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Nonlinear semigroup associated with maximizing operator and large deviation

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ABSTRACT

We consider a class of uniformly elliptic second order differential operators and also its maximizing operator. In this paper, we obtain a variational formula for the principal eigenvalue associated with nonlinear semigroup, defined by M.Nisio ([11]), whose infinitesimal generator corresponds to the maximizing operator. Our result is an extension of [1] and [2], in which they considered the problems relative to linear operators. Moreover, as applications, we shall discuss large deviation, rate function and other properties relative to the maximizing operator. Our proofs are almost relied upon stochastic control method which developped by N.V.Krylov [7], [8], W.Fleming [5], P.L.lions [9] and others.

RESUMEN

Consideramos una clase de operadores diferenciales de segundo orden uniformemente elípticos así como su operador maximizante. En este artículo obtenemos una formulación variacional para el autovalor principal asociado al semigrupo nolineal definido por M.Nisio ([11]), cuyo generador infinitesimal corresponde al operador maximizante. Como aplicación discutiremos la desviación mayor, función de cuociente y otras propiedades relativas al operador maximizante. Nuestras demostraciones casi se basan en el método de control estocástico desarrollado por N.V.Krylov [7], [8], W.Fleming [5] y P.L.lions [9] entre otros.

Key words and phrases: *uniformly elliptic second order differential operator, maximizing operator, principal eigenvalue, nonlinear semigroup, large deviation.*
Math. Subj. Class.: *47H20, 60F10, 65C30, 93E20*

1 Nonlinear semigroup

Let A be a compact convex subset in a Euclidean space R^ℓ. For each element α of A, let L^α be the operator given by the following;

$$L^\alpha = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\alpha,x)\partial^2/\partial x_i\partial x_j + \sum_{i=1}^{d} b_i(\alpha,x)\partial/\partial x_i \tag{1.1}$$

where the coefficients $a = (a_{ij}(\alpha,x))_{1\le i,j\le d}$ and $b = (b_i(\alpha,x))_{1\le i\le d}$, defined on $A\times R^d$, are symmetric $d\otimes d$-matrix and d-vector valued functions, respectively. Assume the following conditions relative to a and b;

(A.1) (a) $a(\alpha,x)$ and $b(\alpha,x)$ are C^2-functions of $x\in R^d$, and their partial derivatives $\partial_i\partial_j a, \partial_i\partial_j b, \partial_i a$ and $\partial_i b$ † including a and b themselves, are all bounded on $A\times R^d$.
(b)

$$|f(x,\alpha) - f(y,\beta)| \le \gamma|x-y| + \rho(|\alpha-\beta|), f = a, b,$$

where ρ is a concave, strictly increasing and continuous function on $[0,\infty)$ and $\rho(0) = 0$, and γ is a positive constant.
(c) a is uniformly positive definite, i.e. there exist positive constants η and K such that

$$\eta\sum_{i=1}^{d}\xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(\alpha,x)\xi_i\xi_j \le K\sum_{i=1}^{d}\xi_i^2, \tag{1.2}$$

uniformly for all $\alpha\in A, x\in R^d$, and $\xi\in R^d$.

We denote by L the operator defined by the formula;

$$Lu(x) = \sup_{\alpha\in A} L^\alpha u(x) \tag{1.3}$$

or simbolically, we often write

$$L = \sup_{\alpha\in A} L^\alpha, \tag{1.3$'$}$$

to which we refer as the *maximizing operator*. In (1.3), the supremum is taken for each x. Define a nonlinear semigroup associated with L, following M.Nisio ([11], see also [5]). Let $\{\alpha_t(\omega), 0\le t<\infty\}$, be a process, defined on a probability space

† $\partial_i\partial_j a = \partial^2 a/\partial x_i\partial x_j$ e.t.c.

$(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ satisfying the usual conditions[††]. Assume that α_t is progressively measurable with respct to $\{\mathcal{F}_t\}$, and that $\alpha_t(\omega) \in A$ for all $(t, \omega) \in [0, \infty) \times \Omega$. Denote by $\mathcal{A}$ the collection of all such processes.

Definition 1.1 $\mathcal{A}$ is called an *admissible control class*, and its arbitrary element $\{\alpha_t, \; t \geq 0\}$ (α_t, for short) an *admissible control*, respectively.

For each $\alpha \in \mathcal{A}$ and $x \in R^d$, consider the following Ito-type stochastic differential equation;

$$\begin{cases} dX_t = \sigma(\alpha_t, X_t)d\beta_t + b(\alpha_t, X_t)dt, \\ X_0 = x, \end{cases} \tag{1.4}$$

where $(\beta_t), 0 \leq t < \infty$, is a R^d-valued Brownian motion with respect to $(\mathcal{F}_t, P)$, defined on a probability space on which the process (α_t) is given. $\sigma = \sigma(\alpha, x), \alpha \in A, x \in R^d$, is a $d \otimes d$ matrix-valued function such that $\sigma \cdot \sigma^* = a$ here, σ^* means the tranposed matrix of σ and $\sigma(\alpha, \cdot)$ is Lipshitz continuous uniformly with respect to α. Then, it is well known that under the assumption **(A.1)**, for each $\alpha_t \in \mathcal{A}$ and $x \in R^d$ (and also β_t), there exists a unique solution of Eq.(1.2), which we denote by $\{X_t^{\alpha,x}, t > 0\}$ or $X_t^{\alpha,x}$, for short.

Definition 1.2 Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a probability space satisfying the usual conditions. Let α_t and β_t be an admissible control and a Brownian motion respectively, given on this space. Let also X_t be the unique solution of Eq.(1.4) associated with these (α_t, β_t). Then the 7-tuple $(\Omega, \mathcal{F}, P, \mathcal{F}_t, \alpha_t, \beta_t, X_t)$ is called an *admissible system*. Moreover, we refer to the process X_t as the *response* associated with α_t and x.

Denote by C the space consisiting of all R^1-valued bounded and uniformly continuous functions on R^d, then the space C is a Banach space with the sup-norm $\|\cdot\|$. For each $\varphi \in C$ and for $(t, x) \in (0, \infty) \times R^d$, put

$$T_t\varphi(x) = \sup_{\alpha \in \mathcal{A}} E[\varphi(X_t^{\alpha,x})], \tag{1.5}$$

where $X_t^{\alpha,x}$ denotes the response given by Eq.(1.4) associated with $\alpha \in \mathcal{A}$ and $x \in R^d$, and the supremum is taken among all admissible systems. Then M.Nisio proved that $\{T_t, t \geq 0\}$ defines contractive, monotone and strongly continuous nonlinear semigroup of operators from C to C. Denote by G the infinitesimal generator of T_t and by $\mathcal{D}(\subset C)$ its domain respectively, then $C^2 \subset \mathcal{D}$ and

$$Gu(x) = \sup_{\alpha \in A} L^\alpha u(x) \equiv Lu(x), \tag{1.6}$$

where $u \in C^2$ and $C^2 = \{\psi \in C; \exists\partial_i\psi, \exists\partial_i\partial_j\psi \in C\}$ (see M.Nisio [11] or [5]).

[††] $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\forall t \geq 0, \mathcal{F}_t$ is an increasing family of sub-σ fields of $\mathcal{F}$, right continuous, $\bigvee_{t>0} \mathcal{F}_t = \mathcal{F}$

Let $T > 0$. Denote by Q_T^0 the cylindrical set $(0, T) \times R^d$ and put $\bar{Q}_T^0 = [0, T] \times R^d$. Consider the following Cauchy problem relative to $L = \sup_\alpha L^\alpha$;

$$\begin{cases} \partial u/\partial t + Lu = 0, & (t, x) \in Q_T^0, \\ u(T, x) = \varphi(x), & x \in R^d. \end{cases} \tag{1.7}$$

Then it is known ([5] p.169, Theorem 4.4.2, e.g.) that if $\varphi \in C^{2,\alpha\,\dagger}$, $0 < \alpha < 1$, then $u(t, x) = T_{T-t}\varphi(x)$ is the unique solution of Eq.(1.7) in $C^{1,2}(\bar{Q}_T^0)$ (for the definition, see §2).

Set $\mathcal{C}_+ = \{\varphi \in \mathcal{C}; \exists c_1, \exists c_2$ such that $0 < c_1 \le \varphi(x) \le c_2 < \infty, \forall x \in R^d\}$ and $C_+^{2,\alpha} \equiv \mathcal{C}_+ \cap C^{2,\alpha}$ e.t.c. Note that if $\varphi \in C_+^{2,\alpha}$ then for all $t > 0$, $T_t\varphi(x) \in C_+^2 (= \mathcal{C}_+ \cap C^2)$. Let $\mathcal{D}$ be the domain of G, and let also $\mathcal{D}_+$ be the subset of $\mathcal{D}$ defined in the analogous way to $\mathcal{C}_+$. Denote by $\mathcal{M}$ the space of all probability measures on R^d. For each $\mu \in \mathcal{M}$, define $I(\mu)$ by the formula;

$$I(\mu) = -\inf_{u \in \mathcal{D}_+} \int_{R^d} (Gu/u)(x)\mu(dx). \tag{1.8}$$

Note that $I(\mu)$ is nonnegative. Moreover, for each $\mu \in \mathcal{M}$, define $\tilde{I}(\mu)$ by the following;

$$\tilde{I}(\mu) = -\inf_{u \in C_+^2} \int_{R^d} (Lu/u)(x)\mu(dx). \tag{1.9}$$

Then we have the following.

Proposition 1.1 *For any $\mu \in \mathcal{M}$, $I(\mu) = \tilde{I}(\mu)$.*

Proof It is sufficient to show that $\tilde{I}(\mu) \ge I(\mu)$. Suppose that $\varphi \in C_+^\infty$. Then it follows that $T_t\varphi(\cdot) \in C_+^2$ for all $t \ge 0$ and that $T_{T-t}\varphi$ is the unique solution of Eq.(1.7). Hence one obtains

$$\frac{\partial}{\partial t}\int_{R^d} \log (T_t\varphi/\varphi)(x)\mu(dx) = \int_{R^d} \frac{\partial T_t\varphi}{\partial t}(x)\mu(dx)/T_t\varphi(x) \tag{1.10}$$

$$= \int_{R^d} LT_t\varphi(x)\mu(dx)/T_t\varphi(x) \ge -\tilde{I}(\mu).$$

because of (1.9). Since $T_0\varphi(x) = \varphi(x)$, $\log T_0\varphi/\varphi = 0$. Integrating the both terms of (1.10) from 0 to t with respect to t, we get

$$\int_{R^d} \log (T_t\varphi/\varphi)(x)\mu(dx) \ge -t \cdot \tilde{I}(\mu), \tag{1.11}$$

for all $\varphi \in C_+^\infty$ and $t \ge 0$. Let $\varphi \in \mathcal{C}_+$, then there exists a sequence $\{\varphi_n\} \subset C_+^\infty$ which converges to φ uniformly on compact sets in R^d. Therefore, the inequality

† $\psi \in C^{2,\alpha}$ iff $\psi \in C^2, \psi, \partial_i\psi, \partial_i\partial_j\psi, \forall i, j$, are all α-order Hölder continuous

(1.11) holds for any $\varphi \in \mathcal{C}_+$ and $t \geq 0$. Thus, if $\varphi \in \mathcal{D}_+$ then $\varphi \in \mathcal{C}_+$, and it follows from (1.11) that for all $t \geq 0$,

$$\int_{R^d} \log\left(T_t\varphi/\varphi\right)(x)\mu(dx) \geq -t \cdot \tilde{I}(\mu).$$

Dividing this by t, and letting $t \downarrow 0$, one sees;

$$\int_{R^d} \left(G\varphi/\varphi\right)(x)\mu(dx) \geq -\tilde{I}(\mu). \tag{1.12}$$

This inequality being true for all $\varphi \in \mathcal{D}_+$, we can conclude that $I(\mu) \leq \tilde{I}(\mu)$ because of (1.8) and (1.12). ∎

2 Nonlinear semigroup on a bounded domain

Let D be a bounded open set in R^d with smooth boundary ∂D. Put $\bar{D} = D \bigcup \partial D$ and $C(D) = \{f : D \to \mathrm{R}$, bounded on D, continuous in $D\}$. For any fixed $T > 0$, let $Q_T = (0,T) \times D$, $\bar{Q}_T = [0,T] \times \bar{D}$, and $\partial^* Q_T = [0,T] \times \partial D \bigcup \{T\} \times D$. $[0,T] \times \partial D$ and $\{T\} \times D$ are called the *lateral boundary* and *terminal boundary*, respectively. We write $\psi \in C^{1,2}(Q_T)$ if ψ is real valued function defined on Q_T such that its partial derivatives $\partial_t\psi, \partial_i\psi, \partial_i\partial_j\psi, 1 \leq i,j \leq d$, and ψ itself are all continuous in Q_T. For any $\alpha \in (0,1)$, we also write $\psi \in C^{1,2,\alpha}(Q_T)$ if $\psi \in C^{1,2}(Q_T)$ and its partial derivatives $\partial_t\psi, \partial_i\partial_j\psi, \partial_i\psi, 1 \leq i,j \leq d$, and itself are α-order Hölder continuous uniformly with respect to (t,x).

Consider the following parabolic Bellman equation with the Dirichlet boundary condition;

$$\partial u/\partial t + Lu + V(x)u = 0, \text{ on } Q_T, \tag{2.1}$$

$$\begin{cases} u(t,x) = 0, & 0 < t < T, \ x \in \partial D \\ u(T,x) = \varphi(x) & x \in \bar{D} \end{cases}, \tag{2.2}$$

where L is given by (1.3) and V is a real valued function defined on R^d such that $V \in C^2(R^d)(= C^2$ in §1) and its partial derivatives up to the second order are all bounded. Then we obtain the following.

Theorem 2.1 *Assume the hypothesys (A.1). Then:*
(a) *if $\varphi \in C^{2,\alpha}(\bar{D})$ for some $\alpha \in (0,1)$, then Eq.(2.1) with (2.2) has a unique solution $u(t,x) \in C^{1,2}(\bar{Q}_T)$.*
(b) *if $\varphi \in C^{2,\alpha}(\bar{D})$ and $\varphi(x)$ vanishes at the boundary ∂D, then Eq.(2.1) with (2.2) has a unique solution $u(t,x) \in C^{1,2}(Q_T) \bigcap C(\bar{Q}_T)$.*

The theorem is a special case of general theorem about fully nonlinear parabolic equation with Dirichlet boundary condition ([4],[9],e.g.), but we can also prove it directly using stochastic control methods (see [6]).

For each $\varphi \in C(\bar{D}), t \geq 0$ and $x \in \bar{D}$, define $T_t^V\varphi(x)$ by the following formula:

$$T_t^V\varphi(x) = \sup_{\alpha\in\mathcal{A}} E[\exp\{\int_0^t V(X_r^{\alpha,x})dr\}\varphi(X_t^{\alpha,x}); t < \tau_D^{\alpha,x}], t > 0, x \in D,$$

$$T_0^V = I, T_t^V\varphi(x) = 0 \ \text{ if } x \in \partial D \text{ and } t > 0. \tag{2.3}$$

Then T_t^V defines nonlinear semigroup on $C(\bar{D})$ (see Remark 2.2 below). Put $\varphi(x) \equiv 1$ in (2.3), and define $\|T_t^V\|$ by the formula:

$$\|T_t^V\| = \sup_{x\in D} |T_t^V 1(x)|. \tag{2.4}$$

Since $\|T_t^V\|$ is submultiplicative with respect to t in view of the semigroup property, $T_{t+s}^V = T_t^V T_s^V$, the following limit exists, we denote it by λ_V,

$$\lambda_V = \lim_{t\to\infty} \log \|T_t^V\|/t. \tag{2.5}$$

Definition 2.1 λ_V is called the *principal eigenvalue* of $L + V$.

Remark 2.2 (a) $\lambda_V > -\infty$. *Indeed, it follows from (2.3) that*

$$\log \|T_t^V\|/t \geq \log \|T_t^{V,\alpha}\|/t$$

for any $\alpha \in A$ and $t > 0$, where $T_t^{V,\alpha}$ means linear semigroup associated with the operator $L^\alpha + V$. But the right side converges to the principal eigenvalue of $L^\alpha + V$ as t tends to infinity (see [6],e.g.). Moreover, λ_V is not necessarily positive. In fact, if V is nonpositive, then λ_V might be nonpositive.

(b)*If λ is an arbitrary real number such that $\lambda > \lambda_V$, then it is easy to see from (2.5) that for any $\varphi \in C(D)$ such that $\|\varphi\| \leq 1$,*

$$\int_0^\infty \exp\{-\lambda t\}T_t^V\varphi(x)dt \leq \int_0^\infty e^{-\lambda t}\|T_t^V\|dt < \infty, \tag{2.6}$$

because $\|T_t^V\varphi\| \leq \|T_t^V\|\|\varphi\|$. These facts suggest us that for any λ such that $\lambda > \lambda_V$, there exists resolvent in a sense, while λ_V itself is in the spectrum. For further discussions about these topics, see [6] or [10].

(c) *Although we do not know if λ_V is an eigenvalue of G^V, the infinitesimal generator of T_t^V, it will be shown ([6]) that if there exists the largest eigenvalue (denote it by λ_0) of G^V, then $\lambda_V = \lambda_0$, so that Definition 2.1 can be justified in this case.*

Remark 2.3 *It is not difficult to show that T_t^V, given by (2.3), defines nonlinear semigroup operator on $C(\bar{D})$ having the following properties. For the proof, see [6] (cf.[5], Th.5.2.1, p.219).*

(a) *T_t^V maps $C(\bar{D})$ into $C(D)$. Especially it maps $C_0(\bar{D}) = \{\varphi \in C(\bar{D}); \varphi(x) = 0, \forall x \in \partial D\}$ into itself.*

(b) *$\|T_t^V\varphi\| \leq M\|\varphi\|$ for all $\varphi \in C(\bar{D})$ and $0 \leq t < T$, where M is a positive constant depending only on T and the dominant of V.*

(c) *(monotonicity)* $T_t^V \varphi \leq T_t^V \psi$ *if* $\varphi \leq \psi, 0 \leq t$.
(d) *(continuity in t) For any* $\varphi \in C_0(\bar{D})$, $\|T_t^V \varphi - T_s^V \varphi\| \to 0$ *as* $t \to s$.
(e) *For any* $\varphi \in C^2(D) \cap C(\bar{D})$,

$$\lim_{t \to 0} \{T_t^V \varphi(x) - \varphi(x)\}/t = (L + V)\varphi(x), x \in D. \tag{2.7}$$

(f) *For each* $T > 0$, *and for any* $t \in (0, T)$, $\varphi \in C^{2,\alpha}(\bar{D})$ *for some* $\alpha \in (0,1)$, $T_{T-t}^V \varphi(x)$ *is the unique solution of Eq.(2.1) with the boundary condition (2.2).*
(g) *(semigroup property) For any* $\varphi \in C^{2,\alpha}(\bar{D})$ *and* $t, s > 0$,

$$T_{t+s}^V \varphi(x) = T_t^V T_s^V \varphi(x), x \in D. \tag{2.8}$$

(h) *Denote by* G^V *the infinitesimal generator of* T_t^V, *then the domain of* G^V *contains* $C^2(D) \cap C(\bar{D})$. *Moreover,* $G^V \varphi = L\varphi + V\varphi$ *for* $\varphi \in C^2(D) \cap C(\bar{D})$.

3 Variational formula

The following theorem is the main result of this paper.

Theorem 3.1 *Given* $V \in C(\bar{D})$, *we have*

$$\lambda_V = \sup_{\mu:\mu(\bar{D})=1} \left[\int_{R^d} V(x)\mu(dx) - I(\mu) \right]. \tag{3.1}$$

Denote by ℓ_V the right side of (3.1),i.e.,

$$\ell_V = \sup_{\mu;\mu(\bar{D})=1} \left[\int_{R^d} V(x)\mu(dx) - I(\mu) \right]. \tag{3.2}$$

Then We can show the inequality that $\lambda_V \leq \ell_V$ by using the way almost similar to that of [2] , while the reversed one will be done by using the stochastic control method. For the detail, see [6].

Let μ be any probability measure on R^d such that $\mu(\bar{D}) = 1$, then we can obtain a variational formula for $I(\mu)$ in terms of λ_V, which means the uniqueness of the rate function in the large deviation principle ([12]).

Theorem 3.2 *For each* μ *with supp.* $\mu \subset \bar{D}$,

$$I(\mu) = -\inf_{V \in C^\infty(R^d)} \left[\lambda_V - \int_{R^d} V(x)\mu(dx) \right]. \tag{3.3}$$

Proof On account of (3.1) in Theorem 3.1, the right side of (3.3) is equal to

$$-\inf_{V \in C^\infty(R^d)} \left[\sup_{\gamma:\gamma(\bar{D})=1} \left\{ \int_{R^d} V(x)\gamma(dx) - I(\gamma) \right\} - \int_{R^d} V(x)\mu(dx) \right]$$

$$= -\inf_{V \in C^\infty(R^d)} \sup_{\gamma:\gamma(\bar{D})=1} \left[-I(\gamma) + \int_{R^d} V(x)\gamma(dx) - \int_{R^d} V(x)\mu(dx) \right].$$

Since we can exchange the order of sup and inf in the above formula by the same reason as in Theorem 3.1, the rest of the proof is the same as [2](Theorem 2.3). ∎

Remark 3.3 *Using these results we can show that λ_V is equal to the largest eigenvalue of G^V if it exists. We will show this. Consider the following eigenvalue problem with Dirichlet boundary condition;*

$$\begin{cases} (L+V)u(x) = \lambda u(x), & x \in D, \\ u(x) > 0, & x \in D, \\ u(x) = 0, & x \in \partial D \end{cases} \tag{3.4}$$

Suppose that there exists a solution of Eq.(3.4), (λ_0, u_0), where λ_0 is an eigenvalue and u_0 is a corresponding smooth eigenfunction. It is easily seen that $\lambda_0 \le \lambda_V$ because, for each element α of A, $\lambda_\alpha \le \lambda_V$ from the definition of λ_V and $\lambda_0 = \sup_\alpha \lambda_\alpha$. On the otherhand, note that for all $u \in C^2_+(R^d)$, we get the inequality.

$$-I(\mu) \le \int_{R^d} (Lu/u)\mu(dx),$$

by means of (1.8) and Proposition 1.1. Using this inequality, we can show that for each $\mu \in \mathcal{M}$ and u,

$$\int V(x)\mu(dx) - I(\mu) \le \int V(x)\mu(dx) + \int (Lu/u)(x)\mu(dx) = \int \{(L+V)u/u\}(x)\mu(dx).$$

Here, note that we may extend this result for any $u \in C^{2,\alpha}$ and $u > 0$ in D (cf.[6]). Therefore, if λ_0 and u_0 are solution of Eq.(3.4), then

$$\int_{R^d} V(x)\mu(dx) - I(\mu) \le \lambda_0,$$

from which follows the inequality $\lambda_V \le \lambda_0$ immediately.

4 Large deviation

In this section we shall discuss some topics of large deviation principle in connection with the variational formula (3.1). Indeed, we can interpretate that (3.1) is a special form of Laplace principle and $I(\cdot)$ is the rate function (see [3],e.g.). We shall set up the problem following [12]. Let $X_t^{\alpha,x}$ be the response associated with $\alpha \in \mathcal{A}$ and $x \in D$. For each $t > 0$, define $L_t^{\alpha,x} : \Omega \to \mathcal{M}$ such that

$$L_t^{\alpha,x}(\Gamma,\omega) = \frac{1}{t}\int_0^t \chi_\Gamma(X_s^{\alpha,x}(\omega))ds, \tag{4.1}$$

where Γ is a Borel set in R^d. For each (α, t, x), put

$$Q_t^{\alpha,x}(\cdot) = P(L_t^{\alpha,x} \in \cdot, t < \tau_D^{\alpha,x}),$$

then $Q_t^{\alpha,x}$ is a measure on $(\mathcal{M}(D), \mathcal{B}(\mathcal{M}(D))$, where $\mathcal{M}(D) = \{\mu \in \mathcal{M}; \mu(\bar{D}) = 1\}$ and $\mathcal{B}(\mathcal{M}(D))$ means Borel σ-field in the sense of weak topology.

Theorem 4.1 *$I(\cdot)$ is given by (1.8). Then,*
(1) for all closed subset F of $\mathcal{M}(D)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(\sup_{x\in D} \sup_{\alpha\in\mathcal{A}} Q_t^{\alpha,x}(F)) \le - \inf_{\nu\in F} I(\nu), \tag{4.2}$$

(2) for all open subset G of $\mathcal{M}(D)$,

$$\liminf_{t\to\infty} \frac{1}{t} \log(\sup_{x\in D} \sup_{\alpha\in\mathcal{A}} Q_t^{\alpha,x}(G)) \ge - \inf_{\nu\in G} I(\nu). \tag{4.3}$$

Proof We first prove the upper bound (4.2) for closed subsets F in $\mathcal{M}(D)$. Since this can be done in the same manner as [12](Theorem 8.1), we shall describe only the outline. Put $\ell = \inf_{\nu\in F} I(\nu)$. For any $\epsilon > 0$ and $\nu \in F$, there exists $V_\nu \in C(\bar{D})$ such that

$$\int V_\nu(x)d\nu(x) - \lambda_{V_\nu} \ge \ell - \epsilon, \tag{4.4}$$

by means of (3.3) in Theorem 3.1. On the otherhand, for each $\nu \in F$, choose an open neighbourhood B_ν of ν such that

$$\sup_{\mu\in B_\nu} | \int V_\nu d\mu - \int V_\nu d\nu| < \epsilon. \tag{4.5}$$

Since F is compact, we can select a finite sequence $\{\nu_1, \nu_2, \cdots, \nu_N\}$ from F so that $F \subset \cup_{i=1}^N B_{\nu_i}$. Then it is shown that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\sup_x \sup_\alpha Q_t^{\alpha,x}(F)) \le \max_i \limsup_{t\to\infty} \log(\sup_x \sup_\alpha Q_t^{\alpha,x}(B_{\nu_i}).$$

But it is not difficult to see that for $\nu \in F$,

$$\sup_x \sup_\alpha Q_t^{\alpha,x}(B_\nu) \le \sup_x \sup_\alpha E[e^{\int_0^t V_\nu(X_s^{\alpha,x})ds}; t < \tau_D^{\alpha,x}] \times \sup_{\mu\in B_\nu} e^{-t\int_D V_\nu d\mu}.$$

Due to (4.4),(2.3) and (2.4),

$$\sup_x \sup_\alpha Q_t^{\alpha,x}(B_\nu) \le \|T_t^{V_\nu}\| \times e^{t(\epsilon - \int V_\nu d\nu)}.$$

Thus,

$$\limsup_{t\to\infty} \frac{1}{t} \log(\sup_x \sup_{\alpha\in\mathcal{A}} Q_t^{\alpha,x}(B_\nu) \le \lambda_{V_\nu} + \epsilon - \int V_\nu d\nu \le -\ell + 2\epsilon,$$

because of (2.5) and (4.4). Since $\epsilon > 0$ is arbitrary, we get the conclusion.

Next, it is hard to prove directly the lower bound (4.3). However, as in many references ([12] e.g.), one can still show (4.3) by the following way. Firstly under some appropriate conditions, it will be seen that there exists an another rate function $\tilde{I}$, which satisfies (4.2) and (4.3). Then we can show that $\tilde{I} = I$. For the detail, see [12] (Chapter 6). ∎

Next we shall consider an analogous problem as [12](Corollary 7.26), in which $I(\nu) = 0$ if and only if ν is invariant. Here it is said that $\nu \in \mathcal{M}(D)$ is *invariant* if $\nu(V) = \nu(T_t V)$ for any $t > 0$ and $V \in \mathcal{D}$, where T_t is given by (1.5) and $\nu(V) = \int V(x)d\nu(x)$.

Theorem 4.2 *Let $\nu \in \mathcal{M}(D)$.*
(a) *Suppose that there exists $\alpha \in \mathcal{A}$ such that $\nu(V) \leq \nu(T_t^\alpha V)$ for all $V \in \mathcal{C}$ and $t > 0$, then $I(\nu) = 0$, where $T_t^\alpha V(x) = E[V(X_t^{\alpha,x})]$.*
(b) *$I(\nu) = 0$ implies that $\nu(V) \leq \nu(T_t V)$ for any $t > 0$ and $V \in \mathcal{C}$.*

Proof Since the second assertion can be shown in the same manner as [12](Theorem 7.25), we will prove only the first one. Let $X_t^{\alpha,x}$ be the response associated with α, x. Then, given any $t > 0$ and bounded V, we have the following inequality.

$$\log \int E[e^{\int_0^t V(X_s^{\alpha,x})ds}]d\nu(x) \geq \int \log E[e^{\int_0^t V(X_s^{\alpha,x})ds}]d\nu(x)$$

$$\geq \int_0^t ds \int T_s^\alpha V(x)d\nu(x) = t \int V d\nu,$$

due to Jensen's inequality and the assumption. Let $\{V_n\}$ be a sequence of bounded C^2 functions on R^d such that $V_n = V$ on D and $V_n \to -\infty$ as n tends to ∞ outside D. Then the above inequaliy yields that for any $t > 0$ and n,

$$\log \int_{R^d} \sup_{\alpha \in \mathcal{A}} E[e^{\int_0^t V_n(X_s^{\alpha,x})ds}]d\nu \geq t \int V d\nu,$$

since $V_n = V$ on D. Letting $n \to \infty$, one sees that

$$\log \int T_t^V 1(x)d\nu(x) \geq t \int V d\nu(x),$$

by means of the definition of T_t^V. Dividing the both sides by t, then letting $t \to \infty$, we can see that for all V, $\lambda_V \geq \nu(V)$. In virtue of Theorem 3.1, this means that $I(\nu) \leq 0$. But from the definition, $I(\nu) \geq 0$ for any ν, so that $I(\nu) = 0$. ∎

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