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A topological definition of the Maslov bundle

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ABSTRACT

We give a definition of the Maslov fibre bundle for a lagrangian submanifold of the cotangent bundle of a smooth manifold. This definition generalizes the definition given, in homotopic terms, by Arnol'd for lagrangian submanifolds of $T^*\mathbb{R}^n$. We show that our definition coincides with the one of Hörmander in his works about Fourier Integral Operators.

RESUMEN

Definimos el ramo de fibras de Maslow para una subvariedad lagrangiana del ramo cotangente de una variedad suave. Esta definición generaliza la definición dada por Arnold, en términos homotópicos, para subvariedades lagrangianas de $T^*\mathbb{R}^n$. Probaremos que nuestra definición coincide con la dada por Hörmander en sus trabajos sobre Operadores Integrales de Fourier.

Key words and phrases: *fourier integral operators, Maslov bundle, Hörmander's index*
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1 Introduction

The Maslov index appears as the phase term when one tries to define the symbol of a Fourier Integral Operator (FIO). This symbol is then defined as a section of the Maslov bundle contructed on a lagrangian submanifold of T^*X. In his historical paper [7], Hörmander proposes a construction of this bundle in terms of cocycles and tries to make the links with the strictly topological presentation (representation of the fundamental group) proposed by Arnol'd [3], originally in an appendix of the book of Maslov [12]. This link is established only for the lagrangian submanifolds of $T^*\mathbb{R}^n$. I propose in this work a new construction (1.2) for the lagrangian submanifolds of T^*X, X a smooth manifold, based on a definition of the Maslov index (1.1) which generalize the one of Arnol'd, and satisfies the cocycles conditions of Hörmander. These correspondances are established in the sections 2 and 3.

1.1 Arnol'd's definition of the Maslov index

Recall first the construction of Arnol'd [3]. The space $T^*\mathbb{R}^n$ has a symplectic structure by the standard symplectic form

$$\omega = \sum_{j=1}^{j=n} d\xi_j \wedge dx_j.$$

Let $\mathbb{L}(n)$ be the Grassmannian manifold of the Lagrangian subspaces of $T^*\mathbb{R}^n$; we identify $\mathbb{L}(n) = U(n)/O(n)$. The map Det^2 is well defined on $\mathbb{L}(n)$. it is showed in [3] that every path $\gamma : \mathbb{S}^1 \to \mathbb{L}(n)$ such that $Det^2 \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1$ is a generator of $\Pi_1(\mathbb{S}^1)$, gives a generator of $\Pi_1(\mathbb{L}(n))$. It follows that $\Pi_1(\mathbb{L}(n)) \simeq \mathbb{Z}$ and that the cocycle μ_0 defined by

$$\forall \gamma \in \Pi_1(\mathbb{L}(n)) \quad \mu_0(\gamma) = \text{Degree } (Det^2 \circ \gamma)$$

is a generator of the group $H^1(\mathbb{L}(n)) \simeq \mathbb{Z}$. It is then possible to define a *Maslov bundle* $\mathbb{M}(n)$ on $\mathbb{L}(n)$ by the representation $\exp(i\frac{\pi}{2}\mu_0) = i^{\mu_0}$ of $\Pi_1(\mathbb{L}(n))$. It is a flat bundle with torsion because $\mathbb{M}(n)^{\otimes 4}$ is trivial.

Now the Maslov bundle of a submanifold $\mathcal{L}$ of $T^*\mathbb{R}^n$ is the pullback of $\mathbb{M}(n)$ by the natural map

$$\begin{aligned} \varphi_n : \mathcal{L} &\to \mathbb{L}(n) \\ \nu &\mapsto T_\nu\mathcal{L}. \end{aligned}$$

Arnol'd precisely shows that $\mu = \varphi_n{}^*\mu_0$ is the Maslov index of $\mathcal{L}$. One can write

$$\begin{aligned} \mu : \Pi_1(\mathcal{L}) &\to \mathbb{Z} \\ [\gamma] &\mapsto < \mu_0, \varphi_n \circ \gamma >= \text{Degree } (Det^2 \circ \varphi_n \circ \gamma). \end{aligned} \qquad (1.1.1)$$

We have to take care of the structural group of this bundle. As a $U(1)-$bundle it is always trivial. But it is concidered as a $\mathbb{Z}_4 = \{1, i, -1, -i\}$-bundle. In fact one can see, using the expression of the Maslov cocycle σ_{jk} given by [7] (3.2.15) that the Chern classes of this bundle are null but σ_{jk} can not be writen in general as the coboundary of a *constant* cochain.

We recall now the theorem of symplectic reduction as it is presented in [6] Proposition 3.2. p.132 .

Proposition 1.1 (Guillemin, Sternberg) *Let Δ be an isotropic subspace of dimension m in $T^*\mathbb{R}^{(n+m)}$. Define $S_\Delta = \{\lambda \in \mathbb{L}(n+m)/\ \lambda \supset \Delta\}$. Then S_Δ is a submanifold of $\mathbb{L}(n+m)$ of codimension $(n+m)$, if we define ρ to be the map*

$$\begin{array}{ccc} \mathbb{L}(n+m) & \xrightarrow{\rho} & \mathbb{L}(n) \\ \lambda & \mapsto & \lambda \cap \Delta^\omega / \lambda \cap \Delta \end{array}$$

(Δ^ω is the orthogonal of Δ for the canonical symplectic form ω), then the map ρ, which is continue on the all $\mathbb{L}(n+m)$, is smooth in restriction to $\mathbb{L}(n+m) - S_\Delta$ and defines on this space a fibre structure with base $\mathbb{L}(n)$ and fibre $\mathbb{R}^{(n+m)}$.

Moreover the image by ρ of the generator of $\Pi_1(\mathbb{L}(n+m))$ is a generator of $\Pi_1(\mathbb{L}(n))$.

1.2 Hörmander's definition of the Maslov bundle

Let X be a smooth manifold, then $T^*X \xrightarrow{\pi_0} X$ is endowed with a canonical symplectic structure by $\omega = d\xi \wedge dx$. Let $\mathcal{L}$ be a lagrangian (homogeneous) submanifold of T^*X. Hörmander, in [7] p.155, defines the Maslov bundle of $\mathcal{L}$ by its sections.

A Lagrangian manifold owns an atlas such that the cards (C_ϕ, D_ϕ) are defined by non degenerated phase functions ϕ defined on $U \times \mathbb{R}^N$ U open in a domain diffeomorphic to a ball of a card of X and

$$\begin{array}{ccc} C_\phi = \left\{(x,\theta);\ \phi'_\theta(x,\theta) = 0\right\} & \xrightarrow{D_\phi} & \mathcal{L}_\phi \subset \mathcal{L} \\ (x,\theta) & \longmapsto & (x, \phi'_x(x,\theta)). \end{array}$$

For the function ϕ, to be non degenerate means that ϕ'_θ is a submersion and thus C_ϕ is a submanifold and D_ϕ an immersion.
A section is then given by a family of functions

$$z_\phi : C_\phi \to \mathbb{C}$$

satisfying the change of cards formulae :

$$z_{\tilde\phi} = \exp i\frac{\pi}{4}\left(\text{sgn}\phi''_{\theta\theta} - \text{sgn}\tilde\phi''_{\tilde\theta\tilde\theta}\right) z_\phi. \tag{1.2.2}$$

In fact $(\text{sgn}\phi''_{\theta\theta} - \text{sgn}\tilde\phi''_{\tilde\theta\tilde\theta})$ is even (see below, proposition 3.4) and we have indeed constructed by this way a $\mathbb{Z}_4$-bundle.

1.3 Definition of the Maslov index and results

In the same situation as before, we can construct on any lagrangian submanifold $\mathcal{L}$ of T^*X (and in fact on all T^*X) the following fibre bundle

$$\begin{array}{ccc} \mathbb{L}(n) & \xrightarrow{i} & \mathbb{L}(\mathcal{L}) \\ & & \downarrow \pi \\ & & \mathcal{L} \end{array}$$

of the lagrangian subspaces of $T_\nu(T^*X), \nu \in \mathcal{L}$.

This bundle has two natural sections :

$$\lambda(\nu) = T_\nu(\mathcal{L}), \text{ and } \lambda_0(\nu) = \text{vert}(T_\nu(T^*X))$$

defined by the tangent to $\mathcal{L}$ and the tangent to the vertical $T^*_{\pi_0(\nu)}X$.

To a fibre bundle is associated a long exact sequence of homotopy groups, here :

$$...\Pi_2(\mathcal{L}) \to \Pi_1(\mathbb{L}(n)) \xrightarrow{i_*} \Pi_1(\mathbb{L}(\mathcal{L})) \xrightarrow{\pi_*} \Pi_1(\mathcal{L}) \to \Pi_0(\mathbb{L}(n)) = 0.$$

But our fibre bundle possesses a section (two in fact), as a consequence the maps $\Pi_k(\mathbb{L}(\mathcal{L})) \xrightarrow{\pi_*} \Pi_k(\mathcal{L})$ are onto and the maps $\Pi_{k+1}(\mathcal{L}) \to \Pi_k(\mathbb{L}(n))$ are null ; this gives a split exact sequence

$$0 \to \Pi_1(\mathbb{L}(n)) \xrightarrow{i_*} \Pi_1(\mathbb{L}(\mathcal{L})) \xrightarrow{\pi_*} \Pi_1(\mathcal{L}) \to 0.$$

Take a base point $\nu_0 \in \mathcal{L}$ and fix a path σ from $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$ lying in the fibre $\mathbb{L}(\mathcal{L})_{\nu_0}$. For $\gamma \in \Pi_1(\mathcal{L})$ we denote $\lambda_0{}^\sigma{}_*(\gamma)$ the composition of σ, $\lambda_{0*}\gamma$ and finaly σ^{-1} (we use here the conventions of writing of [11]).

Then $\forall\gamma \in \Pi_1(\mathcal{L})$, $\pi_*\Big(\lambda_*\gamma * (\lambda_0{}^\sigma{}_*(\gamma^{-1}))\Big) = 0$ and $\lambda_*\gamma * (\lambda_0{}^\sigma{}_*(\gamma^{-1}))$ is in $\Pi_1(\mathbb{L}(n))$. Let us take the

Definition 1.1 *The Maslov index of $\mathcal{L}$ is the map μ :*

$$\forall\gamma \in \Pi_1(\mathcal{L}),\ \mu(\gamma) = \mu_0\Big(\lambda_*\gamma * \lambda_0{}^\sigma{}_*(\gamma^{-1})\Big).$$

Proposition 1.2 *This definition does not depend on the path σ that we have chosen to joint $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$; moreover μ is a morphism of group, that is : $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.*

First remark : in the case where $X = \mathbb{R}^n$ the fibre bundle $\mathbb{L}(\mathcal{L})$ can be trivialized in such a way that the section λ_0 is constant. In this case our definition coincide with the one of [3]. A natural consequence of the proposition is the following definition:

Definition 1.2 *The Maslov bundle $\mathbb{M}(\mathcal{L})$ over $\mathcal{L}$ is defined as in section 1.1 by the representation $\exp(i\frac{\pi}{2}\mu) = i^\mu$ of $\Pi_1(\mathcal{L})$ in $\mathbb{C}$.*

This means that the sections of the bundle are identified with functions f on the universal cover of $\mathcal{L}$ with compex values and satisfying the relation :

$$\forall\gamma \in \Pi_1(\mathcal{L}), \quad f(x.\gamma) = i^{-\mu(\gamma)} f(x), \tag{1.3.3}$$

like in [2] formula (2.19).

Theorem 1.1 *The sections of the Maslov bundle of a Lagrangian (homogeneous) submanifold as defined by the definition 1.2 satisfy the gluing conditions of Hörmander, it means that our definition coincides with the one of Hörmander.*

2 Study of the index μ.

2.1 The index μ_0 on $\mathbb{L}(n)$ is also an intersection number.

For $\alpha \in \mathbb{L}(n)$ et $k \in \mathbb{N}$ one defines $\mathbb{L}^k(n)(\alpha) = \{\beta \in \mathbb{L}(n);\ \dim \alpha \cap \beta = k\}$. Since [3] we know that $\mathbb{L}^k(n)(\alpha)$ is an open submanifold of codimension $\frac{k(k+1)}{2}$, in particular $\overline{\mathbb{L}^1(n)(\alpha)}$ is an oriented cycle of codimension 1 and his intersection number coincides with μ_0.

2.2 Proof of the proposition 1.2.

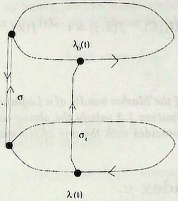
It is a consequence of the two following lemmas. Provide $\mathbb{L}(\mathcal{L})$ with a connection of $U(n)$-bundle. Indeed any symplectic manifold (M,ω), like T^*X, can be provided with an almost complex structure J which is compatible with the symplectic structure(see [1] p.102), it means such that $g(X,Y) = \omega(JX,Y)$ is a riemannian metric. By this way the tangent bundle of M is provided with an hermitian form $g_{\mathbb{C}} = g + i\omega$, and its structural group restricts to $U(n)$ it is also the case for the grassmannian of Lagrangians or its restriction to a submanifold.

We will denote by $\tau(\gamma)_{x\to y}$ the parallel transport for this connection from $\mathbb{L}(\mathcal{L})_x$ to $\mathbb{L}(\mathcal{L})_y$ along the path γ joining x to y in $\mathcal{L}$.

Let's now $\gamma : \mathbb{S}^1 \to \mathcal{L}$ be a closed path such that $\gamma(0) = \nu_0$, we define $\lambda(t) = \lambda_*(\gamma)(t)$ and in the same way ${\lambda_0}^{-1}(t) = \lambda_{0*}(\gamma^{-1})(t)$.

If, as before, σ is a path from $\lambda(0)$ to $\lambda_0(0)$ in the fibre $\mathbb{L}(\mathcal{L})_{\gamma(0)}$; then the path of $\mathbb{L}(\mathcal{L}) : \lambda * \sigma * {\lambda_0}^{-1} * \sigma^{-1}$ is homotopic to a path in the fibre, we have to calculate the Maslov index μ_0 of this last one. For this we use the parallel transport along γ to deform $\lambda * \sigma * {\lambda_0}^{-1}$.

Definition 2.1 *For $t \in [0,1]$ let's σ_t denote the path included in the fibre $\mathbb{L}(\mathcal{L})_{\gamma(t)}$ joining $\lambda(t)$ to $\lambda_0(t)$ and obtained by the parallel transport of $\lambda_{|[t,1]} * \sigma * ({\lambda_0}_{|[t,1]})^{-1}$.*



This path has three distinct parts: first $\tilde{\lambda}(t,s) = \tau(\gamma^{-1})_{\gamma(s)\to\gamma(t)}\lambda(s)$ then $\tilde{\sigma}(t,s) = \tau(\gamma^{-1})_{\gamma(1)\to\gamma(t)}\sigma(s)$ and finally $\tilde{\lambda}_0^{-1}(t,s) = \tau(\gamma^{-1})_{\gamma(s)\to\gamma(t)}(\lambda_0{}^{-1}(t))$.

By the definition (1.2)

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}).$$

Lemma 2.1 *This definition does not depend on the path σ chosen to link $\lambda(0)$ to $\lambda_0(0)$ staying in the fibre above $\gamma(0)$.*

The index μ_0 is defined on the free homotopy group so

$$\mu_0(\sigma_0 * \sigma^{-1}) = \mu_0(\sigma^{-1} * \sigma_0) = \mu_0(\sigma^{-1} * \tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1})$$

if, here, $\tilde{\lambda}(s) = \tilde{\lambda}(0,s)$ and the same notations for λ_0 and σ.

If σ' is an other path from $\lambda(0)$ to $\lambda_0(0)$, then by the preceding remark and the fact that μ_0 is a morphism of group, one has:

$$\mu_0(\sigma_0' * \sigma'^{-1}) - \mu_0(\sigma_0 * \sigma^{-1}) = \mu_0(\sigma'^{-1} * \sigma_0') - \mu_0(\sigma^{-1} * \sigma_0) =$$

$$\mu_0(\sigma'^{-1} * \sigma_0') + \mu_0(\sigma_0{}^{-1} * \sigma) = \mu_0(\sigma'^{-1} * \sigma_0' * \sigma_0{}^{-1} * \sigma) =$$

$$\mu_0(\sigma'^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\lambda}_0^{-1} * (\tilde{\lambda}_0^{-1})^{-1} * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1} * \sigma) = \mu_0(\sigma'^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1} * \sigma) =$$

$$\mu_0(\sigma * \sigma'^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1}) = \mu_0\Big((\sigma * \sigma'^{-1}) * \tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1} * \tilde{\lambda}^{-1}\Big) =$$

$$\mu_0(\sigma * \sigma'^{-1}) + \mu_0(\tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1} * \tilde{\lambda}^{-1}) = \mu_0(\sigma * \sigma'^{-1}) + \mu_0(\tilde{\lambda}^{-1} * \tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1}) =$$

$$\mu_0(\sigma * \sigma'^{-1}) + \mu_0((\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1}) = \mu_0(\sigma * \sigma'^{-1}) - \mu_0(\tilde{\sigma} * \tilde{\sigma}'^{-1}) = 0$$

because $\tilde{\sigma} * \tilde{\sigma}'^{-1}$ is the image of $\sigma * \sigma'^{-1}$ by the parallel transport $\tau(\gamma)$ along γ ; but $\tau(\gamma) \in U(n)$ preserves the Maslov index μ_0. ∎

Lemma 2.2 *μ is a morphism of groups.*

Indeed, if α and β are two elements of $\Pi_1(\mathcal{L})$ it is suffisant to calculate $\mu(\alpha)+\mu(\beta)$ beginning the first circle at $\bar{\sigma}^{-1}(1)\tau(\alpha)\sigma(0)$ and applying $\tau(\alpha)$ to the second circle which was chosen to begin at $\sigma(0)$. ■

3 Links with the definition of Hörmander

To make the link of this definition with signature terms of the formula in [7] we follow the calculation from [4].

3.1 Maslov's index in term of signature.

Let $\gamma\in\mathbb{L}^k(n)(\alpha)$ and $\beta\in\mathbb{L}^0(n)(\alpha)\cap\mathbb{L}^0(n)(\gamma)$. Then α and β are transversal and γ can be presented as a graph: there exists a unique linear map $C:\alpha\to\beta$ such that $\gamma=\{(x,Cx),x\in\alpha\}$. [4] p. 181,defines a quadratic form in α by:

$$Q(\alpha,\beta;\gamma)=\omega(C.,.)\in\mathcal{Q}(\alpha). \qquad (3.1.4)$$

One sees easily that $\ker Q(\alpha,\beta;\gamma)=\ker C=\alpha\cap\gamma$. and if we choose a basis on α such that $Q(\alpha,\beta;\gamma)$ has the form $\begin{vmatrix} B_0 & 0\\ 0 & 0\end{vmatrix}$, the null part corresponds to $\alpha\cap\gamma$.

Let now $\gamma(t)$ be a path in $\mathbb{L}^0(n)(\beta)$ such that $\gamma(0)=\gamma$. The goal of the following calculations is to control the jump of the signature of the quadratic form $Q(\alpha,\beta;\gamma(t))$ in the neighbourhood of $t=0$.

Proposition 3.1 *Let $\gamma(t)$ be a path in $\mathbb{L}^0(n)(\beta)$ such that $\gamma(0)=\gamma$. If*

$$Q(\alpha,\beta;\gamma(t))=\begin{vmatrix} B(t) & C(t)\\ C^t(t) & D(t)\end{vmatrix}$$

with $D(t)$ in $\alpha\cap\gamma$. Then, if $D'(t)$ is invertible in the neighbourhood of 0, there exists $\varepsilon>0$ such that

$$\forall t,\ 0<t<\varepsilon\ \ sgn\ Q(\alpha,\beta;\gamma(t))-sgn\ Q(\alpha,\beta;\gamma(-t))=2sgn\ D'(0).$$

Proof. We know that $B(t)$ is invertible and $C(t)$, $D(t)$ are small. The identity

$$\begin{vmatrix} B & C\\ C^t & D\end{vmatrix}=\begin{vmatrix} 1 & 0\\ C^tB^{-1} & 1\end{vmatrix}\cdot\begin{vmatrix} B & 0\\ 0 & (D-C^tB^{-1}C)\end{vmatrix}\cdot\begin{vmatrix} 1 & B^{-1}C\\ 0 & 1\end{vmatrix} \qquad (3.1.5)$$

gives sgn $Q(\alpha,\beta;\gamma(t))$ =sgn$(B(t))$+sgn$(D(t)-C(t)^tB(t)^{-1}C(t))$. When t is small sgn $B(t)=$ sgn $Q(\alpha,\beta;\gamma)$ and sgn$\Big(D(t)-C(t)^tB(t)^{-1}C(t)\Big)=$ sgn(t) sgn$(D'(0))$ by the mean value theorem. ■

Now if γ is a path which cross *transversally* $\mathbb{L}^1(n)(\alpha)$ at $\gamma(0)$ then the assumption on D' is satisfied.

Theorem 3.1 *Let $\alpha \in \mathbb{L}(n)$ and γ a closed path in $\mathbb{L}(n)$ which cross $\mathbb{L}^1(n)(\alpha)$ transversally, then for all $\beta \in \mathbb{L}(n)$ transversal to α and to $\gamma(t)$ one has*

$$\mu_0(\gamma) = \frac{1}{2} \sum_{t, \gamma(t) \in \mathbb{L}^1(n)(\alpha)} \Big(sgn\ Q(\alpha, \beta; \gamma(t^+)) - sgn\ Q(\alpha, \beta; \gamma(t^-)) \Big).$$

Indeed, in this case $T_\gamma \mathbb{L}(n)/T_\gamma \mathbb{L}^1(n)(\alpha) \sim S^2(\alpha \cap \gamma)$ which is oriented by the positive-definite quadratic forms and sgn $D'(0) = \pm 1$, we use then the previous formula.

Remark 3.1 *This formula allows to define index of path not necessarely closed, see [13].*

3.2 Hörmander's index.

Let α, β, β' be three elements of $\mathbb{L}(n)$ such that $\beta, \beta' \in \mathbb{L}^0(n)(\alpha)$. For any path σ joining β to β' one defines

$$[\sigma, \alpha] = \mu_0(\hat{\sigma})$$

where $\hat{\sigma}$ is the closed path obtained from σ by linking its endpoints staying in $\mathbb{L}^0(n)(\alpha)$:

$$\hat{\sigma} = \sigma * \sigma_\alpha \text{ and } \sigma_\alpha \subset \mathbb{L}^0(n)(\alpha).$$

The theorem (3.1) shows that $[\sigma, \alpha]$ does not depend on the way σ is closed staying in $\mathbb{L}^0(n)(\alpha)$. Let now α' be a point in $\mathbb{L}^0(n)(\beta) \cap \mathbb{L}^0(n)(\beta')$. The *index of Hörmander* is the number

$$s(\alpha, \alpha'; \beta, \beta') = [\sigma, \alpha'] - [\sigma, \alpha] = \mu_0(\sigma * \sigma_{\alpha'} * (\sigma * \sigma_\alpha)^{-1}) = \mu_0(\sigma_{\alpha'} * {\sigma_\alpha}^{-1})$$

because the calculation of μ_0 does not depend on the base point in $\mathbb{S}^1$.

This index depends only on the four points in $\mathbb{L}(n)$ and not on the paths:

Proposition 3.2 *Let β, $\beta' \in \mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\alpha')$ then*

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(sgn\ Q(\alpha, \beta'; \alpha') - sgn\ Q(\alpha, \beta; \alpha') \Big).$$

Indeed, first suppose that α and α' are transversal ; the theorem (3.1) can be applied and also the proposition (3.1) ; this gives

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(\text{sgn}\ Q(\alpha, \alpha'; \beta) - \text{sgn}\ Q(\alpha, \alpha'; \beta') \Big).$$

On the other hand $\beta \in \mathbb{L}^0(n)(\alpha)$ can be writen as the graph of $C \in \text{End}(\alpha, \alpha')$ and so $Q(\alpha, \alpha'; \beta) = \omega(C., .)$. But also α' is the graph of $D \in \text{End}(\alpha, \beta)$ with $\forall x \in \alpha$, $D(x) = -(x + C(x))$, then $Q(\alpha, \beta; \alpha') = \omega(D., .) = -\omega(C., .) = -Q(\alpha, \alpha'; \beta)$. As a consequence

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(\text{sgn}\ Q(\alpha, \beta'; \alpha') - \text{sgn}\ Q(\alpha, \beta; \alpha') \Big).$$

This formula can be generalized by the symplectic reduction (1.1). ■

Let us recall finally the

Proposition 3.3 *Let $\alpha, \alpha', \beta, \beta'$ be four points in $\mathbb{L}(n)$ such that β and β' are in $\mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\alpha')$ then*

$$s(\alpha, \alpha'; \beta, \beta') = -s(\alpha', \alpha; \beta, \beta') = -s(\alpha, \alpha'; \beta', \beta) = -s(\beta, \beta'; \alpha, \alpha').$$

Only the third equality is not obvious. It can be shown by the formula of proposition 3.2. Choose symplectic coordinates (x, ξ) such that $\alpha = \{x = 0\}$ and $\beta = \{\xi = 0\}$. By the transversality hypothesis there exist homomorphisms A and B such that

$$\alpha' = \{x = A\xi\} \quad \beta' = \{\xi = Bx\}.$$

If α' is the graph of $A' \in \mathrm{Hom}(\alpha, \beta')$, then for all $\xi \in \alpha$ we must find $\xi' \in \alpha$ and $x \in \beta$ with

$$A'\xi = (x, Bx) \text{ and } (A\xi', \xi') = (x, Bx + \xi).$$

This gives $x = A\xi'$ and $\xi' = Bx + \xi = BA\xi' + \xi$ so $\xi' = (1 - BA)^{-1}\xi$ and

$$A'\xi = (A(1 - BA)^{-1}\xi, (1 - BA)^{-1}\xi - \xi).$$

We remark that $(1 - BA)$ is indeed invertible : if $\xi \in \ker(1 - BA)$ then $(A\xi, \xi) = (A\xi, BA\xi) \in \alpha' \cap \beta' = \{0\}$ so $\xi = 0$.

Therefore by the proposition (3.2)

$$2s(\alpha, \alpha'; \beta, \beta') = \text{ sgn } \omega(A(1 - BA)^{-1}.,.) - \text{ sgn } \omega(A.,.) \text{ sgn } \begin{vmatrix} A & 0 \\ 0 & -A(1 - BA)^{-1} \end{vmatrix}.$$

Suppose now that A is inversible then, because a symmetric matrix and its inverse have same signature:

$$\text{sgn} \begin{vmatrix} A & 0 \\ 0 & -A(1 - BA)^{-1} \end{vmatrix} = \text{sgn} \begin{vmatrix} A & 0 \\ 0 & -(1 - BA)A^{-1} \end{vmatrix} =$$

$$= \text{sgn} \begin{vmatrix} A & 0 \\ 0 & B - A^{-1} \end{vmatrix} = \text{sgn} \begin{vmatrix} A & 1 \\ 1 & B \end{vmatrix}$$

by formula (3.1.5). By the same calculus, and because ω is skewsymmetric, one has:

$$2s(\beta, \beta'; \alpha, \alpha') = \text{ sgn } Q(\beta, \alpha'; \beta') - \text{ sgn } Q(\beta, \alpha; \beta')) = - \text{ sgn } \begin{vmatrix} B & 1 \\ 1 & A \end{vmatrix}.$$

■

3.3 Proof of theorem 1.1

Following [7], we denote by $\mathcal{T}(\mathcal{L}) \subset \mathbb{L}(\mathcal{L})$ the set of the $\alpha \in \mathbb{L}(\mathcal{L})$ transversal to $\lambda(\pi(\alpha))$ and to $\lambda_0(\pi(\alpha))$. If $p : \mathcal{T}(\mathcal{L}) \to \mathcal{L}$ is the associated projection, then for all $\nu \in \mathcal{L}$

$$p^{-1}(\nu) = \mathbb{L}^0(n)(\lambda(\nu)) \cap \mathbb{L}^0(n)(\lambda_0(\nu)).$$

n.b. On the neighbourhood of points where the two Lagrangian are not transversal this map is not a fibration.

Lemma 3.1 *Let $\alpha : \mathbb{S}^1 \to \mathcal{T}(\mathcal{L})$ satisfying $p \circ \alpha = \gamma$ and σ be a path as before. The index $[\sigma_t, \alpha(t)]$ is constant in t.*

Indeed the index is a continuous map: let $t_0 \in [0,1]$ and β a path in the fibre over the point $\gamma(t_0)$ and linking $\lambda_0(t_0)$ to $\lambda(t_0)$ staying tranversal to $\alpha(t_0)$; by definition $[\sigma_{t_0}, \alpha(t_0)] = \mu_0(\sigma_{t_0} * \beta)$ but the property of transversality is open: if we denote β_t the path in the fibre over the point $\gamma(t)$ resulting of the parallel transport of $\lambda_{0|[t,t_0]} * \beta * \lambda^{-1}{}_{|[t,t_0]}$, then there exists $\varepsilon > 0$ such that for all $|t - t_0| < \varepsilon$ one has β_t is transversal to $\alpha(t)$. This parallel transport realizes an homotopy, so for all $|t - t_0| < \varepsilon$ one has $\mu_0(\sigma_{t_0} * \beta) = \mu_0(\sigma_t * \tilde{\beta}_t)$. ■

Corollary 3.1 *The induced fibrex bundle $p^*\mathbb{M}(\mathcal{L})$ is trivial.*

Proof. We have to show that for all path $\alpha : \mathbb{S}^1 \to \mathcal{T}(\mathcal{L})$ continuous, if we define $\gamma = p \circ \alpha$, then $\mu(\gamma) = 0$. To this goal take σ as before, a path in the fibre over $\gamma(0)$ linking $\lambda(0)$ to $\lambda_0(0)$. Choose σ transversal to $\alpha(1)$ and do the same constrution as before, then

$$[\sigma, \alpha(1)] = [\sigma_0, \alpha(0)] = 0$$

by the definition of $[\sigma, \alpha(1)]$ and lemma 3.1. But $\alpha(0) = \alpha(1)$ so

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}) = [\sigma_0, \alpha(1)] = 0.$$

■

Corollary 3.2 *Let s be a section of the Maslov bundle over $\mathcal{L}$, and $\gamma : \mathbb{S}^1 \to \mathcal{L}$ a closed path such that $\gamma(0) = \nu_0 = \pi(\lambda_0)$. Let $\alpha : [0,1] \to \mathcal{T}(\mathcal{L})$ be a continuous path satisfying $\gamma = p \circ \alpha$. Then*

$$p^* s(\alpha(1)) = i^{s(\lambda_0(0), \lambda(0); \alpha(1), \alpha(0))} p^* s(\alpha(0)).$$

Proof. Let σ be a path linking $\lambda(0)$ to λ_0 staying transversal to $\alpha(1)$. By lemma (3.1), $[\sigma_0, \alpha(0)] = [\sigma, \alpha(1)] = 0$ and

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}) = [\sigma_0, \alpha(1)] = [\sigma_0, \alpha(1)] - [\sigma_0, \alpha(0)] = s(\alpha(0), \alpha(1); \lambda(0), \lambda_0(0))$$

and $s(\alpha(0), \alpha(1); \lambda, \lambda_0) = -s(\lambda_0, \lambda; \alpha(1), \alpha(0))$ by the proposition 3.3. Therefore

$$-\mu(\gamma) = s(\lambda_0(0), \lambda(0); \alpha(1), \alpha(0)).$$

This gives the result by the equivalent relation (1.3.3). ■

From these two corollaries one obtains

Corollary 3.3 *The sections of $\mathbb{M}(\mathcal{L})$ are identified with functions f on $\mathcal{T}(\mathcal{L})$ satisfying the relation:* $\forall\alpha, \tilde{\alpha} \in \mathcal{T}(\mathcal{L})$

$$p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^{s(\lambda_0,\lambda;\tilde{\alpha},\alpha)} f(\alpha).$$

This result gives the gluing condition of Hörmander, in view of the theorem 3.3.3, [7] and finish the proof of the theorem. For completness we recall this last step.

Proposition 3.4 *The functions f on $\mathcal{T}(\mathcal{L})$ which satisfy:* $\forall\alpha, \tilde{\alpha} \in \mathcal{T}(\mathcal{L})$

$$p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^{s(\lambda_0,\lambda;\tilde{\alpha},\alpha)} f(\alpha).$$

are the sections defined by the gluing conditions of the section 1.2.

Proof. Let ϕ be a non degenerated phase function as in section 1.2 and $\nu_0 = (x_0, \xi_0) = (x_0, \phi'_x(x_0, \theta_0))$ a point in $\mathcal{L}_\phi$. For each $\alpha \in \mathcal{T}(\mathcal{L})$ such that $p(\alpha) = \nu_0$, there exists a function ψ defined on an open set U such that the graph $L_\psi = \{(x, d\psi(x)),\ x \in U\}$ of the differential $d\psi$ intersect transversally $\mathcal{L}_\phi$ at ν_0, one has $\xi_0 = d\psi(x_0)$ and $T_{\nu_0} L_\psi = \alpha$.

Or equivalently one can say: the following quadratic form defined on $\mathbb{R}^{n+N}$ by the matrix

$$Q_\psi = \begin{vmatrix} \phi''_{xx} - \psi''_{xx} & \phi''_{x\theta} \\ \phi''_{\theta x} & \phi''_{\theta\theta} \end{vmatrix} \tag{3.3.6}$$

is non degenerated.

The restriction of this quadratic form to the tangent W of $\mathcal{L}_\phi$ at ν_0 only depends on $\mathcal{L}$ and ψ (and not on ϕ). Indeed ϕ defines a card in which

$$\lambda(\nu_0) = T_{\nu_0}(\mathcal{L}) = \{(X, \phi''_{xx}X + \phi''_{x\theta}A); (X, A) \in \mathbb{R}^{n+N},\ \phi''_{\theta x}X + \phi''_{\theta\theta}A = 0\};$$

if now $(X, A), (X', A')$ define two tangent vectors V and $V' \in \lambda(\nu_0)$

$$\begin{aligned} Q_\psi\Big((X, A), (X', A')\Big) &= \ < X, (\phi''_{xx} - \psi''_{xx})X' + \phi''_{x\theta}A' > \\ < -\psi''_{xx}X, X' > - < -X, \phi''_{xx}X' + \phi''_{x\theta}A' > &= \ Q\Big(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)\Big)(V, V') \end{aligned}$$

by definition (3.1.4). More precisely α is transverse to the two lagrangians $\lambda(\nu_0)$ and $\lambda_0(\nu_0)$ so the vertical $\lambda_0(\nu_0)$ is the graph of an homomorphism A_ψ from $\lambda(\nu_0)$ to $\alpha = T_{\nu_0} L_\psi$:

$$\forall(0, \Xi) \in \lambda_0(\nu_0),\ \exists(X, A)\text{unique such that } \Xi = \phi''_{xx}X + \phi''_{x\theta}A \text{ et } \phi''_{\theta x}X + \phi''_{\theta\theta}A = 0$$

because Q_ψ is non degenerated, and one can write

$$(0, \Xi) = (X, \phi''_{xx}X + \phi''_{x\theta}A) - (X, \psi''_{xx}X),$$

it means that $A_\psi(X, \phi''_{xx}X + \phi''_{x\theta}A) = (-X, -\psi''_{xx}X)$.

We see now that the orthogonal W^{Q_ψ} of W with respect to Q_ψ is $\mathbb{R}^N = \{(0, A)\}$ and that $Q_{\psi|W^{Q_\psi}} = \phi''_{\theta\theta}$. But the lemma 3.2 below gives sgn Q_ψ = sgn $Q_{\psi|W}$ + sgn $Q_{\psi|W^{Q_\psi}}$, so:

$$\text{sgn } Q_\psi = \text{sgn } Q(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)) + \text{sgn } \phi''_{\theta\theta}. \tag{3.3.7}$$

Let now z_ϕ be a section in the sens of Hörmander. For any $\alpha \in \mathcal{T}(\mathcal{L}), p(\alpha) = \nu_0$, if ϕ and $\tilde{\phi}$ are two phase functions defining $\mathcal{L}$ in a neighbourhood of ν_0 and if ψ is a function on X satisfying $\alpha = T_{\nu_0} L_\psi$, we denote by Q_ψ and $\tilde{Q}_\psi$ the respective quadratic forms defined by (3.3.6). Put

$$f(\alpha) = \exp(i\frac{\pi}{4}\text{sgn } Q_\psi) z_\phi(\nu_0).$$

By the relation (3.3.7) one has sgn $\phi''_{\theta\theta}$ − sgn $\tilde{\phi}''_{\tilde{\theta}\tilde{\theta}}$ = sgn Q_ψ − sgn $\tilde{Q}_\psi$; the compatibility condition 1.2.2 gives then

$$\exp(i\frac{\pi}{4}\text{sgn } Q_\psi) z_\phi(\nu_0) = \exp(i\frac{\pi}{4}\text{sgn } \tilde{Q}_\psi) z_{\tilde{\phi}}(\nu_0)$$

and the function f is well defined on $\mathcal{T}(\mathcal{L})$. On the other hand if $\tilde{\alpha}$ is an other point in $\mathcal{T}(\mathcal{L})$ such that $p(\tilde{\alpha}) = \nu_0$ and if $\tilde{\psi}$ is an adapted function, then

$$\begin{aligned}
f(\tilde{\alpha}) &= \exp(i\frac{\pi}{4}(\text{sgn } \tilde{Q}_\psi - \text{sgn } Q_\psi)) f(\alpha) \\
&= \exp\left(i\frac{\pi}{4}\left(\text{sgn } Q(\lambda(\nu_0), \tilde{\alpha}; \lambda_0(\nu_0)) - \text{sgn } Q(\lambda(\nu_0), \alpha; \lambda_0(\nu_0))\right)\right) f(\alpha) \\
&= \exp\left(i\frac{\pi}{2} s(\lambda(\nu_0), \lambda_0(\nu_0); \alpha, \tilde{\alpha})\right) f(\alpha) \\
&= \exp\left(i\frac{\pi}{2} s(\lambda_0(\nu_0), \lambda(\nu_0); \tilde{\alpha}, \alpha)\right) f(\alpha)
\end{aligned}$$

So it is a section of the Maslov bundle and the theorem 1.1 is proved. ∎

Lemma 3.2 *Let Q be a non degenerated quadratic form defined on $\mathbb{R}^n$, V be a subspace of $\mathbb{R}^n$ and V^Q its orthogonal for Q, then*

$$sgn\ Q = sgn\ Q_{|V} + sgn\ Q_{|V^Q}.$$

Proof. This lemma can be showed using an induction on $\dim V \cap V^Q$. If $\dim V \cap V^Q = 0$ there is nothing to do, if not let $v_1, \ldots, v_k$ be a base of $V \cap V^Q$. We complete this base with $v_{k+1}, \cdots, v_p$ to obtain a base of $V + V^Q$. Because Q is non degenerated there exists $w_1 \in \mathbb{R}^n$ such that $Q(v_1, w_1) = 1$, and eventually after a modification with a linear combination of the v_j one can suppose $Q(w_1) = 0$ and $Q(v_j, w_1) = 0$ for $j > 1$. One remarks that the signature of Q in restriction to $\mathbb{R}v_1 \oplus \mathbb{R}w_1$ is zero and applies the induction hypotheses to $(\mathbb{R}v_1 \oplus \mathbb{R}w_1)^Q$. ∎

4 Topological comments

Let's have a look to the exact sequence: $0 \to \Pi_1(\mathbb{L}(n)) \xrightarrow{i_*} \Pi_1(\mathbb{L}(\mathcal{L})) \xrightarrow{\pi_*} \Pi_1(\mathcal{L}) \to 0$.

The group $\Pi_1(\mathbb{L}(\mathcal{L}))$ is the semidirect product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(\mathcal{L})$. It means that $\Pi_1(\mathcal{L})$ acts on $\Pi_1(\mathbb{L}(n))$ by conjugation. More precisely for all $\gamma \in \Pi_1(\mathcal{L})$ let's define

$$\begin{array}{rcl} \rho_\gamma : \Pi_1(\mathbb{L}(n)) & \to & \Pi_1(\mathbb{L}(n)) \\ \sigma & \mapsto & \lambda_0(\gamma) * i_*(\sigma) * (\lambda_0(\gamma))^{-1} \end{array}$$

Lemma 4.1 *This representation is trivial and $\Pi_1(\mathbb{L}(\mathcal{L}))$ is in fact the direct product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(\mathcal{L})$.*

Proof. As was seen in paragraph 2, the parallel transport along γ defines an homotopy of $\lambda_0(\gamma) * i_*(\sigma) * (\lambda_0(\gamma))^{-1}$ to a path which can be writen $\tilde{\lambda}_0 * \tilde{\sigma} * (\tilde{\lambda}_0)^{-1}$ where $\tilde{\sigma}$ is the image of σ by $\tau(\gamma)$. But

$$\mu_0(\tilde{\lambda}_0 * \tilde{\sigma} * (\tilde{\lambda}_0)^{-1}) = \mu_0((\tilde{\lambda}_0)^{-1} * \tilde{\lambda}_0 * \tilde{\sigma}) = \mu_0(\tilde{\sigma}) = \mu_0(\sigma).$$

As a consequence of the works of Arnol'd recalled above, a generator of $\Pi_1(\mathbb{L}(n))$ is caracterized by $\mu_0(\sigma) = 1$. ■

Theorem 4.1 *Let $\mathbb{L}^1(\mathcal{L})$ be the set of the points $l \in \mathcal{L}$ which are not transversal to $\lambda_0(\pi(l))$. It is an oriented cycle of $\mathcal{L}$ of codimension 1 ; if m is its Poincaré dual form, then*

$$\mu = \lambda^* m.$$

Proof. We keep the notations of paragraph 2. By choosing the starting point one can suppose that *the two lagrangians $\lambda_0 = \lambda_0(0)$ and $\lambda(0)$ are transversal.* We will use a deformation of the path $\tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1}$ joining $\lambda(0)$ to $\lambda_0(0)$. Recall that $\tilde{\sigma}(t) = \tau(\gamma)(\sigma(t))$.

There exists a (continuous) path $u(t) \in U(n)$ such that $u(0)$ =I and

$$\forall t \in [0,1]\ \tilde{\lambda}_0(t) = u(t)(\lambda_0).$$

But $\tilde{\lambda}_0(1) = \tau(\gamma)(\lambda_0)$, so $\tau(\gamma)$ and $u(1)$ differ by an element of $O(n)$:

$$\exists a \in O(n)\ ;\ \tau(\gamma) = u(1) \circ a.$$

Let's construct the following homotopy of $\tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1}$ by the concatenation of $u(st)^{-1}\tilde{\lambda}(t)$, next $u(s)^{-1}\tilde{\sigma}$ and finally the inverse of $u(st)^{-1}\tilde{\lambda}_0(t)$. The end of this homotopy is a path, result of the concatenation of $\bar{\lambda}(t) = u(t)^{-1}\tilde{\lambda}(t)$ and $u(1)^{-1}\tilde{\sigma} = a\sigma$ because $u(t)^{-1}\tilde{\lambda}_0(t) = \lambda_0$ is a constant path.

We have now to calculate $\mu_0(\sigma^{-1} * \bar{\lambda} * a\sigma)$. Because $a \in O(n)$

$$Det^2(\sigma(t)) = Det^2(a\sigma(t));$$

$Det^2 \circ \bar{\lambda}$ is a closed path even if $\bar{\lambda}$ is not, so $\mu(\gamma) = \text{Degree } (Det^2 \circ \bar{\lambda})$.

Considering the results of section 2.1, we have obtained

Proposition 4.1 *$\mu(\gamma)$ is the intersecting number of the submanifold $\overline{\mathbb{L}^1(n)(\lambda_0)}$ and the cycle obtained from $\tilde{\lambda}$, by closing it with a path staying transversal to λ_0.*

Remark that $\tilde{\lambda}(0) = \lambda(0)$ and $\tilde{\lambda}(1) = a\lambda(0)$ are both transversal to λ_0. Let's now

$$\mathbb{L}^1(\mathcal{L}) = \left\{ l \in \mathbb{L}(\mathcal{L}) \; ; \; \lambda_0(\pi(l)) \cap l \neq \{0\} \right\}.$$

It is a fibration above $\mathcal{L}$ with fibre $\overline{\mathbb{L}^1(n)(\lambda_0)}$, so it is an oriented cycle of codimension 1 in $\mathcal{L}$. If $\lambda \circ \gamma$ cuts $\mathbb{L}^1(\mathcal{L})$ transversally at $\lambda \circ \gamma(t)$ then $\tilde{\lambda}$ cuts transversally $\overline{\mathbb{L}^1(n)(\lambda_0)}$ at $\tilde{\lambda}(t)$ and conversely. Moreover the transformations which permit to pass from $\lambda \circ \gamma$ to $\tilde{\lambda}$ realise a continuous deformation of $\mathbb{L}^1(\mathcal{L})$ to $\overline{\mathbb{L}^1(n)(\lambda_0)}$ above γ. This argument finishes the proof of the theorem 4.1. ∎

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