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Further reduction of Poincaré-Dulac normal forms in symmetric systems

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ABSTRACT

The Poincaré-Dulac normalization procedure is based on a sequence of coordinate transformations generated by solutions to homological equations; in the presence of resonances, such solutions are not unique and one has to make somewhat arbitrary choices for elements in the kernel of relevant homological operators, different choices producing different higher order effects. The simplest, and usual, choice is to set these kernel elements to zero; here we discuss how a different prescription can lead to a further simplification of the resulting normal form, in a completely algorithmic way.

RESUMEN

El procedimiento de normalización de Poincaré-Dulac se basa en una sucesión de transformaciones de coordenadas generadas por soluciones de las ecuaciones homológicas. En la presencia de resonancias, esas soluciones no son únicas y se hacen elecciones arbitrarias de los elementos en el kernel de relevantes operadores de homología. Diferentes elecciones producen diferentes efectos de orden superior. La elección más simple y común es anular los elementos del kernel. Aquí discutimos cómo una elección diferente puede conducir a una simplificación mayor de la forma normal resultante, de una manera completamente algorítmica.

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Introduction.

Poincaré-Birkhoff normal forms [1, 12, 15] are central to our understanding of nonlinear dynamics in perturbative terms in many ways.

When the normal form of X does not reduce to its linear part X_0, it is well known that there is some freedom in the normalizing procedure; different choices in this will produce different normal forms. The choices alluded to here can be thought as a choice of an element in the (nontrivial) kernel of the homological operator; in the original Poincaré-Dulac discussion – and in most discussions in later times – this is assumed to be just zero for simplicity.

Already Dulac [6] remarked that it is conceivable that a different choice would produce a somewhat "simpler" normal form, and several authors have been working, also in recent times, in this direction. We will not attempt at reviewing the different approaches present in the literature; a short discussion with references is provided e.g. in [4].

The present author has discussed a "further normalization" of Poincaré-Dulac normal forms [9]; as this is a direct extension of Poincaré-Dulac approach based on higher order homological operators, the corresponding normal forms have been christened "Poincaré renormalized forms" (PRF). This approach has later on been adapted to cases where a Lie algebraic structure is naturally present in the set of resonant vector fields, i.e. in the normal forms corresponding to a given linear part [10]; in this case the name "Lie renormalized forms" (LRF) is used.

The purpose of this note is twofold. On the one hand, we want to provide a brief but comprehensive overview of this approach and of the results that can be obtained within it. On the other hand, in many occurrences, in particular (but not only) for systems of physical significance, we deal with nonlinear dynamical systems with given symmetry properties; thus we want to discuss how this approach deals with symmetric systems.

In this note, we deal with dynamics described by a vector field X on a smooth real n-dimensional manifold M; we want to study this around an equilibrium position $m_0 \in M$, and reduce to a neighbourhood E of m_0 in M. As considerations are merely local, we identify E with an $\mathbf{R}^n$ space, and introduce cartesian coordinates $(x^1, ..., x^n)$; the equilibrium point m_0 will correspond to the origin of this coordinate system.

Thus our general frame will be to consider a dynamical system in $\mathbf{R}^n$,

$$\dot{x} \;=\; f(x) \tag{1}$$

where $f(0) = 0$ and f is expanded around the origin as a power series

$$\dot{x} \;=\; f(x) \;=\; \sum_{k=0}^{\infty} f_k(x) \tag{2}$$

with $f_k(ax) = a^{k+1} f_k(x)$. We could equivalently consider vector fields $X = f^i(x)\partial/\partial x^i$ or vector power series $\sum_k f_k(x)$.

Notice that we will *not* consider the convergence of the f expansion [5, 16], so all the series met here will be formal ones.

Similarly, we will not consider the specific features of hamiltonian systems and of Birkhoff-Gustavsson normal forms [4]; a similar theory also exists in this case, dealing with simpler objects (hamiltonian functions rather than vector fields).

1 Normal forms and redundancy of their classification

The Poincaré normalization procedure [1, 12, 15] is based on considering a sequence $(m = 1, 2, ...)$ of near-identity changes of coordinates

$$x = \widetilde{x} + h_m(\widetilde{x}) \tag{3}$$

with $h_m(a\widetilde{x}) = a^{m+1}h_m(\widetilde{x})$. We can also consider Lie-Poincaré transformations [13] (which has several conceptual and computational advantages [2]): we define a vector field $H = h^i_m\, \partial/\partial x^i$ on $\mathbf{R}^n$ and consider the flow under it,

$$dx/d\lambda \;=\; h_m(x) \tag{4}$$

which, with $x(0) = x_0$, we denote by $x(\lambda) = \Phi_m(\lambda; x_0)$; then the changes of coordinates are given by

$$x \;=\; \Phi_m(1; \widetilde{x})\ . \tag{5}$$

The two procedures give the same result up to order x^{m+1}.

Under (3) or (5), (2) is changed to a new system (we drop the tilde over x for ease of notation)

$$\dot{x} \;=\; \sum_{k=0}^{\infty} \widetilde{f}_k(x) \tag{6}$$

where $\widetilde{f}_k(x) = f_k(x)$ for $k < m$, and $\widetilde{f}_m = f_m - \mathcal{L}_0(h_m)$ with $\mathcal{L}_0$ the homological operator associated to f_0, i.e. $\mathcal{L}_0(h) := \{f_0(x), h(x)\}$ where we use the Lie-Poisson bracket $\{\varphi, \psi\} := (\varphi \cdot \nabla)\psi - (\psi \cdot \nabla)\varphi$; this corresponds to the commutator of vector fields having components φ and ψ when expressed in x coordinates.

The action on $\widetilde{f}_k$ with $k > m$ is different for Poincaré and Lie-Poincaré transformations; for the latter, we have (with $[k/m]$ the integer part of k/m) [9, 13]

$$\widetilde{f}_k \;=\; \sum_{s=0}^{[k/m]} \frac{1}{s!}\, \mathcal{H}^s_m(f_{k-sm}) \tag{7}$$

where the operator $\mathcal{H}_m$ is defined as $\mathcal{H}_m(.) := \{h_m, .\}$.

Notice that if V_k denotes vector polynomials homogeneous of degree $(k+1)$, we have $\{.,.\} : V_k \times V_m \to V_{k+m}$, so that $\mathcal{L}_0 : V_k \to V_k$ and $\mathcal{H}_m : V_k \to V_{k+m}$.

By introducing the projection π_0 on the range of $\mathcal{L}_0$, and choosing at each step a h_m solution to the *homological equation*

$$\mathcal{L}_0(h_m) \ = \ \pi_0 f_m \tag{8}$$

(here f_m is the one obtained after the previous $m-1$ transformations), we can eliminate all the nonlinear terms which are not in $[\mathrm{Ran}(\mathcal{L}_0)]^\perp = \mathrm{Ker}(\mathcal{L}_0^+)$, and set the system in Poincaré-Dulac normal form,

$$\dot{x} \ = \ \sum_{k=0}^{\infty} g_k(x) \tag{9}$$

where $g_0(x) = f_0(x) = Ax$ and all the terms g_k with $k \geq 1$ are resonant ones (see standard references [1, 12] for details, as well as for the meaning of the adjoint operator mentioned above).

For ease of discussion, we will consider A to be semisimple, so that we can – with a linear transformation – take it to be diagonal, $A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$; the resonant terms are then a sum of resonant monomials $|\mu, r\rangle$: these are vectors having only the r component different from zero, and equal to

$$x^\mu \ \equiv \ x_1^{\mu_1} ... x_n^{\mu_n} \quad \mu_i \in \mathbf{N} \tag{10}$$

with μ satisfying a resonant relation of order $|\mu| = \sum_i \mu_i$:

$$(\mu \cdot \lambda) \ := \ \sum_{i=1}^{n} \mu_i \lambda_i \ = \ \lambda_r \ ; \tag{11}$$

see usual references [1, 12] for more general $A = (Df)(0)$.

It is obvious from (8) that: (i) we cannot eliminate terms in $\mathrm{Ker}(\mathcal{L}_0^+)$, which are indeed the resonant ones; (ii) the h_m is determined up to a $\delta h_m \in \mathrm{Ker}(\mathcal{L}_0)$. In particular, choosing different δh_m will not affect $\widetilde{f}_m$ but will produce different effects on higher order terms, see (7). Thus, different normal forms can be (formally) conjugated to the same system (2), and therefore (formally) conjugated among themselves.

We would like therefore to reduce this redundancy in normal forms classification; this would also have a computational advantage [10], and is thus relevant to obtain algorithmic – and easily implementable – procedures, as the standard Poincaré one.

2 Poincaré renormalized form

The idea behind the approach I proposed in previous work [9] is to use the freedom in the choice of δh_m in order to control – i.e. if possible eliminate – higher order resonant terms.

I should say immediately that the "Poincaré renormalized forms" (PRF) obtained in this way are, in general, *not* unique; on the other side, they: i) can represent a considerable simplification over standard normal forms; ii) can be obtained algebraically,

with the same kind of computations needed for standard normal forms (NF), by a simple and well defined recursive algorithm. I will now briefly define PRFs and describe their construction, following [9].

We define higher order homological operators

$$\mathcal{L}_m := \{f_m, .\} \tag{12}$$

(notice this will be useful only after f_m has stabilized, see below), and the spaces

$$H^{(p)} := \cap_{s=0}^{p-1} \mathrm{Ker}(\mathcal{L}_s) \text{ and } F^{(p)} := \cap_{k=0}^{p} \mathrm{Ker}(\mathcal{M}_k) \equiv \cap_{k=0}^{p} [\mathrm{Ran}(\mathcal{M}_k)]^{\perp} \tag{13}$$

where $\mathcal{M}_p : H^{(p)} \to V$ is the restriction of $\mathcal{L}_p : V \to V$ to $H^{(p)}$.

We will operate sequentially for $k = 1, 2, ...$ as follows. Suppose we have already operated the transformation at orders up to $k-1$, and denote by $f_k^{(0)}$ the resulting term $\widetilde{f}_k$; we will then operate a sequence of Lie-Poincaré transformations with generators $h_k^{(0)}, h_{k-1}^{(1)}, ..., h_1^{(k-1)}$, where $h_p^{(s)} \in H_p^{(s)} = H^{(s)} \cap V_p$: this condition guarantees, see (7), that the f_m – and thus the $\mathcal{L}_m$ – with $m < k$ will not be changed and can be considered as stabilized. The transformation with generator $h_{k-p}^{(p)}$ will change $f_k^{(p)}$ into

$$f_k^{(p+1)} = f_k^{(p)} - \mathcal{M}_p \left[h_{k-p}^{(p)} \right] . \tag{14}$$

The $h_k^{(0)}$ – which operates the standard normalization at order k – is chosen as solution to the standard homological equation (8), while the $h_{k-p}^{(p)} \in H_{k-p}^{(p)}$ should be chosen as solution to the higher order homological equations

$$\mathcal{M}_p \left[h_{k-p}^{(p)} \right] = \pi_p f_k^{(p)} , \tag{15}$$

where π_p is the projection on the range of $\mathcal{M}_p$.

In this way we arrive in the end to a system $\dot{x} = \sum_{k=0}^{\infty} f_k^*(x)$ in which the renormalized terms satisfy $f_k^{(k)} = f_k^*$ satisfy $f_k^* \in F_k^{(k)} = F^{(k)} \cap V_k$. We define a system to be in PRF up to order n when this is satisfied – i.e. $f_k \in F^{(k)}$ – for $k < n$, and the above discussion shows that *any dynamical system (vector field, formal power series) in* $\mathbf{R}^n$ *can be formally brought into PRF up to any desired order* n *by means of a formal series of Lie-Poincaré transformations.*

Example. Let $n = 2$ with $A = (Df)(0)$ given by $A_{ij} = -\epsilon_{ij}$ (the standard rotation matrix), and denote by I the two dimensional identity matrix. The standard NF is then $f(x) = Ax + \sum_{k=1}^{\infty} (x_1^2 + x_2^2)^k [a_k I + b_k A]x$.

If the first nonzero a_k is a_μ, and the first nonzero b_k is b_ν, the corresponding PRF (with $r^2 = x_1^2 + x_2^2$) is $f^*(x) = Ax + r^{2\mu}\alpha x + r^{4\mu}\beta x + r^{2\nu}\gamma Ax$, where $\alpha = a_\mu$ and $\beta, \gamma \in \mathbf{R}$; if $\nu > \mu$, then $\gamma = 0$. The proof is given – by explicit computation – in [9]; from it also results that no small denominators can appear in the procedure. This result corresponds to a well known one [14, 8] for hamiltonian systems, but applies also to non-hamiltonian ones.

3 Further normalization and symmetries

It is well known that when (1) is symmetric under the linear action of a Lie group G, the whole normalizing procedure can be made compatible with the symmetry, and the NF will also be G-symmetric [4, 7, 12].

More precisely, let G act in $\mathbf{R}^n$ by a matrix representation M_g: then (1) is G-equivariant if, for all $x \in \mathbf{R}^n$ and all $g \in G$,

$$f(M_g x) \;=\; M_g\, f(x) \tag{16}$$

[here we are identify $\mathrm{T}_x\mathbf{R}^n$ and $\mathrm{T}_{gx}\mathbf{R}^n$ in the standard way], and the NF (9) will also satisfy the analogue of (16). If we write the NF as $\dot{x} = \widehat{f}(x) = \sum_{k=0}^{\infty} \widehat{f}_k(x)$ (we use $\widehat{f}$ to avoid confusion among different meanings of g), we have then $\widehat{f}(M_g x) = M_g \widehat{f}(x)$, and actually

$$\widehat{f}_k(M_g x) \;=\; M_g\, \widehat{f}_k(x) \tag{17}$$

due to the linearity of the group action.

This result is better recast in terms of Lie algebras, following Broer's approach [3]. Let us focus on connected Lie groups, and let S be an infinitesimal generator for G (for ease of discussion I will assume this is semisimple); then the equivariance condition (16) reads (as can be seen by choosing $M_g = I + \varepsilon S$) $\{f(x), Sx\} = 0$ or, introducing the linear operator $\mathcal{S} := \{Sx, .\}$ we have equivalently that f is equivariant if and only if

$$f \in \mathrm{Ker}(\mathcal{S}) \tag{18}$$

and the equivariance of the NF means that actually[2]

$$\widehat{f}_k \;\in\; \mathrm{Ker}(\mathcal{L}_0) \cap \mathrm{Ker}(\mathcal{S}) \;=\; F^{(0)} \cap \mathrm{Ker}(\mathcal{S})\;. \tag{19}$$

I will now show that (19) generalizes to PRF. First of all, (18) implies that $f_k \in \mathrm{Ker}(\mathcal{S})$, and thus in particular $f_0(x) = Ax$ must satisfy $\{Sx, Ax\} = 0$. But for linear fields we have – from Jacobi identity – that $\{Sx, Ax\} = [A, S]x$: therefore we must have $[A, S] = 0$, from which it also follows $[\mathcal{L}_0, \mathcal{S}] = 0$.

It follows from this that $\mathcal{S} : \mathrm{Ker}(\mathcal{L}_0) \to \mathrm{Ker}(\mathcal{L}_0)$, and conversely $\mathcal{L}_0 : \mathrm{Ker}(\mathcal{S}) \to \mathrm{Ker}(\mathcal{S})$. For non semisimple A and S we should consider also adjoint operators, with similar relations holding true.

Now, suppose we have already set the system in NF, so that (19) applies, and consider the transformation to PRF. First of all we notice that (19), together with Jacobi relation, implies that $\{\mathcal{L}_m, \mathcal{S}\} = 0$: indeed,

$$\{\mathcal{L}_m, \mathcal{S}\}\,(h) = \{\widehat{f}_m, \{Sx, h\}\} - \{Sx, \{\widehat{f}_m, h\}\} = \{\{\widehat{f}_m, Sx\}, h\} = -\{\mathcal{S}(\widehat{f}_m), h\}\;.$$

Thus, $\mathcal{L}_0, \mathcal{L}_m$ and $\mathcal{S}$ all commute with each other (note $[\mathcal{L}_m, \mathcal{L}_n] \neq 0$ in general); recalling that we actually operate only on h's which belong to $\mathrm{Ker}(\mathcal{L}_0)$, i.e. with

[2] The case where $G(x)$ does not coincide with its closure in $\mathbf{R}^n$ presents some subtleties, see chapter XVI of [11]; this case can occur only for non-compact groups. I will not discuss this case here, and suppose $G = \bar{G}$.

operators $\mathcal{M}_k$ rather than $\mathcal{L}_k$, it is immediate to see that all the $f_k^{(k)}$ will also remain in $\mathrm{Ker}(\mathcal{L}_0) \cap \mathrm{Ker}(\mathcal{S})$, and thus in particular in $\mathrm{Ker}(\mathcal{S})$.

Thus we conclude that: *for a G-symmetric system – with G acting linearly – not only the standard NF is G-symmetric, but the PRF is G-symmetric as well.*

Notice also that we can choose the generating functions h_m – both in the standard normalization and in all steps of the renormalization procedure – to be in $\mathrm{Ker}(\mathcal{S})$, and that it is precisely this choice which ensures the G-symmetry of the NF and the PRF.

We stress that here we have actually considered the Lie algebra of symmetries rather than the Lie group, thus not taking into account any genuinely discrete symmetry. The argument generalizes easily, at least in the present setting of linear group action, to discrete group actions and discrete symmetries as well.

Indeed, rewrite (17) by considering the action Λ of G on vector fields, where $\Lambda_g f$ is a vector field with components $(\Lambda_g f)(x) = M_g^{-1} f M_g x$; we say that that f is equivariant if $\Lambda_g f = f$ for all $g \in G$. Now the role of $\mathcal{S}$ is played by Λ; more precisely, we should consider, instead of $\mathrm{Ker}(\mathcal{S})$, the set $C(G)$ of vector fields such that $\Lambda h = h$. The argument goes then along the same lines, showing that $C(G) \cap \mathrm{Ker}(\mathcal{L}_0)$ is invariant under the renormalizing transformation, and we can always choose $h \in C(G)$ so that, again, the PRF will be G-symmetric.

4 Lie renormalized forms

Let us now consider a different further normalization scheme. This makes a consistent use of the Lie algebraic structure of Poincaré-Dulac normal forms [3].

Consider the set of vector fields in $\mathbf{R}^n$ which are in normal form with respect to the given linear part A, i.e. the set of $Y \in \mathcal{V}$ such that $[X_A, Y] = 0$. It is obvious that these form a Lie algebra (the Lie operation being the standard commutator of vector fields); we denote this algebra by $\mathcal{G}$.

Let us recall a general characterization of vector fields in normal form relevant in this context [4, 7, 12, 15]. Consider the linear vector field X_A; we say that the differentiable function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is an invariant for X_A if $X_A(\varphi) = 0$.

Denote by $\mathcal{I}^*(A)$ the set of invariants for X_A which are meromorphic (that is, can be expressed as a quotient of algebraic functions) in the x coordinates; denote by $\mathcal{I}(A) \subset \mathcal{I}^*(A)$ the set of algebraic invariants for X_A, and by $\mathcal{I}_k(A) \subset \mathcal{I}(A)$ the set of algebraic invariants for X_A which are functions homogeneous of degree $k+1$ in the x variables.

Let $G = C(A)$ be the centralizer of A in the algebra of n dimensional matrices; let its Lie algebra be spanned by matrices $\{K_1, ..., K_d\}$ (we can always assume $K_1 = I$, and that $K_\alpha = A$ for some α, provided $A \neq 0$; notice that $d \leq n$). We denote by $X^{(\alpha)}$ the vector fields corresponding to these, i.e. given in the x coordinates by $X^{(\alpha)} = (K_\alpha x)^i \partial_i$.

Then the most general vector field W in $\mathcal{G}$ can be written as

$$W = \sum_{\alpha=1}^{d} \mu_\alpha(x)\, X^{(\alpha)} \tag{20}$$

where $\mu_\alpha(x) \in \mathcal{I}^*(A)$. In other words, $\mathcal{G}$ is contained in a finitely generated module over $\mathcal{I}^*(A)$.

Notice that the vector field W must be algebraic in the x, and $X^{(\alpha)}$ are linear in x, so that functions $\mu_\alpha(x) \in \mathcal{I}^*(A)$ having poles of degree $d \geq 2$ in $x = 0$ cannot appear in (7). That is, only algebraic functions and functions with simple poles in the origin[3] can appear in the actual normal form unfolding: $\mathcal{G}$ is not the full G-generated module over $\mathcal{I}^*(A)$.

In several cases it happens that $\mathcal{G}$ has a more convenient structure, i.e. the μ_α in (7) can actually be taken to be in $\mathcal{I}(A)$, and not just in $\mathcal{I}^*(A)$. In this case we say that all the vector fields in $\mathcal{G}$ are *quasi-linear*, or that we have a *quasi-linear normal form*. In particular, this is the case when A admits only one basic invariant (see below).

If the normal form is quasilinear, we have $\mathcal{G} \cap \mathcal{V}_{k+1} = \mathcal{I}_k(A) \otimes G$, and the analysis of the structure of $\mathcal{G}$ results to be particularly simple, as we now discuss.

Call $\mathcal{X}^*_\alpha$ the algebra spanned by vectors which are written as $X = s(x)X^{(\alpha)}$ with $s \in \mathcal{I}^*(A)$; call $\mathcal{X}_\alpha$ the algebra spanned by vectors as above with $s \in \mathcal{I}(A)$ (this is the module over $\mathcal{I}(A)$ generated by $X^{(\alpha)}$).

As seen before, in general we have $\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_d \subseteq \mathcal{G} \subset \mathcal{X}^*_1 \oplus ... \oplus \mathcal{X}^*_d$, and in the (favorable) quasi-linear case we actually have $\mathcal{G} = \mathcal{X}_1 \oplus ... \oplus \mathcal{X}_d$.

Consider now the commutation relations between elements of the subalgebras $\mathcal{X}^*_\alpha$ and $\mathcal{X}^*_\beta$; it is immediate to check that

$$\begin{aligned} \left[\mu_\alpha(\Psi)X^{(\alpha)}, \sigma_\beta(\Psi)X^{(\beta)}\right] &= \left(\mu_\alpha(\Psi)\left(\partial\sigma_\beta/\partial\psi_i\right) X^{(\alpha)}(\psi_i)\right) X^{(\beta)} \\ &- \left(\sigma_\beta(\Psi)\left(\partial\mu_\alpha/\partial\psi_i\right) X^{(\beta)}(\psi_i)\right) X^{(\alpha)} + \left(\mu_a(\Psi)\sigma_\beta(\Psi)\right) \left[X^{(\alpha)}, X^{(\beta)}\right] . \end{aligned}$$

Notice that when $X^{(\beta)} = X_A$, by definition $X^{(\beta)}(\psi_i) = 0$, and $[X^{(\alpha)}, X^{(\beta)}] = 0$; thus the corresponding subalgebra $\mathcal{X}_\beta$ is always an abelian ideal in $\mathcal{G}$. Note also that, as obvious from the formula above, the union of subalgebras $\mathcal{X}_{\alpha_1} \cup ... \cup \mathcal{X}_{\alpha_s}$ is a subalgebra in $\mathcal{G}$ if and only if $\{X^{(\alpha_1)}, ..., X^{(\alpha_s)}\}$ span a subalgebra in G.

It can happen that we are able to determine a sequence of subalgebras $\mathcal{F}_p \subseteq \mathcal{G}$, each of them being the union of $\mathcal{X}_\alpha$ subalgebras, such that $\mathcal{F}_0 = \mathcal{G}$ and

$$[\mathcal{G}, \mathcal{F}_p] = \mathcal{F}_{p+1} ;$$

if this terminates in zero we say that $\mathcal{G}$ has a *quasi-nilpotent* structure. Notice that the factor algebras $\Gamma_p := \mathcal{F}_p/\mathcal{F}_{p+1}$ are in general *not* abelian.

[3] Let us briefly mention an example where indeed meromorphic functions of the invariants enter in the normal form unfolding. Consider systems in $\mathbb{R}^3$ with coordinates (x, y, z); let the linear part be given by the diagonal matrix $A = \mathrm{diag}(-1, 1, 2)$, so that $X_A = -x\partial_x + y\partial_y + 2z\partial_z$. This has two basic invariants, given by $\Psi_1 = xy$ and $\Psi_2 = x^2z$. We take as $X^{(\alpha)}$ the vectors $X^{(1)} = x\partial_x$, $X^{(2)} = y\partial_y$, and $X^{(3)} = z\partial_z$. It is immediate to check that $(\Psi_1^2/\Psi_2)X^{(3)} = y^2\partial_z$ and $(\Psi_2/\Psi_1)X^{(2)} = xz\partial_y$ are polynomial and resonant with X_A.

By the above remark, $\mathcal{G}$ can have a quasi-nilpotent structure only if G is nilpotent. The chain of subalgebras $\mathcal{F}_p \subset \mathcal{G}$ can then be read off the descending central series G_p of G; recall that the factor algebras $\gamma_p = G_p/G_{p+1}$ for this are abelian. The subalgebras Γ_p introduced above are therefore moduli over $\mathcal{I}(A)$ generated by abelian subalgebras γ_p of G.

Assume now $\mathcal{G}$ is quasi-nilpotent. In this case we can first work with generators in Γ_1 and simplify terms in Γ_1 (e.g. by following the PRF algorithm within the set Γ_1; this allows to work with more familiar projection and homological equations than if setting the problem in a completely Lie algebraic framework), then consider generators in Γ_2 and simplify the corresponding terms being guaranteed that Γ_1 terms are not changed, and so on.

Notice that in this case we are – roughly speaking – just using the nilpotent structure of (the finite dimensional group) G, rather than the one of (the infinite dimensional algebra) $\mathcal{G}$.

Needless to say, this approach is particularly convenient when the Γ_p are generated by a single element of G. The situation depicted above is met in applications: e.g., it applies to any nontrivial two-dimensional case. More generally, it always applies when there is only one basic invariant.

Example Let us consider a system in $\mathbf{R}^3$ (we use coordinates x, y, z) with linear part given by

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is easy to see that this has only one basic invariant $\Psi := (x^2 + y^2)$. As any meromorphic function of Ψ is either algebraic or has a pole of degree $d \geq 2$ in the origin, we deduce that the most general vector field in normal form with respect to this linear part is

$$W = X_A + \sum_{k=1}^{\infty} [a_k X_k + b_k Y_k + c_k Z_k] \ , \tag{21}$$

where (with $k \geq 0$) $X_k := \Psi^k(x\partial_x + y\partial_y)$, $Y_k := \Psi^k(-y\partial_x + x\partial_y)$, $Z_k := \Psi^k(z\partial_z)$.

This form can also be easily deduced by explicit computation applying the definition of resonant vector field. Obviously, $X_A = Y_0 - Z_0$. We denote by μ, ν, σ the first $k \geq 1$ such that a_k, b_k, c_k are nonzero.

The X_k, Y_k, Z_k satisfy the commutation relations

$$\begin{array}{l} [X_k, X_m] = 2(m-k)X_{k+m} \ , \ [Y_k, Y_m] = 0 \ , \ [Z_k, Z_m] = 0 \\ [X_k, Y_m] = 2mY_{k+m} \ , \ [X_k, Z_m] = 2mZ_{k+m} \ , \ [Y_k, Z_m] = 0 \end{array} \tag{22}$$

Denoting by $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ the algebras spanned by the X_k, the Y_k and the Z_k, we have that $\mathcal{G} = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{Z}$, and that $\mathcal{Y} \oplus \mathcal{Z}$ is an abelian ideal in $\mathcal{G}$. We can thus apply the LRF procedure discussed above.

We first operate on $\mathcal{X}$, with generators also in $\mathcal{X}$ (thus $H_k = \alpha_k X_k$); in this way we can eliminate all terms except the X_μ and the $X_{2\mu}$ ones, as implied by (22). In doing this we modify terms in $\mathcal{Y} \oplus \mathcal{Z}$.

Having performed this first step, we pass to consider the $\mathcal{Y}$ and $\mathcal{Z}$ terms, operating with generators $H_k = \beta_k Y_k + \gamma_k Z_k \in \mathcal{Y} \oplus \mathcal{Z}$. It is clear from (22) that we can eliminate all terms with $k > \mu$, but no lowest order ones. Thus we end up with $\widehat{W} = X_A + a_\mu X_\mu + \widehat{a}_{2\mu} X_{2\mu} + \sum_{k=\nu}^{\mu} \widehat{b}_k Y_k + \sum_{k=\sigma}^{\mu} \widehat{c}_k Z_k$; the hat on constants mean that these are not the same as in the initial form (21), and obviously a sum with lower limit greater than the higher limit should just be meant as zero.

As shown by this example, the computations required for the determination of the general LRF are actually very simple.

5 Lie further normalization and symmetries

The Lie renormalization scheme is designed to take full advantage of the Lie algebra structure of Poincaré-Dulac normal forms, and it should be no surprise that taking into account symmetries of the problem results specially simple in this scheme.

Indeed, if the system is symmetric under a Lie symmetry group S with Lie algebra $\mathcal{S}$ generated by vector fields Y_i $(i = 1, ..., r)$, this means that $[X, Y_i] = 0$ for all $i = 1, ..., r$; hence that X is in the centralizer of S. We are specially interested in the case where the Y_i are linear vector fields and S is a group of matrices acting in $\mathbb{R}^n$.

In this case, the full LRF construction is immediately generalized, with now $G = C(A) \cap C(S)$. That is, we just intersect the centralizer of A with the centralizer of the symmetry group S. Correspondingly, we have to consider $\mathcal{I}^*(A, S)$ and $\mathcal{I}(A, S) \subset \mathcal{I}^*(A, S)$; that is, common invariants for A and S.

If G has generators $X^{(\alpha)}$, then the LRF will be

$$W = \sum_{\alpha=1}^{d} \mu_\alpha(x)\ X^{(\alpha)} \tag{23}$$

where now $\mu_\alpha(x) \in \mathcal{I}^*(A, S)$; thus $\mathcal{G}$ is still contained in a finitely generated module, now over $\mathcal{I}^*(A, S)$.

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