CUBO A Mathematical Journal
Vol.19, Nº 02, (01–09). June 2017

On Some Recurrent Properties of Three Dimensional K-Contact Manifolds

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ABSTRACT

In this paper we characterize some recurrent properties of three dimensional K-contact manifolds. Here we study Ricci η-recurrent, semi-generalized recurrent and locally generalized concircularly ϕ-recurrent conditions on three dimensional K-contact manifolds.

RESUMEN

En este paper caracterizamos algunas propiedades recurrentes de variedades K-contacto tridimensionales. Estudiamos las condiciones de Ricci η-recurrencia, recurrencia semi-generalizada y ϕ-recurrencia concircular localmente generalizada en variedades K-contacto tridimensionales.

Keywords and Phrases: K-contact manifold, Ricci η-recurrent, semi-generalized recurrent, locally generalized concircularly ϕ-recurrent, scalar curvature, Einstein manifold.

2010 AMS Mathematics Subject Classification: 53C25, 53D15.

1 Introduction

In 1950, Walker [17] introduced the notion of recurrent manifolds. In the last five decades, recurrent structures have played an important role in the geometry and the topology of manifolds. In [3], the authors De and Guha introduced the idea of generalized recurrent manifold with the non-zero 1-form A and another non-zero associated 1-form B. If the associated 1-form B becomes zero, then the manifold reduces to a recurrent manifold given by Ruse [11]. As a generalization of recurrency, Khan [6] introduced the notion of generalized recurrent Sasakian manifold. Semi-generalized recurrent manifolds were first introduced and studied by Prasad [10]. The notion of recurrency in a Riemannian manifold has been weakened by many authors in several ways to different extent viz., [1, 8, 12] etc.,

A K-contact manifold is a differentiable manifold with a contact metric structure such that ξ is a Killing vector field [2, 13]. These are studied by several authors like [4, 9, 14, 15] and many others. It is well known that every Sasakian manifold is K-contact, but the converse ia not true, in general. However a three-dimensional K-contact manifold is Sasakian [5].

Motivated by the above studies, in this study we consider some recurrent properties of three dimensional K-contact manifolds. The paper is organized in the following way: In Section 2, we give the definitions and some results concerning the K-contact manifolds that will be needed hereafter. In Section 3, we discuss the Ricci η-recurrent property of three dimensional K-contact manifold. In particular, we obtain the 1-form A is η parallel and give the expression for Ricci tensor. The Section 4 is devoted to three dimensional semi-generalized recurrent K-contact manifolds. Here we prove some interesting results, such as the facts that a specific linear combination of the 1-forms A and B is always zero and that the manifold is Einstein. In Section 5, we consider three dimensional locally generalized concircularly ϕ-recurrent K-contact manifolds. In this case the manifold is a space of constant curvature.

2 Preliminaries

A Riemannian manifold M is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it carries a tensor field ϕ of type $(1, 1)$, a vector field ξ, 1-form η and compatible Riemannian metric g on M, such that

$$\phi^2 X = -X + \eta(X)\xi, \;\; \phi\xi = 0, \;\; \eta(\phi X) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \;\; g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \;\; g(\phi X, X) = 0. \tag{2.4}$$

If moreover ξ is Killing vector field, then M is called a K-contact manifold [2, 13]. A K-contact manifold is called Sasakian [2], if the relation

$$(\nabla_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

holds on M, where ∇ denotes the operator of covariant differentiation with respect of metric g.

In a K-contact manifold, the following relations hold:

$$\nabla_X\xi = -\phi X, \tag{2.6}$$

$$(\nabla_X\eta)(Y) = g(\nabla_X\xi, Y). \tag{2.7}$$

Also in a three dimensional K-contact manifold, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= \frac{r-4}{2}[g(Y,Z)X - g(X,Z)Y] - \frac{r-6}{2}[g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{2.8}$$

$$S(X,Y) = \frac{1}{2}[(r-2)g(X,Y) - (r-6)\eta(X)\eta(Y)], \tag{2.9}$$

$$QX = \frac{1}{2}[(r-2)X - (r-6)\eta(X)\xi], \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2\eta(X)\eta(Y), \tag{2.11}$$

where r, S and Q are the scalar curvature, Ricci tensor and Ricci operator respectively.

Definition 1. *A K-contact manifold is said to be Einstein if the Ricci tensor S is of the form*

$$S(X,Y) = a g(X,Y),$$

where a is constant.

3 On three dimensional Ricci η-recurrent K-contact manifold

Definition 2. *The Ricci tensor of an three dimensional K-contact manifold is said to be η-recurrent if its Ricci tensor satisfies the following:*

$$(\nabla_X S)(\phi(Y), \phi(Z)) = A(X)S(\phi(Y), \phi(Z)), \tag{3.1}$$

for all vector fields $X, Y, Z \in TM$, where $A(X) = g(X, \rho)$, ρ is called the associated vector field of 1-form A.

In particular, if the 1-form A vanishes then the Ricci tensor is said to be η-parallel and this notion for Sasakian manifold was first introduced by Kon [18].

Now consider three dimensional Ricci η-Recurrent K-contact manifold. From (3.1), it follows that

$$\nabla_Z S(\phi(X), \phi(Y)) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y) = A(Z)S(\phi(X), \phi(Y)). \tag{3.2}$$

By using (2.5), (2.6) and (2.11) in (3.2), yields

$$\begin{aligned}(\nabla_Z S)(X, Y) &= -\eta(X)[2g(\phi Z, Y) + S(Z, \phi Y)] - \eta(Y)[2g(\phi Z, X) + S(\phi X, Z)] \\ &+ A(Z)[S(X, Y) - 2\eta(X)\eta(Y)].\end{aligned} \tag{3.3}$$

Hence we can state the following:

Theorem 3.1. *In a three dimensional* K*-contact manifold, the Ricci tensor is* η*-recurrent if and only if (3.3) holds.*

By virtue of (3.3), let $\{e_i\}$ is an local orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq 3$, we have

$$dr(Z) = [r - 2]A(Z). \tag{3.4}$$

If the manifold has a constant scalar curvature r ($r \neq 2$ because the 1-form A is definite), then from (3.4) it follows that

$$A(Z) = 0, \ \ \forall \ Z.$$

This leads to the following:

Theorem 3.2. *In a three dimensional Ricci* η*-recurrent* K*-contact manifold* M *if the scalar curvature is constant then the 1-form* A *is* η*-parallel.*

Again putting $X = Z = e_i$ in (3.3), and taking summation over i, $1 \leq i \leq 3$, we get

$$\frac{1}{2}dr(Y) + \mu\eta(Y) = S(Y, \rho) - 2\eta(\rho)\eta(Y), \tag{3.5}$$

where $\mu = \Sigma_{i=1}^{3} S(\phi e_i, e_i)$. By using (3.4) in (3.5), we obtain

$$\frac{1}{2}A(Y)[r - 2] + \mu\eta(Y) = S(Y, \rho) - 2\eta(\rho)\eta(Y), \tag{3.6}$$

Putting $Y = \xi$ in (3.6), yields

$$\mu = \left(1 - \frac{r}{2}\right)\eta(\rho). \tag{3.7}$$

Considering (3.7) in (3.6), we get

$$S(Y, \rho) = \left(\frac{r}{2} - 1\right) g(Y, \rho) + \left(3 - \frac{r}{2}\right) \eta(\rho)\eta(Y). \tag{3.8}$$

Thus we have the following result:

Theorem 3.3. *If the Ricci tensor in a three dimensional* K*-contact manifold is* η*-recurrent, then its Ricci tensor along the associated vector field of the 1-form is given by (3.8).*

Substituting $Y = \phi Y$ in (3.8) and by virtue of (2.1), we obtain

$$S(Y, L) = Kg(Y, L), \tag{3.9}$$

where $L = \phi\rho$, $K = \frac{r}{2} - 1$.
Hence we can state the following:

Theorem 3.4. *If the Ricci tensor in a three dimensional* K*-contact manifold is* η*-recurrent, then* $K = \frac{r}{2} - 1$ *is an eigen value of the Ricci tensor corresponding to the eigen vector* $\phi\rho$*.*

4 On three dimensional semi-generalized recurrent K-contact manifolds

Definition 3. *A Riemannian manifold is said to be semi-generalized recurrent manifold if its curvature tensor* R *satisfies the relation*

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)g(Z, W)Y, \tag{4.1}$$

where A *and* B *are two 1-forms,* B *is non-zero,* ρ_1 *and* ρ_2 *are two vector fields such that*

$$g(X, \rho_1) = A(X), \quad g(X, \rho_2) = B(X), \tag{4.2}$$

for any vector field X *and* ∇ *be the covariant differentiation operator with respect to the metric* g*.*

Definition 4. *A Riemannian manifold* M *is said to be three dimensional semi-generalized Ricci recurrent manifold if:*

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + 3B(X)g(Y, Z). \tag{4.3}$$

Taking cyclic sum of (4.1) with respect to X, Y, Z, and using second Bianchi's identity, we get

$$\begin{aligned} 0 &= A(X)R(Y, Z)W + A(Y)R(Z, X)W + A(Z)R(X, Y)W \\ &+ B(X)g(Z, W)Y + B(Y)g(X, W)Z + B(Z)g(Y, W)X. \end{aligned} \tag{4.4}$$

On contracting above equation with respect to Y, yields

$$\begin{aligned} 0 &= A(X)S(Z, W) - g(R(Z, X)\rho_1, W) - A(Z)S(X, W) \\ &+ 3B(X)g(Z, W) + g(X, W)g(\rho_2, Z) + B(Z)g(X, W). \end{aligned} \tag{4.5}$$

Again putting $Z = W = e_i$ in (4.5), and taking summation over i, $1 \le i \le 3$, we obtain

$$rA(X) + 11B(X) - 2S(X, \rho_1) = 0. \tag{4.6}$$

Putting $X = \xi$ in (4.6) and by virtue of (4.2) and (2.11), we get

$$r = \frac{1}{\eta(\rho_1)}[4\eta(\rho_1) - 11\eta(\rho_2)]. \tag{4.7}$$

Since for a contact metric manifold $\eta(\rho_1) \neq 0$. Hence we can state the following:

Theorem 4.1. *In a three dimensional semi-generalized recurrent K-contact manifold, the scalar curvature r takes the form (4.7).*

Again taking $Z = \xi$ in (4.3), we get

$$(\nabla_X S)(Y, \xi) = A(X)S(Y, \xi) + 3B(X)g(Y, \xi). \tag{4.8}$$

Left hand side of the above equation can be written as

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi). \tag{4.9}$$

In view of (2.2), (2.9) and (4.9) in (4.8), gives

$$-2g(\phi X, Y) + S(\phi X, Y) = 2A(X)\eta(Y) + 3B(X)\eta(Y). \tag{4.10}$$

Plugging $Y = \xi$ in (4.10), we obtain

$$2A(X) + 3B(X) = 0.$$

This leads to the following:

Theorem 4.2. *In a three dimensional semi-generalized Ricci recurrent K-contact manifold, the linear combination $2A + 3B$ is always zero.*

Replace Y by ϕY in (4.10), we get

$$S(X, Y) = 2g(X, Y).$$

Thus we have the following result:

Theorem 4.3. *A three dimensional semi-generalized Ricci recurrent K-contact manifold is Einstein manifold.*

5 On three dimensional locally generalized concircularly ϕ-recurrent K-contact manifolds

Definition 5. *A three dimensional K-contact manifold is called the locally generalized concircularly ϕ-recurrent if its concircular curvature tensor $\widetilde{C}$*

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6}[g(Y, Z)X - g(X, Z)Y], \tag{5.1}$$

satisfies the condition

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{5.2}$$

for all X, Y, Z *and* W *orthogonal to* ξ.

Taking covariant differentiation of (2.8) with respect to W, we get

$$\begin{aligned}(\nabla_W \tilde{R})(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] - \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] - \frac{r-6}{2}[g(Y,Z)(\nabla_W\eta)(X)\xi \\ -g(X,Z)(\nabla_W\eta)(Y)\xi + (\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X \\ -(\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y].\end{aligned} \tag{5.3}$$

Again taking X, Y, Z and W orthogonal to ξ, we obtain

$$\begin{aligned}(\nabla_W \tilde{R})(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] - \frac{r-6}{2}[g(Y,Z)g(\phi X, W)\xi \\ &- g(X,Z)g(\phi Y, W)\xi].\end{aligned} \tag{5.4}$$

From above equation it follows that

$$\phi^2((\nabla_W \tilde{R})(X,Y)Z) = \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X]. \tag{5.5}$$

Taking covariant differentiation of (5.1) with respect to W, we get

$$(\nabla_W \tilde{\tilde{C}})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y], \tag{5.6}$$

from which it follows that

$$\begin{aligned}\phi^2((\nabla_W \tilde{\tilde{C}})(X,Y)Z) &= \phi^2((\nabla_W \tilde{R})(X,Y)Z) \\ &- \frac{dr(W)}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].\end{aligned} \tag{5.7}$$

By virtue of (2.1), (5.2), (5.5) in (5.7), yields

$$R(X,Y)Z = \left[\frac{r}{6} - \left(\frac{B(W)}{A(W)} + \frac{dr(W)}{3A(W)}\right)\right][g(Y,Z)X - g(X,Z)Y]. \tag{5.8}$$

Since in a locally generalized concircularly ϕ-recurrent K-contact manifold $A(W) \neq 0$. On contracting above equation over W, we get

$$R(X,Y)Z = \mu[g(Y,Z)X - g(X,Z)Y], \tag{5.9}$$

where $\mu = \frac{r}{6} - \left(\frac{B(e_i)}{A(e_i)} + \frac{dr(e_i)}{3A(e_i)}\right)$ is a scalar. Then by Schur's theorem [7] μ will be constant on the manifold.

Thus we have the following result:

Theorem 5.1. *A three dimensional locally generalized concircularly* ϕ*-recurrent* K*-contact manifold is a space of constant curvature.*

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