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Selection principles and continuous images

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ABSTRACT

We characterize the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in terms of continuous images in $\mathbb{R}^\omega$.

RESUMEN

Caracterizamos las propiedades de recubrimiento clásicas de Menger, Rothberger, Hurewicz y Gerlits-Nagy en términos de imágenes continuas en $\mathbb{R}^\omega$.

Key words and phrases: *Selection principles, Menger, Hurewicz, Rothberger, property (*), function spaces, ω-cover, γ-cover, γ-set.*
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1 Introduction

It is often the case that the presence of some topological property in a topological space can be detected by the properties of the images of the space under certain maps into "nice" spaces. We are interested here in this schema in connection with covering properties described by classical selection principles. This idea (for selection principles) was initiated by Hurewicz in [10], and then used by many authors (for example, Sierpiński [23], [24], Rothberger [20]). Typical results of that sort assert

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that a set of real numbers has some property if and only if each its continuous image, usually into some special spaces with nice combinatorial properties, has the same or another property.

To describe some of those results we need some definitions.

Endow the set ω of nonnegative integers with the discrete topology. Let ${}^{\omega}\omega$ be the Tychonoff product of countably many copies of this space. There is also a natural pre-order $\leq^*$ defined on ${}^{\omega}\omega$: $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n. A subset D of ${}^{\omega}\omega$ is said to be *dominating* if for each $g \in {}^{\omega}\omega$ there is a function $f \in D$ such that $g \leq^* f$. A subset B of ${}^{\omega}\omega$ is called *bounded* if there is an $g \in {}^{\omega}\omega$ such that $f \leq^* g$ for each $f \in B$.

Hurewicz proved [10]:

- A space X has the Menger property if and only if for each continuous function $f : X \to {}^{\omega}\omega$ the set $f(X)$ is not a dominating family in ${}^{\omega}\omega$.

- A space X has the Hurewicz property if and only if each continuous image of X in ${}^{\omega}\omega$ is a bounded family in ${}^{\omega}\omega$.

A characterization of the Hurewicz property in all finite powers has been found in [26] (in terms of continuous images into ${}^{\omega}\omega$), and γ-sets have been characterized in a similar spirit in [18] (in terms of continuous images into another interesting space – the space $[\mathbb{N}]^\infty$ of all infinite subsets of $\mathbb{N}$). Some classes of spaces related to selection principles have been described by properties of Borel images of spaces into ${}^{\omega}\omega$ or $[\mathbb{N}]^\infty$ (see, for example, [4], [5], [18], [17], [22], [26]).

In this article we give pure topological characterizations of the classical covering properties of Menger, Hurewicz, Rothberger and Gerlits-Nagy (see [12] for more information about selection principles) in terms of continuous images into the space $\mathbb{R}^\omega$.

Our notation and terminology are standard as in [7]. All spaces are assumed to be Tychonoff. In particular, for a space X, $\mathsf{C}_p(X)$ denotes the space of all continuous real-valued functions on X with the topology of pointwise convergence. $\underline{0}$ denotes the constantly zero function from $\mathsf{C}_p(X)$. Recall that a space is said to be an *ϵ-space* [8] if all its finite powers are Lindelöf.

2 The properties of Menger and Rothberger

A space X is said to have the *Menger property* if [15], [9] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and the set $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an open cover of X. Recall also that a space X has *countable fan tightness* [1], [2] if for each $x \in X$ and each countable collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \overline{A_n}$ for each n, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n and x belongs to the closure of the set $\bigcup_{n\in\mathbb{N}} B_n$.

In [3] it was shown the following:

Theorem 2.1 *Let X be a space such that each its continuous separable metrizable image is σ-compact. Then:*

(a) *If X is Lindelöf, then it has the Menger property;*

(b) *If X is Lindelöf in all finite powers, then it has the Menger property in all finite powers.*

Let us mention first that by an appropriate modification in the proof of Theorem 2.1 and using the Arhangel'skiĭ theorem which states that all finite powers of a space X have the Menger property if and only if the space $C_p(X)$ has countable fan tightness [1], [2], one can prove the following two theorems (see the similar proofs of theorems below concerning the Rothberger and Hurewicz properties).

Theorem 2.2 *For a Lindelöf space X the following are equivalent:*

(1) *X has the Menger property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, the space $f(X)$ has the Menger property.*

Theorem 2.3 *For an ϵ-space X the following are equivalent:*

(1) *Each finite power of X has the Menger property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, all finite powers of the space $f(X)$ have the Menger property.*

Arhangel'skiĭ [3] observed that from the well known Katetov's characterization of paracompactness (a T_1 space X is paracompact if and only if for each open cover $\mathcal{U}$ of X there are a metric space M of weight $|\mathcal{U}|$, a continuous mapping f from X onto M and an open cover $\mathcal{V}$ of M such that $f^{\leftarrow}(\mathcal{V})$ refines $\mathcal{U}$, see [7]), it can be obtained the following proposition, which we shall use in what follows.

Proposition 2.4 *If $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of a Lindelöf space X, then there exist a continuous mapping $f : X \to \mathbb{R}^\omega$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of open covers of $f(X)$ such that for each $n \in \mathbb{N}$, $f^{\leftarrow}(\mathcal{V}_n)$ refines $\mathcal{U}_n$.*

Recall that a space X has the *Rothberger property* [19] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(U_n : n \in \mathbb{N})$ such that the family $\{U_n : n \in \mathbb{N}\}$ covers X and for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$. X has *countable strong fan tightness* if for each $x \in X$ and each countable collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \overline{A_n}$ for each n, there are $a_n \in A_n$, $n \in \mathbb{N}$, such that x belongs to the closure of the set $\{a_n : n \in \mathbb{N}\}$ [21]. Sakai [21] proved the following theorem.

Theorem 2.5 *All finite powers of a Tychonoff space X have the Rothberger property if and only if the space $C_p(X)$ has countable strong fan tightness.*

The next two theorems characterize the Rothberger property and the Rothberger property in all finite powers.

Theorem 2.6 *For a Lindelöf space X the following are equivalent:*

(1) *X has the Rothberger property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, the space $f(X)$ has the Rothberger property.*

Proof. Let us prove that (2) implies (1) because the opposite implication follows from the fact that the Rothberger property is preserved by continuous mappings. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By Proposition 2.4 there are a continuous mapping f from X into $\mathbb{R}^\omega$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of open covers of $Y = f(X)$ such that for each $n \in \mathbb{N}$, $f^{\leftarrow}(\mathcal{V}_n)$ refines $\mathcal{U}_n$. Since Y has the Rothberger property there are sets $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, such that the set $\{V_n : n \in \mathbb{N}\}$ is an open cover of Y. For each n, pick an element U_n in $\mathcal{U}_n$ such that $f^{\leftarrow}(V_n) \subset U_n$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of X witnessing that X has the Rothberger property. ∎

Theorem 2.7 *For an ϵ-space X the following are equivalent:*

(1) *Each finite power of X has the Rothberger property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, all finite powers of the space $f(X)$ have the Rothberger property.*

Proof. We prove only non-trivial part (2) $\Rightarrow$ (1). To prove that (2) implies that for each $n \in \mathbb{N}$, X^n has the Rothberger property it is enough, according to Theorem 2.5, to prove that the function space $C_p(X)$ has countable strong fan tightness.

Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_p(X)$ such that $\underline{0} \in \overline{A_n}$ for each $n \in \mathbb{N}$. As X is an ϵ-space, by the well-known Arhangel'skiĭ-Pytkeev theorem [2], the tightness of $C_p(X)$ is countable, so that one can suppose that all A_n's are countable. Let $B = \cup\{A_n : n \in \mathbb{N}\} \cup \{\underline{0}\}$ and let g be the diagonal product of mappings from B. Then g is a continuous mapping from X onto the set $Y = g(X) \subset \mathbb{R}^\omega$. Since by (2) all finite powers of Y have the Rothberger property, the space $C_p(Y)$ has countable strong fan tightness. On the other hand, the set $Z = \{f \circ g : f \in C_p(Y)\}$ is a subset of $C_p(X)$ which is homeomorphic to $C_p(Y)$ [2] and, as it is easily seen, contains B. So, Z has countable strong fan tightness. Thus there exists a sequence $(f_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $f_n \in A_n$ and $\underline{0}$ belongs to the Z-closure, and thus to the $C_p(X)$-closure, of the set $\{f_n : n \in \mathbb{N}\}$. This means that $C_p(X)$ has countable strong fan tightness and completes the proof of the theorem. ∎

3 The Hurewicz and Gerlits-Nagy properties

In [9] (see also [10]), W. Hurewicz introduced the following covering property of a space X, nowadays known as the *Hurewicz property*: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$

of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each element of X belongs to all but finitely many of the sets $\bigcup\mathcal{V}_n$. In [14] the Hurewicz property was characterized by a S_{fin}-type selection principle and shown (see also [13]):

Theorem 3.1 *For an ϵ-space X the following are equivalent:*

(1) *Each finite power of X has the Hurewicz property;*

(2) *The space $\mathsf{C}_p(X)$ has countable fan tightness as well as the Reznichenko property.*

Recall that a space X has the *Reznichenko property* if for each $A \subset X$ and each $x \in \overline{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite, pairwise disjoint subsets of A such that each neighborhood of x meets B_n for all but finitely many n.

We give now characterizations of the Hurewicz property and the Hurewicz property in all finite powers in terms of images into $\mathbb{R}^\omega$.

The first of these theorems is proved in a similar way as Theorem 2.6 was proved.

Theorem 3.2 *For a Lindelöf space X the following are equivalent:*

(1) *X has the Hurewicz property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, the space $f(X)$ has the Hurewicz property.*

Theorem 3.3 *For an ϵ-space X the following are equivalent:*

(1) *All finite powers of X have the Hurewicz property;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, the space $f(X)$ has the Hurewicz property in all finite powers.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that the Hurewicz property in all finite powers is preserved by continuous mappings.

(2) $\Rightarrow$ (1): According to Theorem 3.1 we have to prove that $\mathsf{C}_p(X)$ has (a) the Reznichenko property and (b) countable fan tightness.

(a) Let A be a subset of $\mathsf{C}_p(X)$ and $\underline{0} \in \overline{A}$. Because X is an ϵ-space, again by the Arhangel'skiĭ-Pytkeev theorem, the tightness of $\mathsf{C}_p(X)$ is countable, so that one can suppose that A is countable. Put $B = A \cup \{\underline{0}\}$ and let $g : X \to \mathbb{R}^\omega$ be the diagonal product of mappings from B. By the assumption all finite powers of the set $Y = g(X)$ have the Hurewicz property, so that the function space $\mathsf{C}_p(Y)$ has the Reznichenko property. The set $Z = \{f \circ g : f \in \mathsf{C}_p(Y)\} \subset \mathsf{C}_p(X)$ is homeomorphic to $\mathsf{C}_p(Y)$ and therefore has the Reznichenko property. Since $B \subset Z$ there is a family $(A_n : n \in \mathbb{N})$ of finite, pairwise disjoint subsets of A such that each Z-neighborhood of $\underline{0}$ meets all but finitely many sets A_n. Then each $\mathsf{C}_p(X)$-neighborhood of $\underline{0}$ intersects all but finitely many A_n, i.e. $\mathsf{C}_p(X)$ has the Reznichenko property.

(b) Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_p(X)$ such that $\underline{0} \in \overline{A_n}$ for each $n \in \mathbb{N}$. As in the first part of the proof we may assume that all A_n's are countable. Letting $B = \cup\{A_n : n \in \mathbb{N}\} \cup \{\underline{0}\}$ and $g = \Delta\{f : f \in B\}$ we get again a continuous mapping from X onto the set $Y = g(X) \subset \mathbb{R}^\omega$. Since Y^n has the Hurewicz property for each $n \in \mathbb{N}$, the space $C_p(Y)$ has countable fan tightness. Let Z be as in the first part of the proof. Then Z has countable fan tightness and contains B, so that there exists a sequence $(F_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, F_n is a finite subset of A_n and $\underline{0}$ belongs to the Z-closure, hence to the $C_p(X)$-closure, of the set $\cup\{F_n : n \in \mathbb{N}\}$. This means that $C_p(X)$ has countable fan tightness and the theorem is shown. ∎

Theorem 3.2 allows us to give one more ZFC counterexample to the famous Hurewicz conjecture [10] that a space has the Hurewicz property if and only if it is σ-compact. (Let us say that the first ZFC counterexamples to this conjecture were found only recently; see [11], [6]. See also [16] for the Menger conjecture.) In [25] it was constructed a Lindelöf Σ-space which is not σ-compact but all its continuous images into $\mathbb{R}^\omega$ are Hurewicz. So, by Theorem 3.2, that space is Hurewicz, too.

In 1982, Gerlits and Nagy [8] introduced a covering property denoted $(*)$: a space X has that property if and only if it has the Hurewicz property as well as the Rothberger property. In [14] this property has been characterized by an S_1-type selection principle.

Combining the proofs of Theorems 2.6 and 3.2 it is easy to prove:

Theorem 3.4 *For a Lindelöf space X the following are equivalent:*

(1) *X has the Gerlits-Nagy property $(*)$;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, the space $f(X)$ has the property $(*)$.*

In [14] (see and [13]), it was proved that all finite powers of an ϵ-space X have the property $(*)$ if and only if $C_p(X)$ has countable strong fan tightness as well as the Reznichenko property.

Using this result and combining the proofs of Theorems 2.7 and 3.3, one can prove the following theorem.

Theorem 3.5 *For an ϵ-space X the following are equivalent:*

(1) *For each $n \in \mathbb{N}$, the space X^n has the Gerlits-Nagy property $(*)$;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$, all finite powers of the space $f(X)$ have the property $(*)$.*

Recall that a space X is said to be *ω-simple* [2] if each continuous separable metrizable image Y of X is countable. (For example, all Lindelöf P-spaces and all Lindelöf scattered spaces are ω-simple.) By the previous theorem each ω-simple ϵ-space has the Gerlits-Nagy property $(*)$ in all finite powers. (For each $n \in \mathbb{N}$, Y^n is both Rothberger and Hurewicz being countable.) But, we have something more: each such space is a γ-set, which is stronger than the property $(*)$. In [8], Gerlits and Nagy introduced the notion of *γ-set*: a space X is a γ-set if each ω-cover of X (an open cover which does not contain X, but each finite subset of X is contained in a member of the cover) contains a countable subset $\mathcal{V}$ which is a γ-cover (i.e. $\mathcal{V}$ is infinite and each point of X belongs to all but finitely many elements of $\mathcal{V}$). To conclude that ω-simple ϵ-spaces are γ-sets we should combine the result II.7.10 in [2] (which states that for an ω-simple ϵ-space X the space $C_p(X)$ is Fréchet-Urysohn) and the Gerlits-Nagy theorem stating that $C_p(X)$ is Fréchet-Urysohn if and only if X is a γ-set.

Similarly to the proof of Theorem 2.6 (or to the proof of Theorem 2.7, having in mind the fact that each finite power of a γ-set is also a γ-set), we prove also the following result.

Theorem 3.6 *For an ϵ-space X the following are equivalent:*

(1) *X is a γ-set;*

(2) *For each continuous function $f : X \to \mathbb{R}^\omega$ the space $f(X)$ is a γ-set.*

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