t, y).

Let $y(t) = \psi(t; t_0, y_0)$ be the solution of system (2.2), definite on the interval $[t_0, \tau)$ and noncontinuable behind the point τ, i.e˙ $y(t)$ is not definite for $t = \tau$. Then

$$\overline{\lim}\, \|y(t)\| = +\infty \quad \text{as} \quad t \to \tau - 0. \tag{2.5}$$

Using solution $y(t)$ and the right-hand part of system (2.2) we construct the vector-function

$$f(t, x) = Y(t,\, x + \psi(t)) - Y(t, \psi(t)) \tag{2.6}$$

and consider the system

$$\frac{dx}{dt} = f(t, x). \tag{2.7}$$

It is easy to verify that the solutions of systems (2.2) and (2.7) are correlated as

$$x(t) = y(t) - \psi(t)$$

on the general interval of existence of solutions $y(t)$ and $\psi(t)$. It is clear that system (2.7) has a trivial solution $x(t) \equiv 0$. This solution corresponds to the solution $y(t) = \psi(t)$ of system (2.2). Obviously, the reduction of system (2.2) to system (2.7) is possible only when the solution $y(t) = \psi(t)$ is known.

Qualitative investigation of solutions of system (2.2) relatively solution $\psi(t)$ is reduced to the investigation of behaviour of solution $x(t)$ to system (2.7) which differs "little" from the trivial one for $t = t_0$.

In case when stability of unperturbed motion is discussed with respect to some continuously differentiable functions $Q_s(t, \psi_1, \ldots, \psi_n)$ the perturbed motion equations are found by the system of equations

$$\frac{dx_i}{dt} = \frac{\partial Q_i}{\partial t} + \sum_{s=1}^{n} \frac{\partial Q_i}{\partial \psi_s} Y_s(t, \psi_1, \ldots, \psi_n) - \dot{F}(t),$$

where $x_i = Q_i(t, \psi_1(t), \ldots, \psi_n(t)) - F_s(t)$, and $F_i(t) = Q_i(t,\, \psi_1(t),\, \ldots,\, \psi_n(t))$ are some known time functions. The system of equations of (2.7) type obtained hereat satisfies the condition $f(t, 0) = 0$ for all $t \in \mathcal{T}$ and therefore these system has a trivial solution in this case as well.

In motion stability theory system (2.7) is called the system of perturbed motion equations.

Since equations (2.7) can generally not be solved analytically in closed from, the qualitative properties of the equilbrium state are of great practical interest. We begin with a series of definitions.

A very large number of definitions of stability exist for the system (2.7). Of course, the various definitions of stability can be broadly classified as those which deal with the trajectory, or a motion, or the equilibrium of the null solution of free or unforced systems and those which consider the dynamic response of systems subject to various classes of forcing functions or inputs. In the following, the equilibrium state of (2.7) can always be set equal to zero by a linear state transformation, so that the equilibrium state and the null solution to (2.7) are considered throughout as equivalent.

Definition 2.1 The equilibrium state $x = 0$ of the system (2.7) is:

(i) *stable* iff for every $t_0 \in \mathcal{T}_i$ and every $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$, such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies

$$\|x(t; t_0, x_0)\| < \varepsilon \quad \text{for all} \quad t \in \mathcal{T}_0;$$

(ii) *uniformly stable* iff both (i) holds and for every $\varepsilon > 0$ the corresponding maximal δ_M obeying (i) satisfies

$$\inf\,[\delta_M(t, \varepsilon)\;\; t \in \mathcal{T}_i] > 0;$$

(iii) *stable in the whole* iff both (i) holds and

$$\delta_M(t, \varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to +\infty \quad \text{for all} \quad t \in R;$$

(iv) *uniformly stable in the whole* iff both (ii) and (iii) hold;

(v) *unstable* iff there are $t_0 \in \mathcal{T}_i$, $\varepsilon \in (0, +\infty)$ and $\tau \in \mathcal{T}_0$, $\tau > t_0$, such that for every $\delta \in (0, +\infty)$ there is x_0, $\|x_0\| < \delta$, for which

$$\|x(\tau; t_0, x_0)\| \geq \varepsilon.$$

Definition 2.2 The equilibrium state $x = 0$ of the system (2.7) is:

(i) *attractive* iff for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for every $\zeta > 0$ there exists $\tau(t_0; x_0, \zeta) \in [0, +\infty)$ such that $\|x_0\| < \Delta(t_0)$ implies $\|x(t; t_0, x_0)\| < \zeta$ for all $t \in (t_0 + \tau(t_0; x_0, \zeta),\, +\infty)$;

(ii) x_0-*uniformly attractive* iff both (i) is true and for every $t_0 \in R$ there exists $\Delta(t_0) > 0$ and for every $\zeta \in (0, +\infty)$ there exists $\tau_u[t_0, \Delta(t_0), \zeta] \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0; x_0, \zeta)\;\; x_0 \in B_\Delta(t_0)] = \tau_u(t_0, x_0, \zeta);$$

(iii) t_0-*uniformly attractive* iff both (i) is true, there is $\Delta > 0$ and for every $(x_0, \zeta) \in B_\Delta \times (0, +\infty)$ there exists $\tau_u(R, x_0, \zeta) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0); x_0, \zeta)\;\; t_0 \in \mathcal{T}_i] = \tau_u(\mathcal{T}_i, x_0, \zeta);$$

(iv) *uniformly attractive* iff both (ii) and (iii) hold, that is, that (i) is true, there exists $\Delta > 0$ and for every $\zeta \in (0, +\infty)$ there is $\tau_u(R, \Delta, \zeta) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0; x_0, \zeta)\ (t_0, x_0) \in \mathcal{T}_i \times B_\Delta] = \tau(\mathcal{T}_i, \Delta, \zeta);$$

The properties (i) – (iv) hold *"in the whole"* iff (i) is true for every $\Delta(t_0) \in (0, +\infty)$ and every $t_0 \in \mathcal{T}_i$.

Definition 2.3 The equilibrium state $x = 0$ of the system (2.7) is:

(i) *asymptotically stable* iff it is both stable and attractive;

(ii) *equi-asymptotically stable* iff it is both stable and x_0-uniformly attractive;

(iii) *quasi-uniformly asymptotically stable* iff it is both uniformly stable and t_0-uniformly attractive;

(iv) *uniformly asymptotically stable* iff it is both uniformly stable and uniformly attractive;

(v) The properties (i) – (iv) hold *"in the whole"* iff both the corresponding stability of $x = 0$ and the corresponding attraction of $x = 0$ hold in the whole;

(vi) *exponentially stable* iff there are $\Delta > 0$ and real numbers $\alpha \geq 1$ and $\beta > 0$ such that $\|x_0\| < \Delta$ implies

$$\|x(t; t_0, x_0)\| \leq \alpha \|x_0\| \exp[-\beta(t - t_0)], \quad \text{for all} \quad t \in \mathcal{T}_0, \quad \text{for all} \quad t_0 \in \mathcal{T}_i.$$

This holds *in the whole* iff it is true for $\Delta = +\infty$.

Let $g\ : R^n \to R^n$ define the time invariant system

$$\frac{dx}{dt} = g(x), \tag{2.8}$$

where $g(0) = 0$ and the components of g are smooth functions of the components of x for x near zero. Every stability property of $x = 0$ of (2.11) is uniform in $t_0 \in R$.

Note that the nonperturbed motion equations of the time invariant system can be reduced to the time invariant system (2.11) iff the solution $\psi(t) = \text{const}$. Otherwise, i.eif $\psi(t) \neq \text{const}$, equations (2.11) can be nonstationary.

In the investigation of both system (2.2) and (2.11) the solution $x(t)$ is assumed to be definite for all $t \in \mathcal{T}$ (for all $t \in \mathcal{T}_0$).

2.2 Classes of Liapunov's functions

Presently the Liapunov direct method (see Liapunov [1]) in terms of three classes of auxiliary functions: scalar, vector and matrix ones is intensively applied in qualitative theory. In this point we shall present the description of the above mentioned classes of functions.

2.2.1 Matrix-valued Liapunov function

For the system (2.7) we shall consider a continuous matrix-valued function

$$U(t,x) = [v_{ij}(t,x)], \quad i,j = 1,2,\dots,m, \tag{2.9}$$

where $v_{ij} \in C(\mathcal{T}_\tau \times R^n, R)$ for all $i,j = 1,2,\dots,m$. We assume that next conditions are fulfilled

(i) $v_{ij}(t,x)$, $i,j = 1,2,\dots,m$, are locally Lipschitzian in x;

(ii) $v_{ij}(t,0) = 0$ for all $t \in R_+$ $(t \in \mathcal{T}_\tau)$, $i,j = 1,2,\dots,m$;

(iii) $v_{ij}(t,x) = v_{ji}(t,x)$ in any open connected neighbourhood $\mathcal{N}$ of point $x = 0$ for all $t \in R_+$ $(t \in \mathcal{T}_\tau)$.

Definition 2.4 All function of the type

$$v(t,x,\alpha) = \alpha^{\mathrm{T}} U(t,x)\alpha, \quad \alpha \in R^m, \tag{2.10}$$

where $U \in C(\mathcal{T}_\tau \times \mathcal{N},\, R^{m\times m})$, are attributed to the *class* SL.

Here the vector α can be specified as follows:

(i) $\alpha = y \in R^m$, $y \neq 0$;

(ii) $\alpha = \xi \in C(R^n,\, R^m_+)$, $\xi(0) = 0$;

(iii) $\alpha = \psi \in C(\mathcal{T}_\tau \times R^n,\, R^m_+)$, $\psi(t,0) = 0$;

(iv) $\alpha = \eta \in R^m_+$, $\eta > 0$.

Note that the choice of vector α can influence the property of having a fixed sign of function (2.13) and its total derivative along solutions of system (2.7).

2.2.2 Comparison functions

Comparison functions are used as upper or lower estimates of the function v and its total time derivative. They are usually denoted by φ, $\varphi : R_+ \to R_+$. The main contributor to the investigation of properties of and use of the comparison functions is Hahn [2]. What follows is mainly based on his definitions and results.

Definition 2.5 A *function* φ, $\varphi : R_+ \to R_+$, belongs to

(i) the *class* $K_{[0,\alpha)}$, $0 < \alpha \leq +\infty$, iff both it is defined, continuous and strictly increasing on $[0,\alpha)$ and $\varphi(0) = 0$;

(ii) the *class* K iff (i) holds for $\alpha = +\infty$, $K = K_{[0,+\infty)}$;

(iii) the *class* KR iff both it belongs to the class K and $\varphi(\zeta) \to +\infty$ as $\zeta \to +\infty$;

(iv) the *class* $L_{[0,\alpha)}$ iff both it is defined, continuous and strictly decreasing on $[0,\alpha)$ and $\lim\,[\varphi(\zeta)\ : \zeta \to +\infty] = 0$;

(v) the *class* L iff (iv) holds for $\alpha = +\infty,\ L = L_{[0,+\infty)}$.

Let φ^{-1} denote the *inverse function* of $\varphi,\ \varphi^{-1}[\varphi(\zeta)] \equiv \zeta$.
The next result was established by Hahn [2].

Proposition 2.1

1. *If $\varphi \in K$ and $\psi \in K$ then $\varphi(\psi) \in K$;*
2. *If $\varphi \in K$ and $\sigma \in L$ then $\varphi(\sigma) \in L$;*
3. *If $\varphi \in K_{[0,\alpha)}$ and $\varphi(\alpha) = \xi$ then $\varphi^{-1} \in K_{[0,\xi)}$;*
4. *If $\varphi \in K$ and* $\lim\,[\varphi(\zeta)\ : \zeta \to +\infty] = \xi$ *then φ^{-1} is not defined on $(\xi, +\infty]$;*
5. *If $\varphi \in K_{[0,\alpha)},\ \psi \in K_{[0,\alpha)}$ and $\varphi(\zeta) > \psi(\zeta)$ on $[0,\alpha)$ then $\varphi^{-1}(\zeta) < \psi^{-1}(\zeta)$ on $[0,\beta]$, where $\beta = \psi(\alpha)$.*

Definition 2.6 A *function* $\varphi,\ \varphi\ : R_+ \times R_+ \to R_+$, belongs to:

(i) the *class* $KK_{[0;\alpha,\beta)}$ iff both $\varphi(0,\zeta) \in K_{[0,\alpha)}$ for every $\zeta \in [0,\beta)$ and $\varphi(\zeta,0) \in K_{[0,\beta)}$ for every $\zeta \in [0,\alpha)$;

(ii) the *class* KK iff (i) holds for $\alpha = \beta = +\infty$;

(iii) the *class* $KL_{[0;\alpha,\beta)}$ iff both $\varphi(0,\zeta) \in K_{[0,\alpha)}$ for every $\zeta \in [0,\beta)$ and $\varphi(\zeta,0) \in L_{[0,\beta)}$ for every $\zeta \in [0,\alpha)$;

(iv) the *class* KL iff (iii) holds for $\alpha = \beta = +\infty$;

(v) the *class* CK iff $\varphi(t,0) = 0,\ \varphi(t,u) \in K$ for every $t \in R_+$;

(vi) the *class* $\mathcal{M}$ iff $\varphi \in C(R_+ \times R^n, R_+),\ \inf \varphi(t,x) = 0,\ (t,x) \in R_+ \times R^n$;

(vii) the *class* $\mathcal{M}_0$ iff $\varphi \in C(R_+ \times R^n, R_+),\ \inf\limits_x \varphi(t,x) = 0$ for each $t \in R_+$;

(viii) the *class* Φ iff $\varphi \in C(K, R_+)$: $\varphi(0) = 0$, and $\varphi(w)$ is increasing with respect to cone K.

Definition 2.7 Two *functions* $\varphi_1, \varphi_2 \in K$ (or $\varphi_1, \varphi_2 \in KR$) are said to be *of the same order of magnitude* if there exist positive constants α, β, such that

$$\alpha\varphi_1(\zeta) \le \varphi_2(\zeta) \le \beta\varphi_1(\zeta) \quad \text{for all} \quad \zeta \in [0,\zeta_1] \quad (\text{or for all} \quad \zeta \in [0,\infty)).$$

2.2.3 Properties of matrix-valued functions

For the functions of the class SL we shall cite some definitions which are applied in the investigation of dynamics of system (2.7).

Definition 2.8 The *matrix-valued function* $U : \mathcal{T}_\tau \times R^n \to R^{m\times m}$ is:

(i) *positive semi-definite on* $\mathcal{T}_\tau = [\tau, +\infty)$, $\tau \in R$, iff there are time-invariant connected neighbourhood $\mathcal{N}$ of $x = 0$, $\mathcal{N} \subseteq R^n$, and vector $y \in R^m$, $y \neq 0$, such that

  (a) $v(t,x,y)$ is continuous in $(t,x) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$;

  (b) $v(t,x,y)$ is non-negative on $\mathcal{N}$, $v(t,x,y) \geq 0$ for all $(t,x,y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$, and

  (c) vanishes at the origin: $v(t,0,y) = 0$ for all $t \in \mathcal{T}_\tau \times R^m$;

  (d) iff the conditions (a)$-$(c) hold and for every $t \in \mathcal{T}_\tau$, there is $w \in \mathcal{N}$ such that $v(t,w,y) > 0$, then v is strictly positive semi-definite on $\mathcal{T}_\tau$;

(ii) *positive semi-definite on* $\mathcal{T}_\tau \times \mathcal{G}$ iff (i) holds for $\mathcal{N} = \mathcal{G}$;

(iii) *positive semi-definite in the whole on* $\mathcal{T}_\tau$ iff (i) holds for $\mathcal{N} = R^n$;

(iv) *negative semi-definite (in the whole) on* $\mathcal{T}_\tau$ *(on* $\mathcal{T}_\tau \times \mathcal{N}$*)* iff $(-v)$ is positive semi-definite (in the whole) on $\mathcal{T}_\tau$ (on $\mathcal{T}_\tau \times \mathcal{N}$) respectively.

The expression "*on* $\mathcal{T}_\tau$" is omitted iff all corresponding requirements hold for every $\tau \in R$.

Definition 2.9 The *matrix-valued function* $U : \mathcal{T}_\tau \times R^n \to R^{m\times m}$ is:

(i) *positive definite on* $\mathcal{T}_\tau$, $\tau \in R$, iff there are a time-invariant connected neighbourhood $\mathcal{N}$ of $x = 0$, $\mathcal{N} \subseteq R^n$ and a vector $y \in R^m$, $y \neq 0$, such that both it is positive semi-definite on $\mathcal{T}_\tau \times \mathcal{N}$ and there exists a positive definite function w on $\mathcal{N}$, $w : R^n \to R_+$, obeying $w(x) \leq v(t,x,y)$ for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$;

(ii) *positive definite on* $\mathcal{T}_\tau \times \mathcal{G}$ iff (i) holds for $\mathcal{N} = \mathcal{G}$;

(iii) *positive definite in the whole on* $\mathcal{T}_\tau$ iff (i) holds for $\mathcal{N} = R^n$;

(iv) *negative definite (in the whole) on* $\mathcal{T}_\tau$ *(on* $\mathcal{T}_\tau \times \mathcal{N} \times R^m$*)* iff $(-v)$ is positive definite (in the whole) on $\mathcal{T}_\tau$ (on $\mathcal{T}_\tau \times \mathcal{N} \times R^m$) respectively;

(v) *weakly decrescent* if there exists a $\Delta_1 > 0$ and a function $a \in CK$ such that $v(t,x,y) \leq a(t, \|x\|)$ as soon as $\|x\| < \Delta_1$;

(vi) *asymptotically decrescent* if there exists a $\Delta_2 > 0$ and a function $b \in KL$ such that $v(t,x,y) \leq b(t, \|x\|)$ as soon as $\|x\| < \Delta_2$.

The expression *"on* $\mathcal{T}_\tau$*"* is omitted iff all corresponding requirements hold for every $\tau \in R$.

Proposition 2.2 *The matrix-valued function* $U : R \times R^n \to R^{m\times m}$ *is positive definite on* $\mathcal{T}_\tau$, $\tau \in R$, *iff it can be written as*

$$y^T U(t,x)y = y^T U_+(t,x)y + a(\|x\|),$$

where $U_+(t,x)$ *is a positive semi-definite matrix-valued function and* $a \in K$.

Definition 2.10 (cf Grujić, *et al.* [1]) Set $v_\zeta(t)$ is *the largest connected neighborhood of* $x = 0$ *at* $t \in R$ which can be associated with a function $U : R \times R^n \to R^{m\times m}$ so that $x \in v_\zeta(t)$ implies $v(t,x,y) < \zeta$, $y \in R^m$.

Definition 2.11 The matrix-valued function $U : R \times R^n \to R^{s\times s}$ is:

(i) *decreasing on* $\mathcal{T}_\tau$, $\tau \in R$, iff there is a time-invariant neighborhood $\mathcal{N}$ of $x = 0$ and a positive definite function w on $\mathcal{N}$, $w : R^n \to R_+$, such that $y^{\mathrm{T}} U(t,x)y \le w(x)$ for all $(t,x) \in \mathcal{T}_\tau \times \mathcal{N}$;

(ii) *decreasing on* $\mathcal{T}_\tau \times \mathcal{G}$ iff (i) holds for $\mathcal{N} = \mathcal{G}$;

(iii) *decreasing in the whole on* $\mathcal{T}_\tau$ iff (i) holds for $\mathcal{N} = R^n$.

The expression *"on* $\mathcal{T}_\tau$*"* is omitted iff all corresponding conditions still hold for every $\tau \in R$.

Proposition 2.3 *The matrix-valued function* $U : R \times R^n \to R^{m\times m}$ *is decreasing on* $\mathcal{T}_\tau$, $\tau \in R$, *iff it can be written as*

$$y^T U(t,x)y = y^T U_-(t,x)y + b(\|x\|), \quad (y \ne 0) \in R^m,$$

where $U_-(t,x)$ *is a negative semi-definite matrix-valued function and* $b \in K$.

Definition 2.12 The matrix-valued function $U : R \times R^n \to R^{m\times m}$ is:

(i) *radially unbounded on* $\mathcal{T}_\tau$, $\tau \in R$, iff $\|x\| \to \infty$ implies $y^{\mathrm{T}} U(t,x)y \to +\infty$ for all $t \in \mathcal{T}_\tau$, $y \in R^m$, $y \ne 0$;

(ii) *radially unbounded*, iff $\|x\| \to \infty$ implies $y^{\mathrm{T}} U(t,x)y \to +\infty$ for all $t \in \mathcal{T}_\tau$ for all $\tau \in R$, $y \in R^m$, $y \ne 0$.

Proposition 2.4 *The matrix-valued function* $U : \mathcal{T}_\tau \times R^n \to R^{m\times m}$ *is radially unbounded in the whole (on* $\mathcal{T}_\tau$*) iff it can be written as*

$$y^T U(t,x)y = y^T U_+(t,x)y + a(\|x\|) \quad \textit{for all} \quad x \in R^n,$$

where $U_+(t,x)$ *is a positive semi-definite matrix-valued function in the whole (on* $\mathcal{T}_\tau$*) and* $a \in KR$.

According to Liapunov function (2.17) is applied in motion investigation of system (2.7) together with its total derivative along solutions $x(t) = x(t; t_0, x_0)$ of system

(2.7). Assume that each element $v_{ij}(t,x)$ of the matrix-valued function (2.16) is definite on the open set $\mathcal{T}_\tau \times \mathcal{N}$, $\mathcal{N} \subset R^n$, i.e. $v_{ij}(t,x) \in C(\mathcal{T}_\tau \times \mathcal{N},\, R)$.

If $\gamma(t;t_0,x_0)$ is a solution of system (2.7) with the initial conditions $x(t_0) = x_0$, i.e. $\gamma(t_0;t_0,x_0) = x_0$, the right-hand upper derivative of function (2.17) for $\alpha = y$, $y \in R^m$, with respect to t along the solution of (2.7) is determined by the formula

$$D^+v(t,x,y) = y^{\mathrm{T}} D^+U(t,x)y, \tag{2.11}$$

where $D^+U(t,x) = [D^+v_{ij}(t,x)]$, $i,j = 1,2,\dots,m$, and

$$\begin{aligned} D^+v_{ij}(t,x) = \limsup \Big\{ &\sup_{\gamma(t,t,x)=x} [v_{ij}(t+\sigma,\, \gamma(t+\sigma,\, t,x)) \\ &- v_{ij}(t,x)]\,\sigma^{-1} \;:\; \sigma \to 0^+ \Big\}, \quad i,j = 1,2,\dots,m. \end{aligned} \tag{2.12}$$

In case when system (2.7) has a unique solution for every initial value of $x(t_0) = x_0$ $((t_0,x_0) \in \mathcal{T}_\tau \times \mathcal{N})$, the expression (2.19) is equivalent to

$$\begin{aligned} &D^+v_{ij}(t,x) = \limsup\{[v_{ij}(t+\sigma,\, \gamma(t+\sigma,\, t,x)) \\ &- v_{ij}(t,x)]\,\sigma^{-1} \;:\; \sigma \to 0^+\}, \quad i,j = 1,2,\dots,m. \end{aligned} \tag{2.13}$$

Further we assume that for all $i,j = 1,2,\dots,m$ the functions $v_{ij}(t,x)$ are continuous and locally Lipschitzian in x, i.e. for every point in $\mathcal{N}$ there exists a neighbourhood Δ and a positive number $L = L(\Delta)$ such that

$$|v_{ij}(t,x) - v_{ij}(t,y)| \le L\|x-y\|, \quad i,j = 1,2,\dots,m,$$

for any $(t,x) \in \mathcal{T}_\tau \times \Delta$, $(t,y) \in \mathcal{T}_\tau \times \Delta$. Besides, the expression (2.12) is equivalent to

$$\begin{aligned} &D^+v_{ij}(t,x) = \limsup\{[v_{ij}(t+\sigma,\, x+\sigma f(t,x)) \\ &- v_{ij}(t,x)]\,\sigma^{-1} \;:\; \sigma \to 0^+\}, \quad i,j = 1,2,\dots,m. \end{aligned} \tag{2.14}$$

If the matrix-valued function $U(t,x) \in C^1(\mathcal{T}_\tau \times \mathcal{N},\, R^{m\times m})$, i.eall its elements $v_{ij}(t,x)$ are functions continuously differentiable in t É x, then the expression (2.14) is equivalent to

$$Dv_{ij}(t,x) = \frac{\partial v_{ij}}{\partial t}(t,x) + \sum_{s=1}^{n} \frac{\partial v_{ij}}{\partial x_s}(t,x)\, f_s(t,x), \tag{2.15}$$

where $f_s(t,x)$ are components of the vector-function $f(t,x) = (f_1(t,x),\dots,f_n(t,x))^{\mathrm{T}}$.

Function (2.15) has the Euler derivative (2.10) at point (t,x) along solution $x(t;t_0,x_0)$ of system (2.7) iff

$$\begin{aligned} D^+v(t,x,y) &= D_+v(t,x,y) = D^-v(t,x,y) \\ &= D_-v(t,x,y) = Dv(t,x,y). \end{aligned} \tag{2.16}$$

Note that the application of any of the expressions (2.12), (2.13) or (2.15) in (2.11) is admissible.

2.2.4 Vector Liapunov function

A vector-valued Liapunov function

$$V(t,x) = (v_1(t,x), v_2(t,x), \dots, v_m(t,x))^{\mathrm{T}} \tag{2.17}$$

can be obtained via matrix-valued function (2.9) in several ways.

Definition 2.13 All vector functions of the type

$$L(t,x,b) = AU(t,x)b, \tag{2.18}$$

where $U \in C(\mathcal{T}_\tau \times R^n,\ R^{s\times s})$, A is a constant matrix $s \times s$, and vector b is defined according to (i) – (iv) similarly to the definition of the vector α, are attributed to the *class VL*.

If in two-index system of functions (2.9) for all $i \neq j$ the elements $v_{ij}(t,x) = 0$, then

$$v(t,x) = \mathrm{diag}\ (v_{11}(t,x),\ \dots,\ v_{mm}(t,x))^{\mathrm{T}},$$

where $v_{ii} \in C(\mathcal{T}_\tau \times R^n,\ R)$, $i = 1,2,\dots,m,$, is a vector-valued function. Besides, the function (2.18) has the components

$$L_k(t,x,b) = \sum_{i=1}^{m} a_{ki} b_i v_{ii}(t,x), \quad k = 1,2,\dots,m.$$

The methods of application of Liapunov's vector functions in motion stability theory are presented in a number of monographs some of which are mentioned in the end of this section.

2.2.5 Scalar Liapunov function

The simplest type of auxiliary function for system (2.7) is the function

$$v(t,x) \in C(\mathcal{T}_0 \times R^n,\ R_+), \quad v(t,0) = 0, \tag{2.19}$$

for which

(a) $v(t,0) = 0$ for all $t \in \mathcal{T}_\tau$;

(b) $v(t,x)$ is locally Lipschitzian in x;

(c) $v(t,x) \in C(\mathcal{T}_\tau \times \mathcal{N},\ R)$.

In stability theory both sign-definite in the sense of Liapunov and semi-definite functions (see Hahn [2]) are applied. We shall set out some examples.

Example 2.1

(i) The function
$$v(t,x) = (1+\sin^2 t)x_1^2 + (1+\cos^2 t)x_2^2$$
is positive definite and decreasing, while the function
$$v(t,x) = (x_1^2 + x_2^2)\sin^2 t$$
is decreasing and positive semi-definite.

(ii) The function
$$v(t,x) = x_1^2 + (1+t)x_2^2$$
is positive definite but not non-decreasing, while the function
$$v(t,x) = x_1^2 + \frac{x_2^2}{1+t}$$
is decreasing but not positive definite.

(iii) The function
$$v(t,x) = (1+t)(x_1 - x_2)^2$$
is positive semi-definite and non-decreasing.

Among the variety of the Liapunov functions the quadratic forms
$$v(t,x) = x^{\mathrm{T}} P(t)x, \quad P^{\mathrm{T}}(t) = P(t), \tag{2.20}$$
are of special importance, where $P(t)$ is $n \times n$-matrix with continuous and bounded elements for all $t \in \mathcal{T}_\tau$.

Proposition 2.5 *For the quadratic form (2.20) to be positive definite it is necessary and sufficient that*
$$\begin{vmatrix} p_{11}(t) & \dots & p_{1s}(t) \\ \dots & \dots & \dots \\ p_{s1}(t) & \dots & p_{ss}(t) \end{vmatrix} > k > 0, \qquad s = 1,2,\dots,n, \tag{2.21}$$
for all $t \in \mathcal{T}_\tau$.

Note that if conditions (2.21) are satisfied, the "quasi-quadratic" form
$$v(t,x) = x^{\mathrm{T}} P(t)x + \psi(t,x), \quad \psi(t,0) = 0, \tag{2.22}$$
is positive definite, provided that some constants a, b $(a > 0,\ b \geq 2)$ exist, such that $|\psi(t,x)| \leq ar^b$, where $r = (x^{\mathrm{T}}x)^{1/2}$.

To calculate total derivative of function (2.22) along solutions of system (2.7) either (2.12)–(2.15) is applied for $i = j = 1$, depending on the assumptions on system (2.7) and function (2.22).

It is well known (see Yoshizawa [1]) that if $D^+v(t,x) \le 0$ and consequently $D^+v(t,x(t)) \le 0$, then the function $v(t,x)$ is nonincreasing function of $t \in \mathcal{T}_\tau$. Further, if $D^+v(t,x) \ge 0$, then $v(t,x(t))$ is nondecreasing along any solution of (2.7) and vice versa.

We shall formulate these observations as follows.

Proposition 2.6 *Suppose $m(t) = v(t,x(t))$ is continuous on (a,b). Then $m(t)$ is nondecreasing (nonincreasing) on (a,b) if and only if $D^+m(t) \ge 0$ (≤ 0) for every $t \in (a,b)$, where*

$$D^+m(t) = \lim\sup\{[m(t+\sigma) - m(t)]\,\sigma^{-1} \;:\; \sigma \to 0^+\}.$$

Further all auxiliary functions allowing the solution of the problem on stability (instability) of the equilibrium state $x = 0$ of system (2.7) are called the Liapunov functions. The construction of the Liapunov functions still remains one of the central problems of stability theory.

2.3 Liapunov's like theorems in general

Functions (2.10), (2.18) and (2.20) together with their total derivatives (2.11) along solutions of system (2.7) allow to establish existence conditions for the motion properties of system (2.7) of various types such as stability, instability, boundedness, etc. Below we shall set out some results in the direction.

Theorem 2.1 *Let the vector-function f in system (2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

1. *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
2. *a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m\times m})$ and a vector $y \in R^m$ such that the function $v(t,x,y) = y^T U(t,x) y$ is locally Lipschitzian in x for all $t \in R$ $(t \in \mathcal{T}_\tau)$;*
3. *functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K$, $\widetilde{\psi}_{i2} \in CK$, $i = 1,2,\dots,m$;*
4. *$m \times m$ matrices $A_j(y)$, $j = 1,2,3$, $\widetilde{A}_2(y)$ such that*
   - (a) *$\psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \le v(t,x,y) \le \widetilde{\psi}_2^T(t,\|x\|)\widetilde{A}_2(y)\widetilde{\psi}_2(t,\|x\|)$ for all $(t,x,y) \in R \times \mathcal{G} \times R^m$ (for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$);*
   - (b) *$\psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \le v(t,x,y) \le \psi_2^T(\|x\|)A_2(y)\psi_2(\|x\|)$ for all $(t,x,y) \in R \times \mathcal{G} \times R^m$ (for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$);*
   - (c) *$D^+v(t,x,y) \le \psi_3^T(\|x\|)A_3(y)\psi_3(\|x\|)$ for all $(t,x,y) \in R \times \mathcal{G} \times R^m$ (for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$).*

Then, if the matrices $A_1(y)$, $A_2(y)$, $\widetilde{A}_2(y)$, $(y \ne 0) \in R^m$ are positive definite and $A_3(y)$ is negative semi-definite, then

(a) *the state* $x = 0$ *of system (2.7) is stable (on* $\mathcal{T}_\tau$*), provided condition (4)(a) is satisfied;*

(b) *the state* $x = 0$ *of system (2.7) is uniformly stable (on* $\mathcal{T}_\tau$*), provided condition (4)(b) is satisfied.*

Corollary 2.1 *Let*

1. *condition (1) of Theorem 2.1 be satisfied;*
2. *there exist at least one couple of indices* $(p, q) \in [1, m]$ *for which* $(v_{pq}(t, x) \neq 0) \in U(t, x)$ *and function* $v(t, x, e) = e^T U(t, x)e = v(t, x)$ *for all* $(t, x) \in R \times \mathcal{G}$ *(for all* $(t, x) \in \mathcal{T}_\tau \times \mathcal{G}$*) satisfy the conditions*
   (a) $\psi_1(\|x\|) \leq v(t, x)$;
   (b) $v(t, x) \leq \psi_2(\|x\|)$;
   (c) $D^+ v(t, x)|_{(2.7)} \leq 0$,

   where ψ_1, ψ_2 *are some functions of the class* K.

Then, the state $x = 0$ *of system (2.7) is stable (on* $\mathcal{T}_\tau$*) under conditions (a) and (c), and uniformly stable (on* $\mathcal{T}_\tau$*) under conditions (a)–(c).*

Theorem 2.2 *Let the vector-function* f *in system (2.7) be continuous on* $R \times R^n$ *(on* $\mathcal{T}_\tau \times R^n$*). If there exist*

1. *a matrix-valued function* $U \in C(R \times R^n, R^{m\times m})$ $(U \in C(\mathcal{T}_\tau \times R^n, R^{m\times m}))$ *and a vector* $y \in R^m$ *such that the function* $v(t, x, y) = y^T U(t, x)y$ *is locally Lipschitzian in* x *for all* $t \in R$ $(t \in \mathcal{T}_\tau)$*;*
2. *functions* $\varphi_{1i}, \varphi_{2i}, \varphi_{3i} \in KR$, $\widetilde{\varphi}_{2i} \in CKR$, $i = 1, 2, \dots, m$*;*
3. $m \times m$ *matrices* $B_j(y)$, $j = 1, 2, 3$, $\widetilde{B}_2(y)$ *such that*
   (a) $\varphi_1^T(\|x\|)B_1(y)\varphi_1(\|x\|) \leq v(t, x, y) \leq \widetilde{\varphi}_2^T(t, \|x\|)\widetilde{B}_2(y)\widetilde{\varphi}_2(t, \|x\|)$ *for all* $(t, x, y) \in R \times R^n \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m$*);*
   (b) $\varphi_1^T(\|x\|)B_1(y)\varphi_1(\|x\|) \leq v(t, x, y) \leq \varphi_2^T(\|x\|)B_2(y)\varphi_2(\|x\|)$ *for all* $(t, x, y) \in R \times R^n \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m$*);*
   (c) $D^+ v(t, x, y) \leq \varphi_3^T(\|x\|)B_3(y)\varphi_3(\|x\|)$ *for all* $(t, x, y) \in R\times R^n \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m$*).*

Then, provided that matrices $B_1(y)$, $B_2(y)$ *and* $\widetilde{B}_2(y)$ *for all* $(y \neq 0) \in R^m$ *are positive definite and matrix* $B_3(y)$ *is negative semi-definite,*

(a) *under condition 3(a) the state* $x = 0$ *of system (2.7) is stable in the whole (on* $\mathcal{T}_\tau$*);*

(b) under condition 3(b) the state $x = 0$ of system (2.7) is uniformly stable in the whole (on $\mathcal{T}_\tau$).

Corollary 2.2 *Let for function $v(t, x, e) = v(t, x)$, mentioned in condition 2 of Corollary 2.1 for all $(t, x) \in R \times R^n$ (for all $(t, x) \in \mathcal{T}_\tau \times R^n$) the following conditions hold*

(a) $\varphi_1(\|x\|) \le v(t, x)$;

(b) $v(t, x) \le \varphi_2(\|x\|)$, *for some function* φ_2;

(c) $D^+ v(t, x)|_{(2.7)} \le 0$,

where φ_1, φ_2 are of class KR.

Then the state $x = 0$ of system (2.7) is stable in the whole (on $\mathcal{T}_\tau$) under conditions (a) and (c) and uniformly stable in the whole (on $\mathcal{T}_\tau$) under conditions (a)–(c).

Theorem 2.3 *Let the vector-function f in system (2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

1. *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
2. *a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m \times m})$ $(U \in C(\mathcal{T}_\tau \times \mathcal{N}, R^{m\times m}))$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for all $t \in R$ $(t \in \mathcal{T}_\tau)$;*
3. *functions $\eta_{1i}, \eta_{2i}, \eta_{3i} \in K$, $\widetilde{\eta}_{2i} \in CK$, $i = 1, 2, \dots, m$;*
4. *$m \times m$ matrices $C_j(y)$, $j = 1, 2, 3$, $\widetilde{C}_2(y)$ such that*
   - *(a)* $\eta_1^T(\|x\|) C_1(y) \eta_1(\|x\|) \le v(t, x, y) \le \widetilde{\eta}_2^T(t, \|x\|) \widetilde{C}_2(y) \widetilde{\eta}_2(t, \|x\|)$ *for all* $(t, x, y) \in R \times \mathcal{G} \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$*);*
   - *(b)* $\eta_1^T(\|x\|) C_1(y) \eta_1(\|x\|) \le v(t, x, y) \le \eta_2^T(\|x\|) C_2(y) \eta_2(\|x\|)$ *for all* $(t, x, y) \in R \times \mathcal{G} \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$*);*
   - *(c)* $D^* v(t, x, y) \le \eta_3^T(\|x\|) C_3(y) \eta_3(\|x\|) + m(t, \eta_3(\|x\|))$ *for all* $(t, x, y) \in R \times \mathcal{G} \times R^m$ *(for all* $(t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$*), where function $m(t, \cdot)$ satisfies the condition*

$$\lim \frac{|m(t, \eta_3(\|x\|))|}{\|\eta_3\|} = 0 \quad \text{as} \quad \|\eta_3\| \to 0$$

   uniformly in $t \in R$ $(t \in \mathcal{T}_\tau)$.

Then, provided the matrices $C_1(y)$, $C_2(y)$, $\widetilde{C}_2(y)$ are positive definite and matrix $C_3(y)$ $(y \neq 0) \in R^m$ is negative definite, then

(a) under condition 4(a) the state $x = 0$ of the system (2.7) is asymptotically stable (on $\mathcal{T}_\tau$);

(b) *under condition 4(b) the state $x = 0$ of the system (2.7) is uniformly asymptotically stable (on $\mathcal{T}_\tau$).*

Corollary 2.3 *Let*

1. *condition 1 of Theorem 2.2 be satisfied;*
2. *for function $v(t, x, e) = v(t, x)$, mentioned in condition 2 of Corollary 2.1 for all $(t, x) \in R \times \mathcal{G}$ (for all $(t, x) \in \mathcal{T}_\tau \times \mathcal{G}$))*
   (a) $\psi_1(\|x\|) \le v(t, x) \le \psi_2(\|x\|)$;
   (b) $D^+v(t, x)|_{(2.7)} \le -\psi_3(\|x\|)$, *where ψ_1, ψ_2, ψ_3 are of class K.*

Then the state $x = 0$ of system (2.7) is uniformly asymptotically stable (on $\mathcal{T}_\tau$).

Theorem 2.4 *Let the vector-function f in system (2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$) and conditions 1 – 3 of Theorem 2.2 be satisfied.*

Then, provided that matrices $B_1(y)$, $B_2(y)$ and $\widetilde{B}_2(y)$ are positive definite and matrix $B_3(y)$ for all $(y \ne 0) \in R^m$ is negative definite,

(a) *under condition 3(a) of Theorem 2.2 the state $x = 0$ of system (2.7) is asymptotically stable in the whole (on $\mathcal{T}_\tau$);*

(b) *under condition 3(b) of Theorem 2.2 the state $x = 0$ of system (2.7) is uniformly asymptotically stable in the whole (on $\mathcal{T}_\tau$).*

Corollary 2.4 *For function $v(t, x, e) = v(t, x)$, mentioned in condition 2 of Corollary 2.1 for all $(t, x) \in R \times R^n$ (for all $(t, x) \in \mathcal{T}_\tau \times R^n$) let*

(a) $\varphi_1(\|x\|) \le v(t, x) \le \varphi_2(\|x\|)$;

(b) $D^+v(t, x)|_{(2.7)} \le -\psi_3(\|x\|)$,

where φ_1, φ_2 are of class KR and ψ_3 is of class K.

Then the state $x = 0$ of system (2.7) is uniformly stable in the whole (on $\mathcal{T}_\tau$).

Theorem 2.5 *Let the vector-function f in system (2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

1. *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
2. *a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m\times m})$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^TU(t, x)y$ is locally Lipschitzian in x for all $t \in R$ ($t \in \mathcal{T}_\tau$);*
3. *functions $\sigma_{2i}, \sigma_{3i} \in K$, $i = 1, 2, \dots, m$, a positive real number Δ_1 and positive integer p, $m \times m$ matrices $F_2(y)$, $F_3(y)$ such that*

(a) $\Delta_1\|x\|^p \le v(t,x,y) \le \sigma_2^T(\|x\|)F_2(y)\sigma_2(\|x\|)$ *for all* $(t,x,y\neq 0)\in R\times\mathcal{G}\times R^m$ *(for all* $(t,x,y\neq 0)\in\mathcal{T}_\tau\times\mathcal{G}\times R^m$*);*

(b) $D^+v(t,x,y) \le \sigma_3^T(\|x\|)F_3(y)\sigma_3(\|x\|)$ *for all* $(t,x,y\neq 0)\in R\times\mathcal{G}\times R^m$ *(for all* $(t,x,y\neq 0)\in\mathcal{T}_\tau\times\mathcal{G}\times R^m$*).*

Then, provided that the matrices $F_2(y)$, $(y\neq 0)\in R^m$ *are positive definite, the matrix* $F_3(y)$, $(y\neq 0)\in R^m$ *is negative definite and functions* σ_{2i}, σ_{3i} *are of the same magnitude, then the state* $x=0$ *of system (2.7) is exponentially stable (on* $\mathcal{T}_\tau$*).*

Corollary 2.5 Let

1. condition (1) of Theorem 2.1 be satisfied;

2. for function $v(t,x,e)=v(t,x)$, mentioned in condition (2) of Corollary 2.1 for all $(t,x)\in R\times\mathcal{G}$ (for all $(t,x)\in\mathcal{T}_\tau\times\mathcal{G}$)

   (a) $c_1\|x\|^p \le v(t,x) \le \varphi_1(\|x\|)$,

   (b) $D^+v(t,x)|_{(2.7)} \le -\varphi_2(\|x\|)$.

Then, if the functions φ_1,φ_2 are of class K and of the same magnitude, the state $x=0$ of system (2.7) is exponentially stable (on $\mathcal{T}_\tau$).

Theorem 2.6 *Let the vector-function* f *in system (2.7) be continuous on* $R\times R^n$ *(on* $\mathcal{T}_\tau\times R^n$*). If there exist*

1. *a matrix-valued function* $U\in C\left(R\times R^n,\, R^{m\times m}\right)$ $(U\in C(\mathcal{T}_\tau\times R^n, R^{m\times m}))$ *and a vector* $y\in R^m$ *such that the function* $v(t,x,y)=y^TU(t,x)y$ *is locally Lipschitzian in x for all* $t\in R$ *(for all* $t\in\mathcal{T}_\tau$*);*

2. *functions* ν_{2i}, $\nu_{3i}\in KR$, $i=1,2,\dots,m$, *a positive real number* $\Delta_2>0$ *and a positive integer q;*

3. $m\times m$ *matrices* H_2, H_3 *such that*

   (a) $\Delta_2\|x\|^q \le v(t,x,y) \le \nu_2^T(\|x\|)H_2(y)\nu_2(\|x\|)$ *for all* $(t,x,y\neq 0)\in R\times R^n\times R^m$ *(for all* $(t,x,y)\in\mathcal{T}_\tau\times R^n\times R^m$*);*

   (b) $D^+v(t,x,y) \le \nu_3^T(\|x\|)H_3(y)\nu_3(\|x\|)$ *for all* $(t,x,y\neq 0)\in R\times R^n\times R^m$ *(for all* $(t,x,y\neq 0)\in\mathcal{T}_\tau\times R^n\times R^m$*).*

Then, if the matrix $H_2(y)$ *for all* $(y\neq 0)\in R^m$ *is positive definite, the matrix* $H_3(y)$ *for all* $(y\neq 0)\in R^m$ *is negative definite and functions* ν_{2i}, ν_{3i} *are of the same magnitude, the state* $x=0$ *of system (2.7) is exponentially stable in the whole (on* $\mathcal{T}_\tau$*).*

Corollary 2.6 For function $v(t,x,e)=v(t,x)$, mentioned in condition (2) of Corollary 2.1 for all $(t,x)\in R\times R^n$ (for all $(t,x)\in R^n\times\mathcal{G}$) let

(a) $c_2\|x\|^q \le v(t,x) \le \psi_1(\|x\|)$,

(b) $D^+v(t,x)|_{(2.7)} \le -\psi_2(\|x\|)$,

where $\psi_1, \psi_2 \in KR$–class and are of the same magnitude.

Then the state $x = 0$ of system (2.7) is exponentially stable in the whole (on $\mathcal{T}_\tau$).

Theorem 2.7 *Let the vector-function f in system (2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

1. *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
2. *a matrix-valued function $U \in C^1(R \times \mathcal{N},\, R^{m\times m})$ $(U \in C^1(\mathcal{T}_\tau \times \mathcal{N}, R^{m\times m}))$ and a vector $y \in R^m$;*
3. *functions $\psi_{1i}, \psi_{2i}, \psi_{3i} \in K$, $i = 1,2,\dots,m$, $m \times m$ matrices $A_1(y)$, $A_2(y)$, $G(y)$ and a constant $\Delta > 0$ such that*
   - *(a) $\psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \le v(t,x,y) \le \psi_2^T(\|x\|)A_2(y)\psi_2(\|x\|)$ for all $(t,x,y) \in R \times \mathcal{G} \times R^m$ (for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$);*
   - *(b) $D^+v(t,x,y) \ge \psi_3^T(\|x\|)G(y)\psi_3(\|x\|)$ for all $(t,x,y) \in R\times \mathcal{G} \times R^m$ (for all $(t,x,y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m$);*
4. *point $x = 0$ belongs to $\partial\mathcal{G}$;*
5. *$v(t,x,y) = 0$ on $\mathcal{T}_0 \times (\partial\mathcal{G} \cap B_\Delta)$, where $B_\Delta = \{x : \|x\| < \Delta\}$.*

Then, if matrices $A_1(y)$, $A_2(y)$ and $G(y)$ for all $(y \ne 0) \in R^m$ are positive definite, the state $x = 0$ of system (2.7) is unstable (on $\mathcal{T}_\tau$).

Corollary 2.7 Let

1. condition (1) of Theorem 2.7 be satisfied;
2. there exist at least one couple of indices $(p,q) \in [1,m]$ such that $(v_{\underline{pq}}(t,x) \ne 0) \in U(t,x)$ and a function $v(t,x,e) = v(t,x) \in C^1(R \times B_\Delta,\, R_+)$, $\overline{B}_\Delta \subset \mathcal{G}$, such that on $\mathcal{T}_0 \times \mathcal{G}$
   - (a) $0 < v(t,x) \le a < +\infty$, for some $a > 0$;
   - (b) $D^+v(t,x)|_{(2.7)} \ge \varphi(v(t,x))$ for some function φ of class K;
   - (c) point $x = 0$ belongs to $\partial\mathcal{G}$;
   - (d) $v(t,x) = 0$ on $\mathcal{T}_0 \times (\partial\mathcal{G} \cap B_\Delta)$.

Then the state $x = 0$ of the system (2.7) is unstable.

We shall pay our attention to some specific features of the functions applied in Corollary 2.7.

Function $v(t,x)$ specifies the domain $v(t,x) > 0$, which is changing for $t \in \mathcal{T}_\tau$. Clearly this domain may cease its existence before the instability of motion is discovered.

If the function $v(t,x)$ is positive definite (strictly positive semi-definite), then the domain $v(t,x) > 0$ exists for all $t \in \mathcal{T}_\tau$.

If the function $v(t,x)$ is constant negative, the domain $v(t,x) > 0$ does not exist.

Example 2.2

(i) Function
$$v(t,x) = \sin t\, x_1 x_2$$
is of variable sign and domain $v(t,x) > 0$ exists but not for all $t \in \mathcal{T}_\tau$.

(ii) For the function
$$v(t,x) = (\cos t - 2)\, x_1^2 x_2$$
the domain $v(t,x) > 0$ exists for all $t \in \mathcal{T}_\tau$.

(iii) For the function
$$v(t,x) = \left(\frac{1}{t} - a\right) x_1 x_2 - x_2^2, \qquad a > 0,$$
the domain $v(t,x) > 0$ exists for all $t \geq t_0$, and for $t_0 > 1/a$.

Corollary 2.8 Let condition (1) of Theorem 2.7 be satisfied. If there exist $t_0 \in \mathcal{T}_0$, $\Delta > 0$, $(\overline{B}_\Delta \subset \mathcal{N})$ and an open set $\mathcal{G} \subset B_\Delta$ and the function $v(t,x,e) = v(t,x) \in C^1(\mathcal{T}_0 \times B_\Delta,\, R)$, mentioned in Corollary 2.7 such that on $\mathcal{T}_0 \times \mathcal{G}$

(a) $0 < v(t,x) \leq \varphi_1(\|x\|)$;

(b) $D^+ v(t,x)|_{(2.7)} \geq \varphi_2(\|x\|)$ for some φ_1, φ_2 of class K;

(c) point $x = 0$ belongs to $\partial\mathcal{G}$;

(d) $v(t,x) = 0$ on $\mathcal{T}_0 \times (\partial\mathcal{G} \cap B_\Delta)$.

Then the state $x = 0$ of (2.7) is unstable.

Corollary 2.9 If in Corollary 2.8 condition (b) is replaced by

(b′) $D^+ v(t,x)|_{(2.7)} \geq k v(t,x) + w(t,x)$

on $\mathcal{T}_0 \times \mathcal{G}$, where $k > 0$ and function $w(t,x) \geq 0$ is continuous on $\mathcal{T}_0 \times \mathcal{G}$, then the state $x = 0$ of system (2.7) is unstable.

3 Formulas of Liapunov Matrix-Valued Functions

The two-index system of functions (2.9) being suitable for construction of the Lyapunov functions allows to involve more wide classes of functions as compared with those ussually applied in motion stability theory. For example, the bilinear forms prove to be natural non-diagonal elements of matrix-valued functions. Another peculiar feature of the approach being of importance is the fact that the application of the matrix-valued function in the investigation of multidimensional systems enables to allow for the interconnections between the subsystems in their natural form, i.eṅot necessarily as the destabilizing factor. Finally, for the determination of the property of having a fixed sign of the total derivative of auxiliary function along solutions of the system under consideration it is not necessary to encorporate the estimation functions with the quasimonotonicity property. Naturally, the awkwardness of calculations in this case is the price.

3.1 A class of large-scale systems

We consider a system with finite number of degrees of freedom whose motion is described by the equations (3.1)

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, \dots, x_m), \qquad i = 1, 2, \dots, m \tag{3.1}$$

where $x_i \in R^{n_i}$, $t \in \mathcal{T}_\tau$, $\mathcal{T}_\tau = [\tau, +\infty)$, $f_i \in C(R^{n_i}, R^{n_i})$, $g_i \in C(\mathcal{T}_\tau \times \mathcal{R}^{\backslash\infty} \times \dots \times \mathcal{R}^{\backslash\updownarrow}, \mathcal{R}^{\backslash\rangle})$.

Introduce the designation

$$G_i(t, x) = g_i(t, x_1, \dots, x_m) - \sum_{j=1,\, j \neq i}^{m} g_{ij}(t, x_i, x_j), \tag{3.2}$$

where $g_{ij}(t, x_i, x_j) = g_i(t, 0, \dots, x_i, \dots, x_j, \dots, 0)$ for all $i \neq j$; $i, j = 1, 2, \dots, m$. Taking into consideration (3.2) system (3.1) is rewritten as

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_{j=1,\, j \neq i}^{m} g_{ij}(t, x_i, x_j) + G_i(t, x). \tag{3.3}$$

Actually equations (3.3) describe the class of large-scale nonlinear nonautonomously connected systems. It is of interest to extend the method of matrix Liapunov functions to this class of equations in view of the new method of construction of nondiagonal elements of matrix-valued functions.

3.2 Formulae for non-diagonal elements of matrix-valued function

In order to extend the method of matrix Liapunov functions to systems (3.3) it is necessary to estimate variation of matrix-valued function elements and their total derivatives along solutions of the corresponding systems. Such estimates are provided by the assumptions below.

Assumption 3.1 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibriums state $x_i = 0$, functions $v_{ii} \in C^1(R^{n_i}, R_+)$, the comparison functions φ_{i1}, φ_{i2} and ψ_i of class $K(KR)$ and real numbers $\underline{c}_{ii} > 0$, $\bar{c}_{ii} > 0$ and γ_{ii} such that

1. $v_{ii}(x_i) = 0$ for all $(x_i = 0) \in \mathcal{N}_i$;
2. $\underline{c}_{ii}\varphi_{i1}^2(\|x_i\|) \le v_{ii}(x_i) \le \bar{c}_{ii}\varphi_{i2}^2(\|x_i\|)$;
3. $(D_{x_i}v_{ii}(x_i))^{\mathrm{T}} f_i(x_i) \le \gamma_{ii}\psi_i^2(\|x_i\|)$ for all $x_i \in \mathcal{N}_i$, $i = 1, 2, \dots, m$.

It is clear that under conditions of Assumption 3.1 the equilibrium states $x_i = 0$ of nonlinear isolated subsystems

$$\frac{dx_i}{dt} = f_i(x_i), \qquad i = 1, 2, \dots, m \tag{3.4}$$

are

(a) *uniformly asymptotically stable in the whole*, if $\gamma_{ii} < 0$ and $(\varphi_{i1},\, \varphi_{i2},\, \psi_i) \in KR$-class;

(b) *stable*, if $\gamma_{ii} = 0$ and $(\varphi_{i1},\, \varphi_{i2}) \in K$-class;

(c) *unstable*, if $\gamma_{ii} > 0$ and $(\varphi_{i1},\, \varphi_{i2},\, \psi_i) \in K$-class.

The approach proposed in this section takes large scale systems (3.3) into consideration, subsystems (3.4) having various dynamical properties specified by conditions of Assumption 3.1

Assumption 3.2 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, functions $v_{ij} \in C^{1,1,1}(\mathcal{T}_\tau \times \mathcal{R}^{\rangle} \times \mathcal{R}^{\rceil}, \mathcal{R})$, comparison functions $\varphi_{i1}, \varphi_{i2} \in K(KR)$, positive constants $(\eta_1, \dots, \eta_m)^{\mathrm{T}} \in R^m$, $\eta_i > 0$ and arbitrary constants $\underline{c}_{ij}$, $\bar{c}_{ij}$, $i,\, j = 1, 2, \dots, m$, $i \ne j$ such that

1. $v_{ij}(t, x_i, x_j) = 0$ for all $(x_i, x_j) = 0 \in \mathcal{N}_i \times \mathcal{N}_j$, $t \in \mathcal{T}_\tau$, $i,\, j = 1, 2, \dots, m$, $(i \ne j)$;
2. $\underline{c}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j1}(\|x_j\|) \le v_{ij}(t, x_i, x_j) \le \bar{c}_{ij}\varphi_{i2}(\|x_i\|)\varphi_{j2}(\|x_j\|)$
   for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_\rangle \times \mathcal{N}_\rceil$, $i \ne j$;
3. $$D_t v_{ij}(t, x_i, x_j) + (D_{x_i}v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_i(x_i) + (D_{x_j}v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_j(x_j) + \frac{\eta_i}{2\eta_j}\,(D_{x_i}v_{ii}(x_i))^{\mathrm{T}} g_{ij}(t, x_i, x_j) + \frac{\eta_j}{2\eta_i}\,(D_{x_j}v_{jj}(x_j))^{\mathrm{T}} g_{ji}(t, x_i, x_j) = 0; \tag{3.5}$$

It is easy to notice that first order partial equations (3.5) are a somewhat variation of the classical Liapunov equation proposed for determination of auxiliary function in the theory of his direct method of motion stability investigation. In a particular case these equations are transformed into the systems of algebraic equations whose solutions can be constructed analytically.

Assumption 3.3 There exist open connected neighbourhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, comparison functions $\psi \in K(KR)$, $i = 1, 2, \dots, m$, real numbers α_{ij}^1, α_{ij}^2, α_{ij}^3, ν_{ki}^1, ν_{kij}^1, μ_{kij}^1 and μ_{kij}^2, $i,\, j,\, k = 1, 2, \dots, m$, such that

1. $(D_{x_i}v_{ii}(x_i))^{\mathrm{T}}G_i(t,x) \le \psi_i(\|x_i\|)\sum\limits_{k=1}^{m}\nu_{ki}^1\psi(\|x_k\|) + R_1(\psi)$ for all $(t,x_i,x_j) \in \mathcal{T}_\tau \times \mathcal{N}_\rangle \times \mathcal{N}_|$;
2. $(D_{x_i}v_{ij}(t,\cdot))^{\mathrm{T}}g_{ij}(t,x_i,x_j) \le \alpha_{ij}^1\psi_i^2(\|x_i\|)+\alpha_{ij}^2\psi_i(\|x_i\|)\psi_j(\|x_j\|)+\alpha_{ij}^3\psi_j^2(\|x_j\|)+ R_2(\psi)$ for all $(t,x_i,x_j) \in \mathcal{T}_\tau \times \mathcal{N}_\rangle \times \mathcal{N}_|$;
3. $(D_{x_i}v_{ij}(t,\cdot))^{\mathrm{T}}G_i(t,x) \le \psi_j(\|x_j\|)\sum\limits_{k=1}^{m}\nu_{ijk}^2\psi_k(\|x_k\|) + R_3(\psi)$ for all $(t,x_i,x_j) \in \mathcal{T}_\tau \times \mathcal{N}_\rangle \times \mathcal{N}_|$;
4. $(D_{x_i}v_{ij}(t,\cdot))^{\mathrm{T}}g_{ik}(t,x_i,x_k) \le \psi_j(\|x_j\|)(\mu_{ijk}^1\psi_k(\|x_k\|) + \mu_{ijk}^2\psi_i(\|x_i\|)) + R_4(\psi)$ for all $(t,x_i,x_j) \in \mathcal{T}_\tau \times \mathcal{N}_\rangle \times \mathcal{N}_|$.

Here $R_s(\psi)$ are polynomials in $\psi = (\psi_1(\|x_1\|,\dots,\psi_m(\|x_m\|))$ in a power higher than three, $R_s(0)=0$, $s=1,\dots,4$.

Under conditions (2) of Assumptions 3.1 and 3.2 it is easy to establish for function

$$v(t,x,\eta) = \eta^{\mathrm{T}}U(t,x)\eta = \sum_{i,j=1}^{m} v_{ij}(t,\cdot)\eta_i\eta_j \tag{3.6}$$

the bilateral estimate

$$u_1^{\mathrm{T}}H^{\mathrm{T}}\underset{\sim}{C}Hu_1 \le v(t,x,\eta) \le u_2^{\mathrm{T}}H^{\mathrm{T}}\bar{C}Hu_2, \tag{3.7}$$

where

$$u_1 = (\varphi_{11}(\|x_1\|,\dots,\varphi_{m1}(\|x_m\|))^{\mathrm{T}},$$
$$u_2 = (\varphi_{12}(\|x_1\|,\dots,\varphi_{m2}(\|x_m\|))^{\mathrm{T}}$$

which holds true for all $(t,x) \in \mathcal{T}_\tau \times \mathcal{N}$, $\mathcal{N} = \mathcal{N}_1 \times \dots \times \mathcal{N}_m$.

Based on conditions (3) of Assumptions 3.1, 3.2 and conditions (1)−(4) of Assumption 3.3 it is easy to establish the inequality estimating the auxiliary function variation along solutions of system (3.3). This estimate reads

$$Dv(t,x,\eta)\big|_{(2.1)} \le u_3^{\mathrm{T}}Mu_3, \tag{3.8}$$

where $u_3 = (\psi_1(\|x_1\|),\dots,\psi_m(\|x_m\|)$ and holds for all $(t,x) \in \mathcal{T}_\tau \times \mathcal{N}$.

Elements σ_{ij} of matrix M in the inequality (3.8) have the following structure

$$\begin{aligned}
\sigma_{ii} &= \eta_i^2\gamma_{ii} + \eta_i^2\nu_{ii} + \sum_{k=1,\,k\ne i}^{m}(\eta_k\eta_i\nu_{kii}^2 + \eta_i^2\nu_{kii}^2) + 2\sum_{j=1,\,j\ne i}^{m}\eta_i\eta_j(\alpha_{ij}^1+\alpha_{ji}^3);\\
\sigma_{ij} &= \tfrac{1}{2}\,(\eta_i^2\nu_{ji}^1 + \eta_j^2\nu_{ij}^1) + \sum_{k=1,\,k\ne j}^{m}\eta_k\eta_j\nu_{kij}^2 + \sum_{k=1,\,k\ne i}^{m}\eta_i\eta_j\nu_{kij}^2 + \eta_i\eta_j(\alpha_{ij}^2+\alpha_{ji}^2)\\
&\quad + \sum_{k=1,\,k\ne i,k\ne j}^{m}(\eta_k\eta_j\mu_{kji}^1 + \eta_i\eta_j\mu_{ijk}^2 + \eta_i\eta_k\mu_{kij}^1 + \eta_i\eta_j\mu_{jik}^2),\\
&\quad i = 1,2,\dots,m, \qquad i \ne j.
\end{aligned}$$

3.3 Theorems on stability

Sufficient criteria of various types of stability of the equilibrium state $x = 0$ of system (3.3) are formulated in terms of the sign definiteness of matrices $\underline{C}$, $\bar{C}$ and M from estimates (3.7), (3.8). We shall show that the following assertion is valid.

Theorem 3.1 *Assume that the perturbed motion equations are such that all conditions of Assumptions 3.1– 3.3 are fulfilled and moreover*

1. *matrices $\underline{C}$ and $\overline{C}$ in estimate (3.7) are positive definite;*

2. *matrix M in inequality (3.8) is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (3.3) is uniformly stable (uniformly asymptotically stable).

If, additionally, in conditions of Assumptions 3.1– 3.3 all estimates are satisfied for $\mathcal{N}_i = R^{n_i}$, $R_k(\psi) = 0$, $k = 1, \dots, 4$ and comparison functions $(\varphi_{i1}, \varphi_{i2}) \in KR$-class, then the equilibrium state of system (3.3) is uniformly stable in the whole (uniformly asymptotically stable in the whole).

Proof If all conditions of Assumptions 3.1– 3.2 are satisfied, then it is possible for system (3.3) to construct function $v(t, x, \eta)$ which together with total derivative $Dv(t, x, \eta)$ satisfies the inequalities (3.7) and (3.8). Condition (1) of Theorem 4.1 implies that function $v(t, x, \eta)$ is positive definite and decreasing for all $t \in \mathcal{T}_\tau$. Under condition (2) of Theorem 4.1 function $Dv(t, x, \eta)$ is negative semi-definite (definite). Therefore all conditions of Theorem 2.3.1, 2.3.3 from Martynyuk [/] are fulfilled. The proof of the second part of Theorem 4.1 is based on Theorem 2.3.4 from the same monograph. ∎

An example of non-linear systems Consider the non-linear system

$$\frac{dx_i}{dt} = a_{ii}x_i + \sum_{j=1, j\neq i}^{n} a_{ij}(x_j)x_j, \tag{3.9}$$

where $x_i \in R$, $a_{ii} < 0$ for $i = 1, 2, \dots, n$.

We assume on functions $a_{ij}(x)$ as follows.

Assumption 3.4 There exist constants $\Delta > 0$, $\varepsilon > 0$ and $Q > 0$ such that

1. $a_{ij}(x) \in C(R \setminus (-\varepsilon, \varepsilon), R)$

2. $|a_{ij}(x)| < Q|\tau|^{\gamma_{ij}+\Delta}$ for all $\tau \in (-\varepsilon, \varepsilon)$ $i, j = 1, 2, \dots, n$, $i \neq j$,

where $\gamma_{ji} = -(a_{ii} + a_{jj})/a_{ii}$, $\gamma_{ij} = -(a_{ii} + a_{jj})/a_{jj}$.

For each scalar subsystem

$$\frac{dx_i}{dt} = a_{ii}x_i, \quad i = 1, 2, \dots, n, \tag{3.10}$$

we take an auxiliary function in the form $v_{ii} = x_i^2$. Non-diagonal elements of matrix-valued function $U(x)$ are found as pseudo-quadratic forms $v_{ij}(x_i, x_j) = p(x_i, x_j)x_i x_j$. Basing on equation (3.5) of Assumption 3.2 for $\eta = (1, 1, \dots, 1)^{\mathrm{T}}$ we get

$$a_{ii}x_i \frac{\partial v_{ij}}{\partial x_i} + a_{jj}x_j \frac{\partial v_{ij}}{\partial x_j} = -[a_{ij}(x_j) + a_{ji}(x_i)]x_i x_j. \tag{3.11}$$

In view that the partial derivatives of functions $v_{ij}(x_i, x_j)$ are

$$\begin{aligned} \frac{\partial v_{ij}}{\partial x_i} &= p_{ij}(x_i, x_j)x_j + \frac{\partial p_{ij}}{\partial x_i} x_i x_j, \\ \frac{\partial v_{ij}}{\partial x_j} &= p_{ij}(x_i, x_j)x_i + \frac{\partial p_{ij}}{\partial x_j} x_i x_j, \end{aligned}$$

we find from equations (3.11)

$$a_{ii}x_i \frac{\partial p_{ij}}{\partial x_i} + a_{jj}x_j \frac{\partial p_{ij}}{\partial x_j} + (a_{ii} + a_{jj})p_{ij}(x_i, x_j) = -a_{ij}(x_j) - a_{ji}(x_i). \tag{3.12}$$

Further function $p_{ij}(x_i, x_j)$ is found as a sum of two functions $p_{ij}(x_i, x_j) = q_1(x_i) + q_2(x_j)$. Besides equation (3.12) becomes

$$a_{ii}x_i \frac{dq_1}{dx_i} + (a_{ii}+a_{jj})q_1(x_i) + a_{ji}(x_i) = -a_{jj}x_j \frac{dq_2}{dx_j} - (a_{ii}+a_{jj})q_2(x_j) - a_{ij}(x_j). \tag{3.13}$$

The right-hand part of (3.13) depends on x_i, while the left-hand part of (3.13) depends on x_j, therefore the right-hand and the left-hand parts equal to a constant which is set equal to zero

$$\begin{aligned} a_{ii}x_i \frac{dq_1}{dx_i} + (a_{ii} + a_{jj})q_1(x_i) + a_{ji}(x_i) &= 0, \\ a_{jj}x_j \frac{dq_2}{dx_j} + (a_{ii} + a_{jj})q_2(x_j) + a_{ij}(x_j) &= 0. \end{aligned} \tag{3.14}$$

The corresponding homogeneous equations

$$\begin{aligned} a_{ii}x_i \frac{d\widetilde{q}_1}{dx_i} + (a_{ii} + a_{jj})\widetilde{q}_1(x_i) &= 0, \\ a_{jj}x_j \frac{d\widetilde{q}_2}{dx_j} + (a_{ii} + a_{jj})\widetilde{q}_2(x_j) &= 0 \end{aligned} \tag{3.15}$$

have general solutions

$$lcl\widetilde{q}_1(x_i) = C_1|x_i|^{\gamma_{ji}}, \quad \widetilde{q}_2(x_j) = C_2|x_j|^{\gamma_{ij}}$$

respectively. To find partial solutions to equations (3.14) the method of variation of a constant is applied. If these solutions are presented as

$$q_1(x_i) = C_1(x_i)|x_i|^{\gamma_{ji}}, \quad q_2(x_j) = C_2(x_j)|x_j|^{\gamma_{ij}}$$

with the initial conditions $C_1(0) = C_2(0) = 0$, it is easy to find that

$$\begin{aligned} q_1(x_i) &= -|x_i|^{\gamma_{ji}} \int\limits_0^{x_i} \frac{a_{ji}(\tau)\,\mathrm{sign}\;\tau}{a_{ii}|\tau|^{1+\gamma_{ji}}}\,d\tau, \\ q_2(x_j) &= -|x_j|^{\gamma_{ij}} \int\limits_0^{x_j} \frac{a_{ij}(\tau)\,\mathrm{sign}\;\tau}{a_{jj}|\tau|^{1+\gamma_{ij}}}\,d\tau, \end{aligned} \tag{3.16}$$

where

$$\mathrm{sign}\;\tau \triangleq \begin{cases} -1, & \text{for} \quad \tau < 0, \\ \in [-1,1], & \text{for} \quad \tau = 0, \\ \;\;1, & \text{for} \quad \tau > 0. \end{cases}$$

In view of the assumption on functions $a_{ij}(x)$ it is easy to show that the functions $q_1(x_i)$ and $q_2(x_j)$ are determined over the whole numerical axis and are differentiable there. Thus, we can choose

$$p_{ij}(x_i, x_j) = -|x_i|^{\gamma_{ji}} \int\limits_0^{x_i} \frac{a_{ji}(\tau)\,\mathrm{sign}\;\tau}{a_{ii}|\tau|^{1+\gamma_{ji}}}\,d\tau - |x_j|^{\gamma_{ij}} \int\limits_0^{x_j} \frac{a_{ij}(\tau)\,\mathrm{sign}\;\tau}{a_{jj}|\tau|^{1+\gamma_{ij}}}\,d\tau \tag{3.17}$$

and setting $p_{ii} = 1$ we present function $v(x, \eta)$ as

$$v(x, \eta) = \eta^{\mathrm{T}} U(x) \eta = x^{\mathrm{T}} P(x) x, \tag{3.18}$$

where $P(x) = [p_{ij}(x_i, x_j)]$, $i, j = 1, 2, \dots, n$.

Calculating the corresponding total derivatives of the components of matrix-valued function $U(x)$ we find

$$Dv(x, \eta)\Big|_{(3.9)} = x^{\mathrm{T}} S(x) x, \tag{3.19}$$

where $S(x) = [\sigma_{ij}(x)]_{i,j=1}^{n}$ is a matrix whose elements have the following structure

$$
\begin{aligned}
\sigma_{ii} = \quad & 2a_{ii} + 2\sum_{j=1, j\neq i}^{n} \left(\frac{a_{ii}}{a_{jj}^2} |x_j|^{\gamma_{ij}} \int\limits_0^{x_j} \frac{a_{ij}(\tau)\,\mathrm{sign}\,\tau}{|\tau|^{1+\gamma_{ij}}}\,d\tau \right.\\
& \left. -|x_i|^{\gamma_{ji}} \int\limits_0^{x_i} \frac{a_{ji}(\tau)\,\mathrm{sign}\,\tau}{a_{ii}|\tau|^{1+\gamma_{ji}}}\,d\tau - \frac{a_{ij}(x_j)}{a_{jj}} \right) a_{ji}(x_i),\\
& i = 1, 2, \dots, n,\\
\sigma_{ij}(x) = \quad & \sum_{k=1,\, k\neq i, k\neq j}^{n} \left[\left(\frac{a_{ii}}{a_{kk}^2} |x_k|^{\gamma_{ik}} \int\limits_0^{x_k} \frac{a_{ik}(\tau)\,\mathrm{sign}\,\tau}{|\tau|^{1+\gamma_{ik}}}\,d\tau \right.\right.\\
& \left. -|x_i|^{\gamma_{ki}} \int\limits_0^{x_i} \tfrac{a_{ki}(\tau)\,\mathrm{sign}\,\tau}{a_{ii}|\tau|^{1+\gamma_{ki}}}\,d\tau - \tfrac{a_{ik}(x_k)}{a_{kk}} \right) a_{kj}(x_j)\\
& + \left(\frac{a_{jj}}{a_{kk}^2} |x_k|^{\gamma_{jk}} \int\limits_0^{x_k} \frac{a_{jk}(\tau)\,\mathrm{sign}\,\tau}{|\tau|^{1+\gamma_{jk}}}\,d\tau \right.\\
& \left.\left. -|x_j|^{\gamma_{kj}} \int\limits_0^{x_j} \frac{a_{kj}(\tau)\,\mathrm{sign}\,\tau}{a_{jj}|\tau|^{1+\gamma_{kj}}}\,d\tau - \frac{a_{jk}(x_k)}{a_{kk}} \right) a_{ki}(x_i) \right],\\
& i \neq j, \quad i, j = 1, 2, \dots, n.
\end{aligned}
$$

Using Theorem 3.1 and estimates (3.18) and (3.19) one can formulate the sufficient conditions of stability, asynptotic stability and asymptotic stability in the whole of system (3.9).

Theorem 3.2 *Let system of equations (3.9) be such that*

1. *matrix $P(x)$ is positive definite;*
2. *matrix $S(x)$ is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (3.9) is stable (asymptotically stable).

If in addition to conditions (1) and (2) one more condition is satisfied, namely

1. *there exist constants $r > 0$, $\varepsilon > 0$ and $L > 0$ such that*

$$
\|P^{-1}(x)\| > \frac{L}{\|x\|^{2-\varepsilon}} \quad \textit{for} \quad \|x\| > r,
$$

then the equilibrium state $x = 0$ of syste (3.9) is asymptotically stable in the whole.

4 4 On polystability of motion analysis

Consider the nonlinear system of differential equations

$$
\begin{aligned}
\frac{dx}{dt} &= A(t,x)x + B(t,y)y + F(t,x,y),\\
\frac{dy}{dt} &= D(x)x + G(t,x,y),
\end{aligned}
\tag{4.1}
$$

where $x \in R^{n_1}$, $y \in R^{n_2}$. Assume that functions $A(x)$, $B(t,y)$, $D(x)$, $F(t,x,y)$ and $G(t,x,y)$ are definite and continuous in the domain

$$D = \{(t,x,y)|t \geq 0, \ \ \|x\| \leq h, \ \ \|y\| \leq h\},$$

and functions $F(t,x,y)$ and $G(t,x,y)$ satisfy the inequalities

$$\|F\| \leq c_1(x,y)\|x\|^{\gamma_1}, \quad \|G\| \leq c_2\|x\|^{\gamma_2} \quad \text{for all} \quad (t,x,y) \in D.$$

Here function $c_1(x,y) \to 0$ as $\|x\| + \|y\| \to 0$, $F(t,0,0) = 0$, $G(t,0,0) = 0$ for all $t \in J_t^+$. According to [10, 11] we introduce the following definition.

Definition 4.1 The equilibrium state $x = 0$ of system (3.9) is called

1. *x-polystable*, iff it is stable and asymptotically x-stable;
2. *uniformly x-polystable*, if it is uniformly stable and uniformly asymptotically x-stable;

Assumption 4.1 The pseudo-linear system

$$\frac{dx}{dt} = A(t,x)x \tag{4.2}$$

satisfies the following conditions

1. the equilibrium state $x = 0$ of system (4.2) is uniformly asymptotically stable;
2. there exists a function $v(t,x)$ continuously differentiable in the domain $H = \{(t,x) \ : \ t \geq 0, \ \|x\| \leq h\}$, positive definite and such that

$$\begin{aligned} \underline{c}(\|x\|)\|x\|^2 &\leq v(t,x) \leq \bar{c}(\|x\|)\|x\|^2, \\ \left.\frac{dv}{dt}\right|_{(4.2)} &\leq -\alpha(\|x\|)\|x\|x^2, \\ \left\|\frac{\partial v}{\partial x}\right\| &\leq \rho\|x\|^{\alpha}, \quad \rho > 0 \quad \alpha > 0, \end{aligned}$$

   where $\underline{c}, \bar{c}, \alpha \in C(R_+, R_+)$.

Consider a pseudo-linear approximation of system (4.1)

$$\begin{aligned} \frac{dx}{dt} &= A(x)x + B(t,y)y, \\ \frac{dy}{dt} &= D(x)x, \end{aligned}$$

and construct a matrix-valued function $U(t,x,y)$. The diagonal elements of this function are taken as

$$v_{11}(x) = v(t,x), \quad v_{22}(y) = y^{\mathrm{T}}y.$$

To construct the non-diagonal elements $v_{12}(t,x,y)$ of the matrix-valued function we consider the equation

$$D_t v_{12} + (D_x v_{12})^{\mathrm{T}} A(t,x)x = -\frac{\eta_1}{2\eta_2}(D_x v(x))^{\mathrm{T}} B(t,y)y - \frac{\eta_2}{\eta_1} y^{\mathrm{T}} D(x)x \tag{4.3}$$

for some $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$. Applying function $U(t,x,y)$ and vector η we construct a scalar function $v(t,x,y) = \eta^{\mathrm{T}} U(t,x,y)\eta$.

Theorem 4.1 *Assume that the perturbed motion equations are such that*

1. *all conditions of Assumption 4.1 are satisfied;*

2. *equation (4.3) has a solution in the form of a continuously differentiable function* $v_{12}(t,x,y)$ *admitting the estimates*

$$\underline{c}_{12}(x,y)\|x\|\|y\| \le v_{12}(t,x,y) \le \bar{c}_{12}(x,y)\|x\|\|y\|$$

$$\|D_x v_{12}\| \le \rho_1 \|x\|^{\alpha_1}\|y\|^{\beta_1}; \quad \|D_x v_{12}\| \le \rho_2 \|x\|^{\alpha_2}\|y\|^{\beta_2}, \quad \rho_1, \rho_2 > 0,$$

where $\underline{c}_{12} \in C(R^{n_1} \times R^{n_2},\, R)$, $\bar{c}_{12} \in C(R^{n_1} \times R^{n_2},\, R)$;

3. *matrices*

$$C(x,y) = \begin{pmatrix} c_{11}(x) & c_{12}(x,y) \\ c_{12}(x,y) & 1 \end{pmatrix}, \quad c_{11}(x) = c(\|x\|),$$

$$\overline{C}(x,y) = \begin{pmatrix} \bar{c}_{11}(x) & \bar{c}_{12}(x,y) \\ \bar{c}_{12}(x,y) & 1 \end{pmatrix}, \quad \bar{c}_{11}(x) = \bar{c}(\|x\|),$$

satisfy in the domain $D = \{(x,y)\,:\, \|x\| \le h,\ \|y\| \le h\}$ *the generalized Silvester conditions;*

4. *there exists a constant* $\varkappa > 0$ *such that*

$$-a(\|x\|)\eta_1^2 + \sup_{\|x\|=1} \frac{(D_y v_{12})^T D(x)x + x^T D^T(x) D_y v_{12}}{\|x\|^2} < -\varkappa$$

for all $(t,x,y) \in D$;

5. $$\sup_{\|y\|=1} \frac{(D_x v_{12})^T B(t,y)y + y^T B^T(t,y) D_x v_{12}}{\|y\|^2} \le 0$$

   for all $t \ge 0$ *and* $\|x\| \le h$;

6. $\alpha + \gamma_1 \ge 2,\ \alpha_1 + \gamma_1 \ge 2,\ \alpha_2 + \gamma_2 \ge 2,\ \beta_1 \ge 0,\ \beta_2 > 0,\ \beta_1 > 0.$

Then the equilibrium state $x = y = 0$ of system (4.1) is unoformly x-polystable

Proof Conditions (2) from Assumption 4.1 and Theorem 4.1 for the components of matrix-valued function $U(t,x,y)$ allow to estimate the scalar function $v(t,x,y,\eta) = \eta^{\mathrm{T}}U(t,x,y)\eta$ as

$$u^{\mathrm{T}}H^{\mathrm{T}}\underline{C}(x,y)Hu \le v(t,x,y,\eta) \le u^{\mathrm{T}}H^{\mathrm{T}}\overline{C}(x,y)Hu,$$

where $u = (\|x\|, \|y\|)^{\mathrm{T}}$, $H = \mathrm{diag}\,(\eta_1,\eta_2)$. Under condition (3) the function $v(t,x,y,\eta)$ is positive definite and decreasent. We shall estimate the total derivative of function $v(t,x,y,\eta)$ along solutions of system (4.1) taking unto account conditions (4) and (5)

$$\begin{aligned}
\frac{dv}{dt}\Big|_{(4.1)} &\le -\varkappa\|x\|^2 + \eta_1^2(D_xv)^{\mathrm{T}}F(t,x,y) + \eta_2^2y^{\mathrm{T}}G(t,x,y)\\
&+ 2\eta_1\eta_2(D_xv_{12})^{\mathrm{T}}F(t,x,y) + 2\eta_1\eta_2(D_yv_{12})^{\mathrm{T}}G(t,x,y)\\
&\le -\varkappa\|x\|^2 + \rho\eta_1^2\|x\|^{\alpha}c_1(x,y)\|x\|^{\gamma_1} + \eta_2^2\rho_2c_2\|x\|^{\alpha_2}\|y\|^{\beta_2}\|x\|^{\gamma_2}\\
&+2\eta_1\eta_2\rho_1c_1(x,y)\|x\|^{\alpha_1}\|y\|^{\beta_1}\|x\|^{\gamma_1} + 2\eta_1\eta_2\rho_2c_2\|x\|^{\alpha_2}\|y\|^{\beta_2}\|x\|^{\gamma_2}\\
&\le -\varkappa\|x\|^2 + c(x,y)\|x\|^2,
\end{aligned}$$

where the function $c(x,y) \to 0$ as $\|x\|+\|y\| \to 0$. Therefore there exists a magnitude $h_1 \le h$ such that $c(x,y) < \varkappa/2$ for $\|x\| + \|y\| \le h_1$. Thus in the domain $\widetilde{D} = \{(t,x,y) : t \ge 0,\ \|x\|+\|y\| \le h_1\}$ the derivative of function $v(t,x,y)$ along solutions of system (4.1) is estimated by the inequality

$$\frac{dv}{dt}\Big|_{(4.1)} \le -\frac{\varkappa}{2}\|x\|^2.$$

In terms of Theorem 2.5.2 from Martynyuk [9] we conclude on uniform asymptotic stability of the equilibrium state $x = y = 0$ of system (4.1). The assertion on uniform asymptotic x-stability follows from Theorem 2.6.1 by Martynyuk [9] (see also Theorem 6.1 from Rumyantsev and Oziraner [1]). ∎

5 Large-Scale Linear Systems

Linear systems of perturbed motion equations are of an essential interest in the description of various phenomena in physical and technical systems. General theory of such systems is developed well because in some cases such systems can be integrated precisely. On the other hand systems of the type are the first approximation of quasi-linear equations in the investigation of which the information on the properties of the first approximation system is encorporated. For this class of systems of equations the construction of the Liapunov functions remains in the focus of attention of many researchers.

5.1 Non-autonomous linear systems

Consider a large-scale system whose motion is described by the equations

$$\frac{dx_i}{dt} = A_{ii}x_i + \sum_{j=1, j\neq i}^{m} A_{ij}(t)x_j, \qquad i = 1, 2, \dots, m. \tag{5.1}$$

Here the state vectors $x_i \in R^{n_i}$ and $A_{ii} \in R^{n_i \times n_i}$ are constant matrices for all $i = 1, 2, \dots, m;$ $A_{ij}(t) \in C(R, R^{n_i \times n_j}),$ $i, j = 1, 2, \dots, m,$ $i \neq j,$ $n = \sum_{i=1}^{m} n_i.$

For the independent subsystems

$$\frac{dx_i}{dt} = A_{ii}x_i, \quad i = 1, 2, \dots, m, \tag{5.2}$$

the auxiliary functions $v_{ii}(x_i)$ are constructed as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1, 2, 3, \tag{5.3}$$

whose constant matrices P_{ii} are determined by the algebraic Liapunov equations

$$A_{ii}^{\mathrm{T}} P_{ii} + P_{ii} A_{ii} = -G_{ii}, \quad i = 1, 2, \dots, m, \tag{5.4}$$

where G_{ii} are pre-assigned matrices of constant sign. For the construction of non-diagonal elements $v_{ij}(t, x_i, x_j)$ of the matrix-valued function $U(t, x)$ we apply equation (3.5). Note that for the system (5.1)

$$f_i(x_i) = A_{ii}x_i, \quad f_j(x_j) = A_{jj}x_j,$$
$$g_{ij}(t, x_i, x_j) = A_{ij}(t)x_j, \quad g_{ji}(t, x_i, x_j) = A_{ji}(t)x_j, \quad G_i(t, x) = 0.$$

Suppose that at least one of the matrices A_{ij} or A_{ji} is not equal to constant. Then we determine function $v_{ij}(t, x_i, x_j)$ as

$$v_{ij}(t, x_i, x_j) = v_{ji}(t, x_j, x_i) = x_i^{\mathrm{T}} P_{ij}(t) x_j, \tag{5.5}$$

where $P_{ij} \in C^1(R, R^{n_i \times n_j})$.

Since for the bilinear forms (5.5)

$$D_t v_{ij}(t, x_i, x_j) = x_i \, \frac{dP_{ij}dt}{} x_j, \quad D_{x_i} v_{ij}(t, x_i, x_j) = x_j^{\mathrm{T}} P_{ij}(t)^{\mathrm{T}},$$

$$D_{x_j} v_{ij}(t, x_i, x_j) = x_i^{\mathrm{T}} P_{ij}(t)$$

the equation (3.5) becomes

$$x_i^{\mathrm{T}} \left(\frac{dP_{ij}}{dt} + A_{ii}^{\mathrm{T}} P_{ij} + P_{ij} A_{jj} + \frac{\eta_i}{\eta_j} \, P_{ii} A_{ij}(t) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(t) P_{jj} \right) x_j = 0.$$

For determination of matrices P_{ij} this correlation yields a system of matrix differential equations

$$\frac{dP_{ij}}{dt} + A_{ii}^{\mathrm{T}} P_{ij} + P_{ij} A_{jj} = -\frac{\eta_i}{\eta_j} P_{ii} A_{ij}(t) - \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(t) P_{jj} \qquad (5.6)$$
$$i,\, j = 1, 2, \dots, m, \quad i \neq j.$$

Note that equations (5.6) can be solved in the explicit form. To this end we consider a linear operator (general information on linear operators can be found, for example, in Daletskii and Krene [1])

$$F_{ij} \ : \ R^{n_i \times n_j} \to R^{n_i \times n_j}, \quad F_{ij} X = A_{ii}^{\mathrm{T}} X + X A_{jj}.$$

Equation (5.6) can be represented as

$$\frac{dP_{ij}}{dt} + F_{ij} P_{ij} = -\frac{\eta_i}{\eta_j} P_{ii} A_{ij}(t) - \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(t) P_{ii}, \quad i \neq j.$$

Consider the homogeneous equations

$$\frac{dP_{ij}}{dt} + F_{ij} P_{ij} = 0, \qquad (5.7)$$

whose general solution is presented as

$$P_{ij}(t) = \exp\{-F_{ij} t\} C_{ij},$$

where C_{ij} is a constant $n_i \times n_j$ matrix and $\exp\{-F_{ij}t\} = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k F_{ij}^k t^k}{k!}$ is an operator exponent.

To find the solution of equation (5.6) the method of variation of a constant is applied. Solution of equation (5.6) is presented in the form

$$P_{ij}(t) = \exp\{-F_{ij} t\} C_{ij}(t), \qquad (5.8)$$

where $C_{ij} \in C^1(R, R^{n_1 \times n_2})$ and $C_{ij}(0) = 0$. Substituting by (5.8) into (5.6) yields

$$\frac{dC_{ij}}{dt} = -\exp\{F_{ij} t\} \left(\frac{\eta_i}{\eta_j} P_{ii} A_{ij} + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}} P_{ii} \right), \quad i \neq j.$$

Integrating the last correlation from 0 to t we determine a partial solution of equation (5.6)

$$P_{ij}(t) = -\int\limits_0^t \exp\{-F_{ij}(t-\tau)\} \left(\frac{\eta_i}{\eta_j} P_{ii} A_{ij}(\tau) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} \right) d\tau, \quad i \neq j. \qquad (5.9)$$

We establish estimates for the function

$$v(t, x, \eta) = \eta^{\mathrm{T}} U(t, x) \eta = \sum_{i,j=1}^{m} v_{ij}(t, .) \eta_i \eta_j,$$

where

$$U(t,x)=\begin{pmatrix} v_{11}(x_1) & \cdots & v_{1m}(t,x_1,x_m)\\ \vdots & \ddots & \vdots\\ v_{1m}(t,x_1,x_m) & \cdots & v_{mm}(x_m)\end{pmatrix}.$$

Introduce the designations $\bar{c}_{ii}=\lambda_M(P_{ii})$ and $\underline{c}_{ii}=\lambda_m(P_{ii})$ and assuming $\sup\limits_{t\ge 0}\|P_{ij}(t)\|<\infty$ denote $\bar{c}_{ij}=\sup\limits_{t\ge 0}\|P_{ij}(t)\|$, $\underline{c}_{ij}=-\bar{c}_{ij}$.

Since for the forms (5.3) and (5.5) the estimates

$$\begin{aligned} \lambda_m(P_{ii})\|x_i\|^2 &\le v_{ii}(x_i)\le\lambda_M(P_{ii})\|x_i\|^2, \quad x_i\in R^{n_i};\\ -\bar{c}_{ij}\|x_i\|\|x_j\| &\le v_{ij}(t,x_i,x_j)\le\bar{c}_{ij}\|x_i\|\|x_j\|, \quad (x_i,x_j)\in R^{n_i}\times R^{n_j}, \end{aligned} \tag{5.10}$$

are valid, for the function $v(t,x,\eta)=\eta^{\mathrm{T}}U(t,x)\eta$

$$w^{\mathrm{T}}H^{\mathrm{T}}\underline{C}Hw\le v(t,x,\eta)\le w^{\mathrm{T}}H^{\mathrm{T}}\overline{C}Hw \quad \text{for all} \quad x\in R^n, \tag{5.11}$$

where $w=(\|x_1\|,\dots,\|x_m\|)^{\mathrm{T}}$, $H=\operatorname{diag}(\eta_1,\eta_2,\dots,\eta_m)$, $\overline{C}=[\bar{c}_{ij}]_{i,j=1}^m$, $\underline{C}=[\underline{c}_{ij}]_{i,j=1}^m$.

In order to estimate the derivative of function $v(t,x,\eta)$ along solutions of system (5.1) we calculate the constants from Assumption 3.3

$$\alpha_{ij}^1=\alpha_{ij}^2=0, \quad \alpha_{ij}^3(t)=\lambda_M(A_{ij}^{\mathrm{T}}(t)P_{ij}(t)+P_{ij}^{\mathrm{T}}(t)A_{ij}(t)),$$

$$\nu_{ki}^1=\nu_{ijk}^2=0, \quad \nu_{ijk}^1(t)=\lambda_M^{1/2}[(P_{ij}^{\mathrm{T}}(t)A_{ik}(t))(P_{ij}^{\mathrm{T}}(t)A_{ik}(t))], \quad \mu_{ijk}^2=0.$$

Therefore the elements σ_{ij} of matrix $M(t)$ in estimate (3.8) for system (5.1) have the structure

$$\begin{aligned} \sigma_{ii}(t) &= -\eta_i^2\lambda_m(G_{ii})+2\sum_{j=1,j\ne i}^m\eta_i\eta_j\alpha_{ij}^3, \quad i=1,\dots,m,\\ \sigma_{ij}(t) &= \sum_{k=1,k\ne i,k\ne j}^m(\eta_k\eta_j\nu_{ijk}^1+\eta_i\eta_k\nu_{kij}^1), \quad i,j=1,\dots,m, \quad i\ne j. \end{aligned}$$

Consequently, the variation of function $Dv(t,x,\eta)$ along solutions of system (5.1) is estimated by the inequality

$$Dv(t,x,\eta)\Big|_{(5.1)}\le w^{\mathrm{T}}M(t)w \tag{5.12}$$

for all $(x_1,\dots,x_m)\in R^{n_1}\times\cdots\times R^{n_m}$.

Remark 5.1 In the partial case when matrices A_{ij} and A_{ji} do not depent on t it is reasonable to choose $P_{ij}(t)=\text{const}$. Then equation (5.6) becomes

$$A_{ii}P_{ij}+P_{ij}A_{jj}=-\frac{\eta_i}{\eta_j}P_{ii}A_{ij}-\frac{\eta_j}{\eta_i}A_{ji}^{\mathrm{T}}P_{jj} \tag{5.13}$$

or in the operator form

$$F_{ij}P_{ij} = -\frac{\eta_i}{\eta_j}\,P_{ii}A_{ij} - \frac{\eta_j}{\eta_i}\,A_{ji}^{\mathrm{T}}P_{jj}.$$

Therefore for the equation (5.13) to have a unique solution it is necessary and sufficient that the operator F_{ij} be nondegenerate.

It is known (see Daletskii and Krene [1]) that the set of eigenvalues of the operator F_{ij} consists of the numbers $\lambda_k(A_{ii}) + \lambda_l(A_{jj})$, where $\lambda_k(\cdot)$ is an eigenvalue of the corresponding matrix. Basing on these speculations one can formulate the following result.

For the equation (5.13) to have a unique solution it is necessary and sufficient that

$$\lambda_k(A_{ii}) + \lambda_l(A_{jj}) \neq 0 \quad \text{for all} \quad k,\, l,$$

and this solution can be presented as

$$P_{ij} = -F_{ij}^{-1}\left(\frac{\eta_i}{\eta_j}\,P_{ii}A_{ij} + \frac{\eta_j}{\eta_i}\,A_{ji}^{\mathrm{T}}P_{jj}\right).$$

This result is summed up as follows.

Theorem 5.1 *Assume that for system (5.1) the following conditions are satisfied*

1. *the sign-definite matrices P_{ii}, $i = 1,2,3$, are the solution of algebraic equations (5.4);*

2. *the bounded matrices $P_{ij}(t)$ for all $i,j = 1,2,\dots,m$, $i \neq j$, are the solution of matrix differential equations (5.6);*

3. *matrices $\overline{C}$ É $\underline{C}$ in estimate (5.11) are positive definite;*

4. *matrix $M(t)$ in estimate (5.12) is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (5.1) is uniformly stable in the whole (uniformly asymptotically stable in the whole).

5.2 Time invariant linear systems

Assume that in the system

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3,\\ \frac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3,\\ \frac{dx_3}{dt} &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3,\end{aligned} \tag{5.14}$$

the state vectors $x_i \in R^{n_i}$, $i = 1,2,3$, and $A_{ij} \in R^{n_i \times n_j}$ are constant matrices for all $i,j = 1,2,3$.

For the independent systems

$$\frac{dx_i}{dt} = A_{ii}x_i, \quad i = 1,2,3, \tag{5.15}$$

we construct auxiliary functions $v_{ii}(x_i)$ as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1,2,3, \tag{5.16}$$

whose matrices P_{ii} are determined by

$$A_{ii}^{\mathrm{T}} P_{ii} + P_{ii} A_{ii} = -G_{ii}, \quad i = 1,2,3, \tag{5.17}$$

where G_{ii} are prescribed matrices of definite sign.

In order to construct non-diagonal elements $v_{ij}(x_i, x_j)$ of matrix-valued function $U(x)$ we employ equation (3.5). Note that for system (5.14)

$$f_i(x_i) = A_{ii}x_i, \quad f_j(x_j) = A_{jj}x_j,$$

$$g_{ij}(x_i, x_j) = A_{ij}x_j, \quad G_i(t,x) = 0, \quad i = 1,2,3.$$

Since for the bilinear forms

$$v_{ij}(x_i, x_j) = v_{ji}(x_j, x_i) = x_i^{\mathrm{T}} P_{ij} x_j \tag{5.18}$$

the correlations

$$D_{x_i} v_{ij}(x_i, x_j) = x_j^{\mathrm{T}} P_{ij}^{\mathrm{T}}, \quad D_{x_j} v_{ij}(x_i, x_j) = x_i^{\mathrm{T}} P_{ij}$$

are true, equation (3.5) becomes

$$x_i^{\mathrm{T}} \left(A_{ii}^{\mathrm{T}} P_{ij} + P_{ij} A_{jj} + \frac{\eta_i}{\eta_j} \, P_{ii} A_{ij} + \frac{\eta_j}{\eta_i} \, A_{ji}^{\mathrm{T}} P_{ii} \right) x_j = 0.$$

¿From this correlation for determining matrices P_{ij} we get the system of algebraic equations

$$\begin{gathered} A_{ii} P_{ij} + P_{ij} A_{jj} = -\frac{\eta_i}{\eta_j} \, P_{ii} A_{ij} - \frac{\eta_j}{\eta_i} \, A_{ji}^{\mathrm{T}} P_{ii}, \\ i \neq j, \quad i,j = 1,2,3. \end{gathered} \tag{5.19}$$

Since for (5.16) and (5.18) the estimates

$$\begin{aligned} v_{ii}(x_i) \quad & \geq \lambda_m(P_{ii}) \|x_i\|^2, \quad x_i \in R^{n_i}; \\ v_{ij}(x_i, x_j) \quad & \geq -\lambda_M^{1/2}(P_{ij} P_{ij}^{\mathrm{T}}) \|x_i\| \|x_j\|, \quad (x_i, x_j) \in R^{n_i} \times R^{n_j}, \end{aligned}$$

hold true, for function $v(x, \eta) = \eta^{\mathrm{T}} U(x) \eta$ the inequality

$$w^{\mathrm{T}} H^{\mathrm{T}} C H w \leq v(x, \eta) \tag{5.20}$$

is satisfied for all $x \in R^n$, $w = (\|x_1\|, \|x_2\|, \|x_3\|)^{\mathrm{T}}$ and the matrix

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) & -\lambda_M^{1/2}(P_{13}P_{13}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) & \lambda_m(P_{22}) & -\lambda_M^{1/2}(P_{23}P_{23}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{13}P_{13}^{\mathrm{T}}) & -\lambda_M^{1/2}(P_{23}P_{23}^{\mathrm{T}}) & \lambda_m(P_{33}) \end{pmatrix}.$$

For system (5.14) the constants from Assumption 3.3 are:

$$\alpha_{ij}^1 = \alpha_{ij}^2 = 0; \quad \alpha_{ij}^3 = \lambda_M(A_{ij}^{\mathrm{T}}P_{ij} + P_{ij}^{\mathrm{T}}A_{ij}),$$

$$\nu_{ki}^1 = \nu_{ijk}^2 = 0; \nu_{ijk}^1 = \lambda_M^{1/2}[(P_{ij}^{\mathrm{T}}A_{ik})(P_{ij}^{\mathrm{T}}A_{ik})], \quad \mu_{ijk}^2 = 0.$$

Therefore the elements σ_{ij} of matrix M in (5.12) for system (5.14) have the structure

$$\sigma_{ii} = -\eta_i^2\lambda_m(G_{ii}) + 2\sum_{j=1,\, j\neq i}^{3}\eta_i\eta_j\alpha_{ij}^3, \qquad i = 1,2,3,$$

$$\sigma_{ij} = \sum_{k=1,\, k\neq i, k\neq j}^{3}(\eta_k\eta_j\nu_{ijk}^1 + \eta_i\eta_k\nu_{kij}^1), \qquad i,j = 1,2,3, \quad i \neq j.$$

Consequently, the function $Dv_(x,\eta)$ variation along solutions of system (5.14) is estimated by the inequality

$$Dv(x,\eta)\big|_{(5.14)} \leq w^{\mathrm{T}}Mw \tag{5.21}$$

for all $(x_1, x_2, x_3) \in R^{n_1} \times R^{n_2} \times R^{n_3}$.

We summarize our presentation as follows.

Corollary 5.1 Assume for system (5.14) the folowing conditions are satisfied:

1. algebraic equations (5.17) have the sign-definite matrices P_{ii}, $i = 1,2,3$, as their solutions;

2. algebraic equations (5.19) have constant matrices P_{ij}, for all $i,j = 1,2,3$, $i \neq j$, as their solutions;

3. matrix C in (5.20) is positive definite;

4. matrix M in (5.21) is negative semi-definite (negative definite).

Then the equilibrium state $x = 0$ of system (5.14) is uniformly stable (uniformly asymptotically stable).

This corollary follows from Theorem 3.1 and hence its proof is obvious.

Example 5.3 We study the motion of two non-autonomously connected oscillators whose behaviour is described by the equations

$$\begin{aligned}\frac{dx_1}{dt} &= \gamma_1 x_2 + v\cos\omega t y_1 - v\sin\omega t y_2,\\ \frac{dx_2}{dt} &= -\gamma_1 x_1 + v\sin\omega t y_1 + v\cos\omega t y_2,\\ \frac{dy_1}{dt} &= \gamma_2 y_2 + v\cos\omega t x_1 + v\sin\omega t x_2,\\ \frac{dy_2}{dt} &= -\gamma_2 y_2 + v\cos\omega t x_2 - v\sin\omega t x_1,\end{aligned} \tag{5.22}$$

where γ_1, γ_2, v, ω, $\omega+\gamma_1-\gamma_2\neq 0$ are some constants.

For the independent subsystems

$$\begin{aligned}\frac{dx_1}{dt} &= \gamma_1 x_2, \qquad & \frac{dx_2}{dt} &= -\gamma_1 x_1\\ \frac{dy_1}{dt} &= \gamma_2 y_2, \qquad & \frac{dy_2}{dt} &= -\gamma_2 y_1\end{aligned} \tag{5.23}$$

the auxiliary functions v_{ii}, $i=1,2$, are taken in the form

$$\begin{aligned} v_{11}(x) &= x^{\mathrm T}x, \qquad & x &= (x_1,x_2)^{\mathrm T},\\ v_{22}(y) &= y^{\mathrm T}y, \qquad & y &= (y_1,y_2)^{\mathrm T}.\end{aligned} \tag{5.24}$$

We use the equation (3.5) (see Assumption 3.2) to determine the non-diagonal element $v_{12}(x,y)$ of the matrix-valued function $U(t,x,y)=[v_{ij}(\cdot)]$, $i,j=1,2$. To this end set $\eta=(1,1)^{\mathrm T}$ and $v_{12}(x,y)=x^{\mathrm T}P_{12}y$, where $P_{12}\in C^1(\mathcal{T}_\tau,\mathcal{R}^{\in\times\in})$. For the equation

$$\begin{aligned}\frac{dP_{12}}{dt} &+ \begin{pmatrix} 0 & -\gamma_1\\ \gamma_1 & 0\end{pmatrix} P_{12} + P_{12}\begin{pmatrix} 0 & \gamma_2\\ -\gamma_2 & 0\end{pmatrix}\\ &+2v\begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix} = 0,\end{aligned} \tag{5.25}$$

the matrix

$$P_{12} = -\frac{2v}{\omega+\gamma_1-\gamma_2}\begin{pmatrix}\sin\omega t & \cos\omega t\\ -\cos\omega t & \sin\omega t\end{pmatrix}$$

is a partial solution bounded for all $t\in\mathcal{T}_\tau$.

Thus, for the function $v(t,x,y)=\eta^{\mathrm T}U(t,x,y)\eta$ it is easy to establish the estimate of (3.7) type with matrices $\underline{C}$ and $\overline{C}$ in the form

$$\underline{C} = \begin{pmatrix}\underline{c}_{11} & \underline{c}_{12}\\ \underline{c}_{12} & \underline{c}_{22}\end{pmatrix}, \qquad \overline{C} = \begin{pmatrix}\bar{c}_{11} & \bar{c}_{12}\\ \bar{c}_{12} & \bar{c}_{22}\end{pmatrix},$$

where $\bar{c}_{11}=\underline{c}_{11}=1$, $\bar{c}_{22}=\underline{c}_{22}=1$, $\bar{c}_{12}=-\underline{c}_{12}=\dfrac{|2v|}{|\omega+\gamma_1-\gamma_2|}$. Besides, the vector $u_1^{\mathrm T}=(\|x\|,\|y\|)=u_2^{\mathrm T}$, since the system (5.22) is linear.

For system (5.22) the estimate (5.12) becomes

$$Dv(t,x,y)\big|_{(5.1)} = 0$$

for all $(x,y) \in R^2 \times R^2$ because $M = 0$.

Due to Theorem 3.1 the motion stability conditions for system (5.22) are established basing on the analysis of matrices $\underline{C}$ and $\overline{C}$ property of having fixed sign.

It is easy to verify that the matrices $\underline{C}$ and $\overline{C}$ are positive definite, if

$$1 - \frac{4v^2}{(\omega + \gamma_1 - \gamma_2)^2} > 0.$$

Consequently, the motion of nonautonomously connected oscillators is uniformly stable in the whole, if

$$|v| < \frac{1}{2}\,|\omega + \gamma_1 - \gamma_2|.$$

5.3 Discussion and Numerical Example

To start to illustrate the possibilities of the proposed method of Liapunov function construction we consider a system of two connected equations that was studied earlier by the Bellman-Bailey approach (see Barbashin [1], Voronov and Matrosov [1], etc.).

Partial case of system (6.14) is the system

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1 + C_{12}x_2 \\ \frac{dx_2}{dt} &= Bx_2 + C_{21}x_1, \end{aligned} \tag{6.1}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, and A, B, C_{12} and C_{21} are constant matrices of corresponding dimensions. For independent subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1, \\ \frac{dx_2}{dt} &= Bx_2 \end{aligned} \tag{6.2}$$

the functions $v_{11}(x_1)$ and $v_{22}(x_2)$ are constructed as the quadratic forms

$$v_{11} = x_1^{\mathrm{T}} P_{11} x_1, \quad v_{22} = x_2^{\mathrm{T}} P_{22} x_2, \tag{6.3}$$

where P_{11} and P_{22} are sign-definite matrices.

Function $v_{12} = v_{21}$ is searched for as a bilinear form $v_{12} = x_1^{\mathrm{T}} P_{12} x_2$ whose matrix is determined by the equation

$$A^{\mathrm{T}} P_{12} + P_{12} B = -\frac{\eta_1}{\eta_2} P_{11} C_{12} - \frac{\eta_2}{\eta_1} C_{21}^{\mathrm{T}} P_{22}, \quad \eta_1 > 0, \quad \eta_2 > 0. \tag{6.4}$$

According to Lankaster [1] equation (6.4) has a unique solution, provided that matrices A and $-B$ have no common eigenvalues.

Matrix C in (/./,/) for system (6.4) reads

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \\ & \\ -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) & \lambda_m(P_{22}) \end{pmatrix}. \tag{6.5}$$

Here $\lambda_m(\cdot)$ are minimal eigenvalues of matrices $P_{11},\ P_{22}$, and $\lambda_M^{1/2}(\cdot)$ is the norm of matrix $P_{12}P_{12}^{\mathrm{T}}$.

Estimate (5.15) for function $Dv(x,\eta)$ by virtue of system (6.1) is

$$Dv(x,\eta)|_{(6.1)} \le w^{\mathrm{T}}\Xi w, \tag{6.6}$$

where $w = (\|x_1\|,\ \|x_2\|)^{\mathrm{T}},\ \Xi = [\sigma_{ij}],\ i,j = 1,2;$

$$\sigma_{11} = \lambda_1\eta_1^2 + \eta_1\eta_2\alpha_{22},$$

$$C\sigma_{22} = \lambda_2\eta_2^2 + \eta_1\eta_2\beta_{22},$$

$$V\sigma_{12} = \sigma_{21} = 0.$$

The notations are

$$\begin{aligned} \lambda_1 &= \lambda_M(A^{\mathrm{T}}P_{11} + P_{11}A), \\ \lambda_2 &= \lambda_M(B^{\mathrm{T}}P_{22} + P_{22}B), \\ \alpha_{22} &= \lambda_M(C_{12}^{\mathrm{T}}P_{12} + P_{12}^{\mathrm{T}}C_{12}), \\ \beta_{22} &= \lambda_M(C_{21}^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}C_{21}), \end{aligned}$$

$\lambda(\cdot)$ is a maximal eigenvalue of matrix $(\cdot)$. Partial case of Assumption 3.1 is as follows.

Corollary 6.1 For system (6.1) let functions $v_{ij}(\cdot),\ i,j = 1,2,$ be constructed so that matrix C for system (6.1) is positive definite and matrix Ξ in inequality (/./,/) is negative definite. Then the equilibrium state $x = 0$ of system (6.1) is uniformly asymptotically stable.

We consider the numerical example. Let the matrices from system (6.1) be of the form

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 & 1 \\ 2 & -1 \end{pmatrix}, \tag{6.7}$$

$$C_{12} = \begin{pmatrix} -0.5 & -0.5 \\ 0.8 & -0.7 \end{pmatrix}, \qquad C_{21} = \begin{pmatrix} 1 & 0.5 \\ -0.6 & -0.3 \end{pmatrix}. \tag{6.8}$$

Functions v_{ii} for subsystems

$$\begin{aligned} \dot{x} &= Ax, & x &= (x_1, x_2)^{\mathrm{T}}, \\ \dot{y} &= Bx, & y &= (y_1, y_2)^{\mathrm{T}} \end{aligned}$$

are taken as the quadratic forms

$$\begin{aligned} v_{11} &= 75x_1^2 + x_1x_2 + 5x_2^2, \\ v_{22} &= 0.35y_1^2 + 0.9y_1y_2 + 0.95y_2^2. \end{aligned} \tag{6.9}$$

Let $\eta = (1,1)^{\mathrm{T}}$. Then $\lambda_1 = \lambda_2 = -1$,

$$P_{12} = \begin{pmatrix} -0.011 & 0.021 \\ -0.05 & -0.022 \end{pmatrix},$$

$$\alpha_{22} = 0.03, \qquad \beta_{22} = -0.002.$$

It is easy to verify that $\sigma_{11} < 0$ and $\sigma_{22} < 0$, and hence all conditions of Corollary 6.1 are fulfilled in view that

$$\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \le (\lambda_m(P_{11})\lambda_m(P_{22}))^{1/2},$$

for the values of $\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) = 0.06$, $\lambda_m(P_{11}) = 08$, $\lambda_m(P_{22}) = 0.115$. This implies uniform asymptotic stability in the whole of the equilibrium state of system (6.1) with matrices (6.7) and (6.8).

Let us show now that stability of system (6.1) with matrices (6.7) and (6.8) can not be studied in terms of the Bailey [1] theorem.

We recall that in this theorem the conditions of exponential stability of the equilibrium state are

1. for subsystems (6.2) functions (6.3) must exist satisfying the estimates
   (a) $c_{i1}\|x_i\|^2 \le v_i(t,x_i) \le c_{i2}\|x_i\|^2$,
   (b) $Dv_i(t,x_i) \le -c_{i3}\|x_i\|^2$,
   (c) $\|\partial v_i/\partial x_i\| \le c_{i4}\|x_i\|$ for $x_i \in R^{n_i}$,
   where c_{ij} are some positive constants, $i = 1,2$, $j = 1,2,3,4$;
2. the norms of matrices C_{ij} in system (/./,/) must satisfy the inequality (see Voronov and Matrosov [1], p. 106)

$$\|C_{12}\|\,\|C_{21}\| < \left(\frac{c_{11}c_{21}}{c_{12}c_{22}}\right)^{1/2}\left(\frac{c_{13}c_{23}}{c_{14}c_{24}}\right). \tag{6.10}$$

We note that this inequality is refined as compared with the one obtained firstly by Bailey [1].

The constants $c_{11},\dots,c_{24}$ for functions (/./,/) and system (6.1) with matrices (/./,/) and (/./,/) take the values

$$c_{11} = 1.08, \quad c_{21} = 0.115, \quad c_{12} = 2.14, \quad c_{22} = 2.14,$$

$$\boxed{c_{22} = 1.135}, \quad c_{13} = c_{23} = 1, \quad c_{14} = 4.83, \quad c_{24} = 2.4.$$

Condition (/./ /) requires that

$$\|C_{12}\|\,\|C_{21}\| < 0.0184 \tag{6.11}$$

whereas for system (/./ /), (/./ /), and (/./ /) we have

$$\|C_{12}\|\,\|C_{21}\| = 75.$$

Thus, the Bailey theorem turns out to be nonapplicable to this system and the condition (6.11) is "super-sufficient" for the property of stability.

6 Problems for Investigations

7.1 To obtain existence conditions for solutions to system (3.5) which satisfy bilinear estimates (condition (2) of Assumption 3.2) or other similar conditions allowing to establish algebraic conditions of sign-definiteness and decrease (radial unboundedness) of function (3.6).
7.2 To construct an algorithm of approximate solution of system (3.5) in terms of the method of perturbed nonlinear mechanics.
7.3 To obtain criterion for exponential stability of system (3.3) in terms of function (3.6) provided that the independent subsystems (3.4) are not exponentially stable.
7.4 To investigate other than stability in the sense of Liapunov dynamical properties of system (3.3) or its partial cases such as stability, boundedness, uniform boundedness in terms of two measures.
7.5 In terms of Liapunov function (3.6) to construct algorithms for estimation of domains of stability, attraction and asymptotic stability of system (3.3) and its partial cases in the phase space or/and in the parameter space.

Hint. For the initial definitions of the corresponding domains of stability, attraction and asymptotic stability see Grujic, *et al.* [1], Krasovskii [1], and Martynyuk [7].
7.6 For the class of autonomous systems

$$\frac{dx}{dt} = X(x(t)), \quad x(t_0) = x_0, \tag{7.1}$$

where $x \in R^n,\ X \in C^1(R^n,\, R^n),\ X(0) = 0,$ admitting decomposition to (3.3) form, to establish conditions of global asymptotic stability under condition that the origin for system (7.1) is an asymptotic attractor.

7 Brief Outline of the References and Remarks

Section 2 Nonlinear dynamics of continuous systems is a traditional domain of intensive investigations starting with the works by Galilei, Newton, Euler, Lagrange, etc. The problem of motion stability arises whenever the engineering or physical problem is formulated as a mathematical problem of qualitative analysis of equations. Poincaré and Liapunov laid a background for the method of auxiliary functions for continuous systems which allow not to integrate the motion equations for their qualitative analysis. The ideas of Poincare and Liapunov were further developed and applied in many branches of modern natural sciences.

The results of Liapunov [1], Chetaev [1], Persidskii [1], Malkin [1], Ascoli [1], Barbasin and Krasovskii [1], Massera [1], and Zubov [1], were base for the Definitions 2.1 – 2.3 (ad hoc see Grujić *et al.* [1], pp. 8 – 12) and cf.Rao Mohana Rao [1], Yoshizawa [1], Rouche *et al.* [1], Antosiewicz [1], Lakshmikantham and Leela [1], Hahn [2], etc. For the Definitions 2.4 – 2.7, and 2.13 see Hahn [2], and Martynyuk [9]. Definitions 2.8 – 2.12 are based on some results by Liapunov [1], Hahn [2], Barbashin and Krasovskii [1] (see and cf.Djordjevic [1], Grujić [2], Martynyuk [3 – 6]). The proofs of Proposalls 2.1 – 2.5 are in Hahn [2], Kuz'min [1], Martynyuk [9], Zubov [2], etc.

Theorems 2.1 – 2.7 are set out according to Martynyuk [10] (see also Martynyuk [13]). For the proof of Corollary 2.1 see Liapunov [1], and Chetaev [1]; for the proof of Corollary 2.2 see Barbashin and Krasovskii [1]; for the proof of Corollary 2.3 – 2.4 see Liapunov [1], Massera [1], Yoshizava [1], Halanay [1], etc; for the proof of Corollary 2.5 – 2.6 see He and Wang [1], Krasovskii [1], and Hahn [2]; for the proof of Corollary 2.8 see Chetaev [1], Rouche, *et al.* [1]; and for the proof of Corollary 2.9 – 2.10 see Liapunov [1], and Rouche, *et al.* [1].

Further results obtained via the Liapunov's methods can be found in Burton [1], Galperin [1], Gruyitch [1], Rama Mohana Rao [1], Coppel [1], Cesari [1], Lakshmikantham and Leela [2], Martynyuk [14], Sivasundaram [1], Vincent [1], Vorotnikov [1], Zubov [3] (see also CD ROM by Kramer and Hofmann [1] for references), etc.

Section 3 The problem of constructing the Liapunov functions for nonlinear nonautonomous system of general type remains still unsolved though its more than one-hundred existence. Meanwhile the efforts of many mathematicians and mechanical scientists have resulted in the efficient approach of constructing the appropriate auxiliary functions for specific classes of systems of equations with reference to many applications.

The approach proposed in this section is based on the idea of matrix-valued function as an appropriate medium for Liapunov function construction. This approach has been developed since 1984 and some of the obtained results are published and summarized by Martynyuk [9, 12], and Kats and Martynyuk [1].

Actually, the problem of constructing the Liapunov functions for the class of nonlinear systems of (3.3) type is reduced to the solution of systems of first order partial equations (3.5) which are more simple than the Liapunov equation for the initial system proposed by in 1892 in his famous dissertation paper.

This section is based on some results by Martynyuk and Slyn'ko [1, 2, 3], and Slyn'ko [2]. Besides, some results by Djordjevic [1, 2], Hahn [1], Krasovskii [1], Lankaster [1], etc.are used.

Section 4 The phenomenon of motion polystability has been investigated in nonlinear dynamics since 1987. As noticed by Aminov and Sirazetdinov [1], and Martynyuk [16] this phenomenon was discovered while developing the notion of stability with respect to a part of variables. In monographs by Martynyuk [9, 12] some results are presented obtained in the development of the theory of motion polystability including sufficient conditions for exponential polystability in the first approximation (see also Martynyuk [14, 15], and Slyn'ko [1]). This section encorporated the results by Martynyuk and Slyn'ko [3].

Section 5 Linear nonautonomous system of (5.1) type or autonomous system (5.14) is of essential interest in context with the problem of constructing the Liapunov function since this allows to investigate stability of the equilibrium state of some quasilinear systems. In spite of the seeming simplicity of linear systems the problem of constructing the appropriate Liapunov function remains open in this case es well (see, e.gḂarbashin [1], Zhang [1], etc.).

In this section for the above-mentioned systems we adopt the algorithm of Liapunov function construction presented in Section 3. Since in this case systems of equations (3.5) turns to be linear differential or algebraic, their exact solutions can sometimes be found. The section is based on the results by Martynyuk and Slyn'ko [1 − 3].

Section 6 The Bellman–Bailey approach (see Bellman [1] and Bailey [1]) to stability investigation of large-scale systems has been developed considerably in many papers. In monographs by Barbashin [1], Michel and Miller [1], Siljak [1], Grujic, Martynyuk and Ribbens-Pavella [1], Voronov and Matrosov [1], etcȧlongside the original results the results of many investigations of dynamics of linear and nonlinear systems in terms of vector Liapunov functions are summarized. An essential deficiency of this approach is the supersufficiency of stability conditions for the systems of motion equations under consideration (see Piontkovskii and Rutkovskaya [1], and Martynyuk and Slyn'ko [1]).

The application of the matrix-valued Liapunov function for the same classes of systems of equations provides wider conditions of stability. The reasons for this were scrutinized by Martynyuk [9,/,12]. In this section by the example of linear system it is shown how supersufficient the stability conditions obtained via the Bellman–Bailey approach are as compared with those obtained via the application of matrix-valued function.

Section 7 The problems set out in this section are addressed first of all to the young researchers in the area of motion stability theory and its application. Solution of any of the problems will be not only a subject of a significant paper but an essential contribution to the development of the method of matrix Liapunov functions which was called by ProfV̇. Lakshmikantham (in Moscow, 2001) one of three outstanding achivements of stability theory in the 20th century.

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