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The V_o property in Banach Lattices *

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Abstract.

Order weakly compact and order unconditionally converging operators are consider on the setting of Banach lattices.

In this paper we characterize the class of Banach lattices on which o-weakly compact operators on it are order unconditionally converging operators. Two class of Banach having the V_o property is showing. We also consider the spaces E on which every $|\sigma|(E', E)$-convergent sequences on E' are $\sigma(E', E'')$ -convergent.

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For notations and terminology concerning Banach lattices, we refer the reader to [1] and [9].

We denote the norm dual of a Banach lattice E by E'. Besides the topologies $\sigma(E', E)$ and $\sigma(E', E'')$ in E' we shall need to consider the absolute weak topology. The absolute weak topology $|\sigma|(E', E)$ in E' is the locally convex-solid topology of uniform convergence on the order intervals of E; and it is generated by the family of Riesz seminorms $\{p_x : x \in E\}$, where $p_x(f) = |f|(|x|)$ for each f in E'.

If $T : E \to F$ is an operator (i. e. a linear continuous mapping) between two Banach spaces, then its adjoint $T' : F' \to E'$ is the operator defined by $< T'f, x > = < f, Tx >$ for each f in E' and x in E.

Let E be a Banach lattice and X be a Banach space, an operator $T : E \to X$ is called o-weakly compact if T maps order bounded sets of E into relatively weakly compact subsets of X. This class of operators was first consider by Dodds, [2],

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who noted its connection with the class of all weakly compact operators defined on $C(K)$ spaces.

The next result describes the o-weakly compact operators in terms of disjoint sequences. Its proof can be obtained from a classical result of A. Grothendieck [3] by using Kakutani representations theorems for Abstract M-spaces.

Lemma 1.1 *Let E be a Banach lattice and X be a Banach space. Then a linear continuous operator $T : E \to F$ is o-weakly compact if and only if T maps order bounded disjoint sequences into norm convergent sequences.* ∎

Let E be a Banach lattice. Following [1], we say that $K \subset E'$ is o-equicontinuous in E if for each $0 \leq x \in E$ and $\epsilon > 0$ there exists some $g \geq 0$ in the ideal generated by K in E' such that $< (| f | -g)^+, x > \leq \epsilon$ holds for all f in K; (see [1, Theorem 20.6]). The next theorem characterize the o-weakly compact operators and it is an easy consequence of lemma 1.1. and Theorem 20.6 of [1].

Theorem 1.2 *Let E be a Banach lattice, X be a Banach space and $T : E \to X$ be a linear continuous operator. Then T is o-weakly compact if and only if T' transform bounded subsets of X' into order-equicontinuous in E subsets of E'.* ∎

Let X be a Banach space, Grothendieck shows in [3], that each linear continuous operator T from X into a separable Banach space is weakly compact if and only if $\sigma(X', X)$-convergent sequences in X' are $\sigma(X', X'')$-convergent.

In the next lemma we consider positive linear operators defined in Banach lattices.

Lemma 1.3 *Let E be a Banach lattice. The following statements are equivalent:*
(a) Each positive operator T from E into a separable Banach lattice is weakly compact.
(b) Each positive operator T from E into c_o is weakly compact.
(c) Every $| \sigma | (E', E)$-convergent sequence is $\sigma(E', E'')$ -convergent.
(d) Each positive operator T from E into a Banach lattice F such that the set $\{y' \in F' :\| y' \| \leq 1\}$ is $\sigma(E', E)$ -relatively sequentially compact is weakly compact.

Proof. Clearly $(a) \Rightarrow (b)$ and $(d) \Rightarrow (a)$
(b) $\Rightarrow$ (c) Let $\{x_n'\}_n$ be a sequence in E' such that $x_n' \to 0 \mid \sigma \mid (E', E)$. Since the operator $T : E \to c_o$ defined by $T(x) = \{| x_n' | (x)\}_n$ is positive, it is weakly compact. By Gantmacher Theorem's its adjoint $T' : l_1 \to E'$ is weakly compact, then if $\{e_n : n \in \mathbb{N}\}$ denotes the usual basis for l_1, the set $\{T'e_n : n \in \mathbb{N}\}$ is $\sigma(E', E'')$-relatively compact, so we have that $| x_n' | \to 0 \; \sigma(E', E'')$ since $T'e_n = | x_n' |$. Thus $x_n' \to 0 \sigma(E', E'')$.

(c) $\Rightarrow$ (d) Let F be a Banach lattice such that the set $\{x' \in F' : \| x' \| \leq 1\}$ is $\sigma(F', F'')$ -relatively sequentially compact and $T : E \to F$ be a positive operator.

If $B(F')$ denotes the unit ball of F', let $\{y_n'\}_n$ be a positive sequence in $B(F')$, since $\{y_n'\}_n$ has a subsequence $\sigma(F', F'')$- convergent, we can assume that the sequence $\{T'y_n'\}_n$ is $| \sigma | (E', E)$-convergent, then by our hypothesis, the sequence

$\{T'y'_n\}_n$ is $\sigma(E', E'')$-convergent. Then $T'(B(F'))$ is $\sigma(E', E'')$-relatively compact since $T'(B(F') \cap F'_+)$ does it. Thus T' is weakly compact and then T is also weakly compact. ■

Every Banach lattice which is a Grothendieck space verify the equivalent conditions of lemma1.3. Moreover every $C(K)$ space has the same property.

Corollary 1.4 *If E is a Banach lattice that verifies the equivalent conditions of Lemma 1.3, then:*
(a) If E is separable, then E is reflexive
(b) If E has an order continuous norm, then E' contains no lattice isomorph to l_1.

Proof.
(a) Note that I_E is weakly compact
(b) Let $\{x'_n\}_n$ be a norm bounded disjoint sequence in E'. By the order-continuity of the norm in E, $x'_n \to 0 \mid \sigma \mid (E', E)$. By Lemma 1.3, $x'_n \to 0\ \sigma(E', E'')$, then E'' has an order continuous norm and by [7], E' contains no lattice isomorph to l_1. ■

2 Order Unconditionally Converging Operators

Let E be a Banach lattice and X be a Banach space, a continuous linear operator $T : E \to X$ is called order unconditionally converging (o.u.c.) if T maps weakly summable sequences of positive elements of E into unconditionally summable sequence in X.

Nicolescu, in [7], obtain the next characterization of o. u. c. operators.

Theorem 2.1 *Let E be a Banach lattice, X be a Banach space and $T : E \to X$ be a continuous linear operator. Then the following assertions are equivalent:*
(a) T is o.u.c.
(b) $0 \leq x_n \uparrow, \| x_n \| \leq K$ in E implies $\{Tx_n\}_n$ is norm convergent in X
(c) If $\{x_n\}_n$ is a weakly summable sequence of pairwise disjoint positive elements of E, then $\| Tx_n \| \to 0$
(d) There exists no sublattice F of E, lattice isomorph to c_0 such that T/F is an isomorphism. ■

Following Pelczynki's ideas, see [8], a Banach lattice E is said to have the V_0 property if every o-weakly compact operator T from E into an arbitrary Banach space X is an o.u.c. operator. The following theorem characterizes the Banach lattices with the V_0 property.

Theorem 2.2 *Let E be a Banach lattice. Then the following statements are equivalent:*
(a) E has the V_0 property
(b) For each subset K of E' which is order equicontinuous in E we have that lim sup $\{\mid x'(x_n) \mid : x' \in K\} = 0$ for all weakly summable sequence $\{x_n\}_n$ in E^+.

Proof.
(a) $\Rightarrow$ (b) Let K be an order equicontinuous in E subset of E'. Then by [1, Theorem 20.6], lim sup $\{| x'(x_n) |: x' \in K\} = 0$ for each order bounded pairwise disjoint sequence $\{x_n\}_n$ in E. But by Lemma 1.1 the continuous linear operator $T : E \rightarrow l_\infty(K)$ defined by $Tx = \{x'(x)\}_{x' \in K}$ is an o-weakly compact operator, then by (a) T is an o.u.c. operator.

If $\{x_n\}_n$ is a weakly summable sequence in E^+, then $\| Tx_n \| \rightarrow 0$. Therefore we have that lim sup $\{| x'(x) |: x' \in K\} = 0$

(b) $\Rightarrow$ (a) Let $T : E \rightarrow X$ be an o-weakly compact operator, then by Theorem 1.2, $K = T'(B(X'))$ is order equicontinuous in E. Let $\{x_n\}_n$ be a weakly summable sequence in E, by condition (b) lim sup $\{| x'(x_n) |: x' \in K\} = 0$.
Since sup $\{| x'(x_n) |: x' \in K\} = sup\{| y'(Tx_n) |: y' \in B(X')\}$, we conclude that $\| Tx_n \|_\infty \rightarrow 0$. ∎

The next corollary follows immediately from the above theorem

Corollary 2.3 *(a) For every compact Hausdorff space K, $C(K)$ have the V_0 property.*
(b) If E is a Banach lattice possessing the V_0 property and F be a closed ideal of E. Then $E/_F$ have the V_0 property.
(c) If E is a Banach lattice with order continuous norm, then E has the V_0 property if and only if E is weakly sequentially complete. ∎

We conclude this paper by showing two class of Banach lattices having the V_0 property.

Theorem 2.4 *(a) Every perfect Banach lattice has the V_0 property*
(b) If E is a Banach lattice which satisfies the equivalent conditions of Lemma 1.3. Then E has the V_0 property.

Proof.
(a) Let E be a perfect Banach lattice and T be an o-weakly compact operator from E into an arbitrary Banach space X.

Let $\{x_n\}_n$ be a positive weakly summable sequence in E. Then the sequence $y_n = \sum_{k=1}^{n} x_k$ is norm bounded and increasing in E. Since E is a perfect Banach lattice, there exists some y in E such that $y = sup\, y_n$. Clearly $0 \leq y_n \leq y$ holds for all n. Thus by Lemma 1.1, $\{Tx_n\}_n$ is a norm null sequence in E. The conclusion follows from Theorem 2.1.

(b) If E does not have the V_0 property, then there exists a Banach space X and some o-weakly compact operator $T : E \rightarrow X$ that is not an o.u.c. operator. Then there exists some $\epsilon > 0$ and a positive weakly summable sequence of pairwise disjoint elements $\{x_n\}_n$ in E such that $\| Tx_n \| \geq \epsilon$ for all n.

Let $y'_n \in B(X')$ be such that $y'_n(Tx_n) \geq \epsilon$ for all n, and let $\{z'_n\}_n$ be a pairwise disjoint sequence in E' with $y'_n(Tx_n) = z'_n(x_n)$ and $| z'_n | \leq | T'y'_n |$ for each n. Since $T'(B(X'))$ ia an order equicontinuous in E subset of E' and z'_n belongs to the solid hull of $T'(B(X'))$, then by [1, Theorem 20.6] $| z'_n | \rightarrow 0$ $\sigma(E', E)$, and by our hypothesis $| z'_n | \rightarrow 0$ $| \sigma | (E', E'')$. Since $\{x_n\}_n$ is a weakly summable sequence, $lim\, | z'_n | (x_n) = 0$ and this implies $lim\, y'_n(Tx_n) = 0$ contrary to our assumption. ∎

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