CUBO A Mathematical Journal
Vol. 8, Nº 2, (47 - 51). August 2006.

Global Solutions of Yang-Mills Equation

Qikeng Lu [1]
Institute of Mathematics, Academy of Mathematics and System Science
Chinese Academy of Sciences, Beijing 100080, China
luqik@public.bta.net.cn

ABSTRACT

Global and explicit solutions of Yang-Mills equations are given in the Minkowski space, conformal space and the de-Sitter spaces of arbitrary cosmology constants. The method used is concluded into a general theorem.

RESUMEN

Soluciones explícitas y globales de la ecuaciones de Yang-Mills son dadas en el espacio de Minkowski, en espacios conformes y en los espacios de De-Sitter con constantes cosmológicas arbitrarias. El método usado concluye en un teorema general.

Key words and phrases: *YM equation, homogeneous time-space*
Math. Subj. Class.: *70S15*

It is known[1] that one of the Dirac's[2] conformal space $\mathcal{M}$ is equivalent to the unitary group $U(2)$, which is equivalent to the compacted space $\overline{M}$ of all 2×2 Hermitian matrices, and an explicit global solution of the Yang-Mills equation was construct in $\overline{M}$. In this article we at first construct other solutions from different Lorentz metrics defined on $\mathcal{M}$.

[1]Partially supported by National Natural Science Foundation of China, Project 10231050/A010109.

We arrange the coordinates $x = (x^0, x^1, x^2, x^3)$ of a point in the Minkowski space $\mathbf{M}$ into a 2×2 Hermitian matrix

$$H_x = x^j \sigma_j = \sum_{j=0}^{3} x^j \sigma_j,$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are to prove that

$$\mathbf{A}(x)_j = \frac{i}{8} \mathrm{tr}[(I + H_x^2)^{-1}(H_x \sigma_\mu + \sigma_\mu H_x)](\delta_j^\alpha \delta^{\beta\mu} - \delta_j^\beta \delta^{\alpha\mu})\delta_{\alpha\beta\gamma}^{123} \tag{1}$$

is a $su(2)$ gauge potential(connection), where the Greek indices run from 1 to 3, and satisfies the Yang-Mills equation

$$\mathbf{F}_{jk;l} \equiv g^{kl}\Big(\frac{\partial \mathbf{F}_{jk}}{\partial x^l} + [\mathbf{A}_l, \mathbf{F}_{jk}] - \left\{ \begin{matrix} r \\ jl \end{matrix} \right\} \mathbf{F}_{rk} - \left\{ \begin{matrix} r \\ kl \end{matrix} \right\} \mathbf{F}_{jr}\Big) = 0, \tag{2}$$

where

$$\mathbf{F}_{jk} = \frac{\partial}{\partial x^j}\mathbf{A}_k - \frac{\partial}{\partial x^k}\mathbf{A}_j + [\mathbf{A}_j, \mathbf{A}_k] \tag{3}$$

and $\left\{ \begin{matrix} r \\ jl \end{matrix} \right\}$ is the Christoffel symbol of a metric $ds^2 = g_{jk} dx^j dx^k$ which in the present case we choose[1]

$$g_{jk} = \det(I + H_x^2)^{-1} \eta_{jk}. \tag{4}$$

We should prove that $\mathbf{A}_j$ is actually a $su(2)$-connection. As the first step we construct a $sl(2, \mathbf{C})$-connection(a 2-component spinor connection) from the tensor (4) and reduce it to a $su(2)$-connection. In fact, $\overline{\mathbf{M}}$ and ds^2 are invariant under the transformation

$$T: \quad H_y = (A + H_x B)^{-1}(-B + H_x A), \tag{5}$$

where A, B are 2×2 complex matrices and satisfy the condition

$$A^\dagger A + B^\dagger B = I, \quad A^\dagger B - B^\dagger A = 0 \tag{6}$$

with $A^\dagger, B^\dagger$ denoted the complex conjugate and transpose matrices of A, B respectively.

Associated to the transformation (5) there is a $SL(2, \mathbf{C})$ matrix

$$\mathfrak{A}_T(x) = \det(A + H_x B)^{\frac{1}{2}} (A + H_x B)^{-1}. \tag{7}$$

Since $\overline{\mathbf{M}}$ is transitive under the group $\mathcal{G}$ formed from the transformations (5), the corresponding $\{\mathfrak{A}_T(x)\}_{T \in \mathcal{G}}$ are the transition functions of the natural principal bundle $P(\overline{\mathbf{M}}, SL(2, \mathbf{C}))$. We apply the following theorem(c.f. [3] Theorem 2.4.2)

Theorem A. *If $\mathfrak{M}$ is a 4-dimensional Lorentz spin manifold, then*

$$\Gamma_j = \frac{1}{4}\Gamma^a_{bj}\eta^{bc}\sigma_a\sigma^*_c$$

($\sigma^*_c = \sigma_2\overline{\sigma}_c\sigma_2^\dagger$) is a $\mathfrak{sl}(2,\mathrm{C})$-connection on the principal bundle $P(\mathfrak{M}, SL(2,\mathrm{C}))$, where

$$\Gamma^a_{bj} = e^{(a)}_k\frac{\partial}{\partial x^j}e^k_{(b)} + \left\{ \begin{matrix} k \\ lj \end{matrix} \right\} e^{(a)}_k e^l_{(b)}$$

and

$$ds^2 = g_{jk}dx^j dx^k = \eta_{ab}\omega^a\omega^b$$

is the Lorentz metric with $\omega^a = e^{(a)}_j dx^j$ and $e^j_{(a)}$ satisfying $e^j_{(a)}e^{(b)}_j = \delta^a_b$.

Since $\sigma^*_0 = \sigma_0$ and $\sigma^*_\alpha = -\sigma_\alpha (\alpha = 1,2,3)$, The $sl(2,\mathrm{C})$-connection in Theorem A can be written into

$$\Gamma_j = \frac{1}{2}\Gamma^\alpha_{0j}\sigma_\alpha + \frac{i}{4}\Gamma^\alpha_{\beta j}\delta^{123}_{\alpha\beta\gamma}\sigma_\gamma, \tag{8}$$

where the Greek indices run from 1 to 3 and $\{\sigma_\alpha, i\sigma_\alpha\}_{\alpha=1,2,3}$ is a basis of the Lie algebra $\mathfrak{sl}(2,\mathrm{C})$ of $SL(2,\mathrm{C})$ and $\{i\sigma_\alpha\}_{\alpha=1,2,3}$ is that of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$. Since $\{U\sigma_\alpha U^{-1}\}_{\alpha=1,2,3}$ for any $U \in SU(2)$ is still a basis of the vector space generated by $\{\sigma_\alpha\}_{\alpha=1,2,3}$, according to the reduction theorem of connections,

$$\mathbf{A}_j = \frac{i}{4}\Gamma^\alpha_{\beta j}\delta^{123}_{\alpha\beta\gamma}\sigma_\gamma \tag{9}$$

is a $\mathfrak{su}(2)$-connection on the reduced principal bundle $P_1(\mathfrak{M}, SU(2))$ of $P(\mathfrak{M}, SL(2,\mathrm{C}))$.

In case that $\mathfrak{M} = \overline{M}$ and ds^2 is defined by (4), $\mathbf{A}_j$ is exactly expressed by (1). It remains to prove that such $\mathbf{A}_j$ satisfies the Yang-Mills equation (2). In fact, according to (1) the elements of the matrices $\mathbf{A}_j (j = 0,1,2,3)$ are all odd functions of x^j. Obviously $[\mathbf{A}_j(x)]_{x=0} = 0$. Hence all elements of $\mathbf{F}_{jk}$ are even functions of x^j. Therefore all elements of $\mathbf{F}_{jk;l}$ are odd functions of x^j and consequently $[\mathbf{F}(x)_{jk;l}]_{x=0} = 0$. Since $\overline{\mathrm{M}}$ is transitive[1] under the group $\mathcal{G}$, for any point x_0 of $\overline{\mathrm{M}}$ there is at least a transformation (5) which carries the point $x = x_0$ to the point $y = 0$. Since both g_{jk} and $\mathbf{F}_{jk;l}$ are covariant under the the transformation (5),

$$0 = [\mathbf{F}(y)_{jk;l}]_{y=0} = \left[\mathfrak{A}_T(x)\mathbf{F}(x)_{pq;r}\mathfrak{A}_T(x)^{-1}\frac{\partial x^p}{\partial y^q}\frac{\partial x^q}{\partial y^k}\frac{\partial x^r}{\partial y^l}\right]_{x=x_0},$$

which implies that $[\mathbf{F}(x)_{pq;r}]_{x=x_0} = 0$. Since x_0 can be an arbitrary point of $\overline{\mathrm{M}}$, we have $\mathbf{F}(x)_{jk;l} = 0$ and obviously it satisfies the Yang-Mills equation $g^{kl}\mathbf{F}_{jk;l} = 0$.

Since $\overline{\mathrm{M}}$ is the compacted Minkowski space M and the Yang-Mills equation is conformal invariant, the $\mathfrak{su}(2)$-connection $\mathbf{A}_j$ defined by (1) also satisfies the Yang-Mills equation

$$\eta^{kl}\left(\frac{\partial}{\partial x^l}\mathbf{F}_{jk} + \mathbf{A}_l\mathbf{F}_{jk} - \mathbf{F}_{jk}\mathbf{A}_l\right) = 0$$

in the Minkowski space M.

Another solution of the Yang-Mills equation on $\overline{\mathbf{M}}$ can be deduced from another Lorentz metric. $\overline{\mathbf{M}}$ is diffeomorphic[4] to $S^1 \times S^3$ and on the later space there is naturally a Lorentz metric

$$ds^2 = (dx^0)^2 - (1 + x^\alpha x^\alpha)^{-2}\delta_{\mu\nu}dx^\mu dx^\nu, \tag{11}$$

which can be regarded as the metric of $\overline{\mathbf{M}}$. By Theorem A and formula (8),

$$\mathbf{A}_0 = \Gamma_0 = 0, \quad \mathbf{A}_\mu = \Gamma_\mu = -\frac{i}{2}(1 + x^\nu x^\nu)^{-1}(\delta_\mu^\alpha x^\beta - \delta_\mu^\beta x^\alpha)\delta_{\alpha\beta\gamma}^{123}\sigma_\gamma \tag{12}$$

is a $su(2)$-connection. Since $S^1 \times S^3$ and ds^2 defined by (11) is invariant under $SU(2) \times S(4)$ which acts on $S^1 \times S^3$ as the transformation group of $S^1 \times S^3$ and the elements of the matrices $\mathbf{A}_j$ are odd functions of x^j, it can be proved that $\mathbf{A}_j$ satisfies the Yang-Mills equation by the same argument as in case that ds^2 is defined by (11). This is a static solution because $\mathbf{A}_j$ are not depend on x^0.

Our method can also be applied to construct solutions of the Yang-Mills equation on the de-Sitter spaces defined by Dirac[5]. The de-Sitter space of cosmology constant Λ is denoted by $dS(\Lambda)$. Some authors call it anti-de-Sitter space when $\Lambda < 0$ and denote it by AdS. In fact, $dS(\Lambda)$ is a domain(connected open set) of the real projective space $\mathbf{RP}^4$. In the local coordinate (non-homogeneous coordinate) $x = (x^0, x^1, x^2, x^3)$ of $\mathbf{RP}^4$ the domain $dS(\Lambda)$ is defined by the inequality[6]

$$1 - \Lambda\eta_{jk}x^j x^k > 0 \tag{13}$$

and there is a Lorentz metric

$$ds_\Lambda^2 = \left[\frac{\eta_{jk}}{1 - \Lambda\eta_{pq}x^p x^q} + \Lambda\frac{\eta_{jr}x^r\eta_{ks}x^s}{(1 - \Lambda\eta_{pq}x^p x^q)^2}\right]dx^j dx^k. \tag{14}$$

The space $dS(\Lambda)$ is invariant under the transformation

$$y^j = \sigma(a)^{\frac{1}{2}}\frac{x^k - a^k}{1 - \Lambda\eta_{pq}a^p a^q}D_k^j, \tag{15}$$

where $a = (a^0, a^1, a^2, a^3)$ and D_k^j satisfy

$$\sigma(a) = 1 - \Lambda\eta_{pq}a^p a^q > 0, \quad \eta_{pq}D_j^p D_k^q = \eta_{jk} + \Lambda\sigma(a)^{-1}\eta_{jp}a^p a^q.$$

Obviously, when $a = 0$, (15) is a Lorentz transformation. The metric ds_Λ^2 is invariant under the transformation (15) and $dS(\Lambda)$ is transitive under the group of all such transformations. In fact this is the group $SO(2,3)$(in case $\Lambda < 0$) or the group $SO(1,4)$(in case $\Lambda > 0$) that acts on $dS(\Lambda)$. Moreover the metric ds_Λ^2 under the coordinate transformation

$$x^j = (1 + \frac{1}{4}\Lambda\eta_{pq}u^p u^q)^{-1}u^j, \quad (j = 0, 1, 2, 3) \tag{16}$$

is changed to be

$$ds_\Lambda^2 = (1 - \frac{1}{4}\Lambda\eta_{pq}u^p u^q)^{-2}\eta_{jk}du^j du^k \quad (17)$$

which is a conformal flat metric.

That the de-Sitter spaces are spin manifolds is implied in the Dirac's construction of the $Spin\frac{1}{2}$ wave equation[5]. Applying Theorem A and the theorem of reduction of connections, we obtain the $su(2)$-connection

$$\mathbf{A}_j = -\frac{i}{8}\Lambda(1 - \Lambda\frac{1}{4}\eta_{pq}u^p u^q)^{-1}(\delta_j^\alpha u^\beta - \delta_j^\beta u^\alpha)\delta_{\alpha\beta\gamma}^{123}\sigma_\gamma, \quad (18)$$

which satisfies the Yang-Mills equation because the elements of $\mathbf{A}_j$ are odd functions of u^j.

We conclude in general that

Theorem B. *If $\mathfrak{M}$ is a 4-dimensional spin manifold and possesses a Lorentz metric which is invariant under a Lie group $\mathfrak{G}$ that acts on $\mathfrak{M}$ transitively, and there is an admissible local coordinate $x^j (j = 0, 1, 2, 3)$ such that the $su(u)$-connection $\mathbf{A}_j$ deduced from Theorem A are odd functions of x^j, then $\mathbf{A}_j$ satisfies the Yang-Mills equation.*

Received: March 2005. Revised: April 2005.

References

[1] Qikeng Lu, *A global solution of the Einstein-Yang-Mills equation on the conformal space*, Science in China(Series A), **45**(2002), 342-355.

[2] P. A. M. Dirac, *Wave equation in the conformal space*, Ann. Math., **37**(1936), 171-201.

[3] Qikeng Lu, *Differential Geometry and its Application to Physics(in Chinese)*, Science Press, Beijing, 1983.

[4] Qikeng Lu, Shikun Wang, Ke Wu, *Global solutions of Einstein-Dirac equation*, Asian J. Math., **8** (2004), 1-26.

[5] P. A. M. Dirac, *The electron wave equation in the de-Sitter space*, Ann. Math., **36** (1935), 657-609.

[6] H. K. Look(=Qikeng Lu), G. L. Tsou(=Zhenlong Zhou), H. Y. Kao(=Hanying Gao), *The kinematic effect in the classical domains and the red-shift phenomena of extra-galatic objects(in Chinese)*, Acta Physica Sinicas, **23** (1974), 225-238.