

Inverse Crack Problem and Probe Method

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ABSTRACT

A problem of extracting information about the location and shape of unknown cracks in a background medium from the Dirichlet-to-Neumann map is considered. An application of a new formulation of the probe method introduced by the author to the problem is given. The method is based on: the blowup property of sequences of special solutions of the governing equation for the background medium which are related to a singular solution of the equation; an explicit lower bound of an L^2 -norm of the gradient of the so-called *reflected solution*.

RESUMEN

Se considera un problema de extracción de información acerca de la ubicación y forma de grietas de forma desconocida sobre un medio de fondo proveniente de la aplicación de Dirichlet a Neumann. Como aplicación se muestra una formulación nueva del método del experimento introducido por el autor. El método se basa en la propiedad de explosión de las sucesiones de funciones especiales de la ecuación que modela el medio de fondo, el cual se relaciona con la solución singular de la ecuación; se encuentra una cota inferior explícita de la norma L^2 del gradiente de la llamada solución reflejada.

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1 Introduction

The aim of this paper is to develop a general method for extracting information about unknown cracks in the domain from the associated Dirichlet-to-Neumann map (or Neumann-to-Dirichlet map).

In [9] the author introduced the *probe method* which gives a general idea to obtain a reconstruction formula of unknown objects embedded in a known background medium from a mathematical counterpart (the Dirichlet-to-Neumann map) of the measured data of some physical quantity on the boundary of the medium. The method has been applied to an inverse boundary value problem in elasticity [12] and inverse obstacle scattering problems [10, 11]. The recent study of [16] also is based on applying the probe method to an inverse boundary value problem related to cracks in an inhomogeneous anisotropic elastic medium.

Recently Erhard-Potthast [6] studied numerically the probe method by using techniques in [17, 18]. It is a quite exciting work and the author gained again interest about the probe method itself. The paper [14] is one of the results of reconsidering the probe method. In the paper the blowup property of the special sequence of solutions of the governing equation for the background medium is clarified (see Lemmas 3.1 and 3.2 of this paper) and a new characterization of an unknown volumetric discontinuity by using the associated Dirichlet-to-Neumann map is given.

In this paper, we continue to reconsider the probe method and introduce an approach in applying the probe method to inverse problems related to cracks (inverse crack problems). The approach is completely different from existing applications [15, 16] of the probe method to the problems, clarifies another side of the probe method and is extremely simple. Since we are interested in methodology in inverse problems, this paper restricts itself to an inverse crack problem for the Laplace equation. However, there is no doubt that one can apply the approach presented in this paper to other inverse crack problems, for example, those in elasticity. Those applications shall be reported in subsequent papers.

2 Formulation of the problem

Now let us formulate the problem precisely. Let Ω be a bounded domain in \mathbb{R}^m ($m = 2, 3$) with Lipschitz boundary. Let Σ be a $(m - 1)$ -dimensional closed submanifold of \mathbb{R}^m of class C^0 with boundary. Σ is divided into two parts the interior and the boundary denoted by $\text{Int } \Sigma$ and $\partial\Sigma$, respectively.

We say that Σ is *extendable* of class $C^{0,1}$, if Σ admits the existence of an open subset D with Lipschitz boundary of Ω , having finitely many connected components and satisfying the following:

$$(\star) \begin{cases} \bar{D} \subset \Omega; \\ \Omega \setminus \bar{D} \text{ is connected}; \\ \Sigma \subset \partial D. \end{cases}$$

Of course, there should be infinitely many D satisfying (\star) for given extendable Σ . In this paper, we always assume that Σ is extendable of class $C^{0,1}$ unless otherwise specified. We denote by ν the unit outward normal relative to D unless otherwise specified. Set $\partial D = \Gamma$. Let $\Omega_+ = \Omega \setminus \bar{D}$ and write $D = \Omega_-$. For a function $v \in L^2(\Omega)$ set $v_+ = v|_{\Omega_+}$, $v_- = v|_{\Omega_-}$.

Define

$$X(\Omega \setminus \Sigma; D) = \{v \in L^2(\Omega) \mid v_+ \in H^1(\Omega_+), v_- \in H^1(\Omega_-), v_+|_{\Gamma \setminus \Sigma} = v_-|_{\Gamma \setminus \Sigma}\};$$

$$\|v\|_{X(\Omega \setminus \Sigma; D)} = \|v_+\|_{H^1(\Omega_+)} + \|v_-\|_{H^1(\Omega_-)}.$$

$X(\Omega \setminus \Sigma; D)$ is complete with respect to the norm $\|\cdot\|_{X(\Omega \setminus \Sigma; D)}$. Define

$$X_0(\Omega \setminus \Sigma; D) = \{v \in X(\Omega \setminus \Sigma; D) \mid v = 0 \text{ on } \partial\Omega \text{ in the sense of trace}\}.$$

Given $f \in H^{1/2}(\partial\Omega)$ we say that $u \in X(\Omega \setminus \Sigma; D)$ is a weak solution of the elliptic problem

$$\Delta u = 0 \text{ in } \Omega \setminus \Sigma,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma, \tag{2.1}$$

$$u = f \text{ on } \partial\Omega$$

if u satisfies $u = f$ on $\partial\Omega$ in the sense of trace and, for all $\varphi \in X_0(\Omega \setminus \Sigma; D)$

$$\int_{\Omega \setminus \Sigma} \nabla u \cdot \nabla \varphi \, dy = 0. \tag{2.2}$$

The starting point is to establish the existence and uniqueness of the weak solution of (2.1) and the invariance of the solution with respect to the choice of D . One can easily prove

Proposition 2.1. *For each fixed D satisfying (\star) there exists a unique weak solution of (2.1). Moreover the solution does not depend on the choice of D .*

For each fixed D satisfying (\star) , define the bounded linear functional $\Lambda_\Sigma f$ on $H^{1/2}(\partial\Omega)$ by the formula

$$\langle \Lambda_\Sigma f, h \rangle = \int_{\Omega \setminus \Sigma} \nabla u \cdot \nabla v \, dy, \quad h \in H^{1/2}(\partial\Omega) \tag{2.3}$$

where u is the weak solution of (2.1) and $v \in X(\Omega \setminus \Sigma; D)$ is an arbitrary function with $v = h$ on $\partial\Omega$ in the sense of the trace. The f in (2.1) denotes a given voltage potential on $\partial\Omega$ and $\Lambda_\Sigma f$ the corresponding current flux. The map $\Lambda_\Sigma : f \mapsto \Lambda_\Sigma f$ is called the Dirichlet-to-Neumann map. We set $\Lambda_\Sigma = \Lambda_0$ in the case when $\Sigma = \emptyset$. From Proposition 2.1 and (2.3) we know that Λ_Σ does not depend on the choice of D satisfying $(*)$. Since Λ_Σ is a mathematical model of the measured data, this can be considered as a mathematical expression of the statement: the measured data are independent of the representation of unknown objects.

We are interested in the following.

Inverse Crack Problem Extract information about the shape and location of Σ from Λ_Σ or its partial knowledge.

This is a mathematical model of electrical impedance tomography and related to a nondestructive evaluation of the material. Σ corresponds to the union of perfectly insulated cracks. The problem raised here is not a uniqueness one. For the study of the uniqueness in several formulations of inverse crack problems see [2, 5, 7] and references therein.

We cite also [4] for an approach to *Inverse Crack Problem* by using Kirsch's factorization method and refer the reader to [1, 3, 13] for inverse problems related to the cracks having special geometry.

3 Two Sides of Probe Method

The purpose of this paper is to give an answer to *Inverse Crack Problem* by using the new formulation of the probe method given in [14].

3.1 Needle, Needle Sequence

Given a point $x \in \Omega$ let N_x denote the set of all piecewise linear curves $\sigma : [0, 1] \rightarrow \bar{\Omega}$ such that : (1) $\sigma(0) \in \partial\Omega$, $\sigma(1) = x$ and $\sigma(t) \in \Omega$ for all $t \in]0, 1[$; (2) σ is injective.

We call $\sigma \in N_x$ a *needle* with tip at x .

Choose an arbitrary fundamental solution G of the Laplace equation in \mathbb{R}^m and fix it in this paper. Let $\sigma \in N_x$. We call the sequence $\xi = \{v_n\}$ of $H^1(\Omega)$ solutions of the Laplace equation a *needle sequence* for (x, σ) if it satisfies, for each fixed compact set K of \mathbb{R}^m with $K \subset \Omega \setminus \sigma([0, 1])$

$$\lim_{n \rightarrow \infty} (\|v_n(\cdot) - G(\cdot - x)\|_{L^2(K)} + \|\nabla\{v_n(\cdot) - G(\cdot - x)\}\|_{L^2(K)}) = 0.$$

The existence of the needle sequence is a consequence of the Runge approximation property for the Laplace equation.

3.2 Indicator Sequence/Function

Definition 3.1. Given $x \in \Omega$, needle σ with tip x and needle sequence $\xi = \{v_n\}$ for (x, σ) define

$$I(x, \sigma, \xi)_n = \langle (\Lambda_0 - \Lambda_\Sigma)f_n, \bar{f}_n \rangle$$

where $f_n(y) = v_n(y)$, $y \in \partial\Omega$.

$\{I(x, \sigma, \xi)_n\}_{n=1,2,\dots}$ is a sequence depending on ξ and $\sigma \in N_x$. We call the sequence the *indicator sequence*.

In short, the probe method is a method of *probing* inside Ω by using the indicator sequence. For the study of the behaviour of the indicator sequence as $n \rightarrow \infty$ we prepare

Definition 3.2. The *indicator function* I is defined by the formula

$$I(x) = \int_{\Omega \setminus \Sigma} |\nabla w_x|^2 dy, \quad x \in \Omega \setminus \Sigma$$

where $w_x \in X_0(\Omega \setminus \Sigma; D)$ is the unique weak solution of the problem:

$$\begin{aligned} \Delta w &= 0 \text{ in } \Omega \setminus \Sigma, \\ \frac{\partial w}{\partial \nu} &= -\frac{\partial}{\partial \nu}(G(\cdot - x)) \text{ on } \Sigma, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The function w_x is called the *reflected solution* by Σ .

3.3 Side A of Probe Method

The following theorem describes the behaviour of the indicator function $I(x)$ as x approaches a point in $\text{Int } \Sigma$ and gives a way of calculating the value by using the indicator sequence for a suitable needle.

Theorem A. We have:

- (A.1) given $x \in \Omega \setminus \Sigma$ and needle σ with tip at x if $\sigma([0, 1]) \cap \Sigma = \emptyset$, then for any needle sequence $\xi = \{v_n\}$ for (x, σ) the sequence $\{I(x, \sigma, \xi)_n\}$ converges to the indicator function $I(x)$;

- (A.2) for each $\epsilon > 0$

$$\sup_{\text{dist}(x, \Sigma) > \epsilon} I(x) < \infty;$$

- (A.3) given point $a \in \text{Int } \Sigma$

$$\lim_{x \rightarrow a} I(x) = \infty.$$

Theorem A is the *essence of the previous formulation* of the probe method.

The proof is based on the *convergence property* of the needle sequence outside the needle; the *divergence property* of the L^2 -norm of the gradient of $G(\cdot - x)$ over an arbitrary finite cone with vertex at x .

Using (A.1) to (A.3), we can define *another* indicator function along a given path joining two points on $\partial\Omega$ and extract the *first hitting parameter* of the path with

respect to Σ from $\Lambda_0 - \Lambda_\Sigma$. This is the original formulation of the probe method. Note that: if $\text{Int } \Sigma$ is smooth, a corresponding fact to Theorem A in terms of the original formulation of the probe method has been established in [15]. The proof therein is completely different from that of this paper.

3.4 Side B of Probe Method

The following theorem is not an improvement in proof technique and a technical weakening of hypotheses of an existing known result. It is new and not covered in [15, 16] (even in the case when $\text{Int } \Sigma$ is smooth). It gives an answer to the natural question: what happens on the indicator sequence when the tip of the needle is just located on the crack or passing through the crack?

Theorem B. *Let $x \in \Omega \setminus \partial\Sigma$ and $\sigma \in N_x$ satisfy $\emptyset \neq \sigma \setminus \{0, 1\} \cap \Sigma \subset \text{Int } \Sigma$. Then for any needle sequence $\xi = \{v_n\}$ for (x, σ) we have $\lim_{n \rightarrow \infty} I(x, \sigma, \xi)_n = \infty$.*

The proof is given in Section 4 and a consequence of Lemmas 3.1 and 3.2 described below, which have been established in [14]. For their description we make a definition. Let \mathbf{b} be a nonzero vector in \mathbb{R}^m . Given $x \in \mathbb{R}^m$, $\rho > 0$ and $\theta \in]0, \pi[$ the set $V = \{y \in \mathbb{R}^m \mid |y - x| < \rho \text{ and } (y - x) \cdot \mathbf{b} > |y - x| |\mathbf{b}| \cos(\theta/2)\}$ is called a *finite cone* of height ρ , axis direction \mathbf{b} and aperture angle θ with vertex at x .

Lemma 3.1. *Let $x \in \Omega$ be an arbitrary point and σ be a needle with tip at x . Let $\xi = \{v_n\}$ be an arbitrary needle sequence for (x, σ) . Then, for any finite cone V with vertex at x we have*

$$\lim_{n \rightarrow \infty} \int_{V \cap \Omega} |\nabla v_n(y)|^2 dy = \infty.$$

Lemma 3.2. *Let $x \in \Omega$ be an arbitrary point and σ be a needle with tip at x . Let $\xi = \{v_n\}$ be an arbitrary needle sequence for (x, σ) . Then for any point $z \in \sigma \setminus \{0, 1\}$ and open ball B centred at z we have*

$$\lim_{n \rightarrow \infty} \int_{B \cap \Omega} |\nabla v_n(y)|^2 dy = \infty.$$

These two lemmas tell us that any needle sequence for any needle blows up on the needle.

4 Proof of Theorems

4.1 The Reflected Solution

Let $v \in H^1(\Omega)$ be a weak solution of the Laplace equation in Ω and $u \in X(\Omega \setminus \Sigma; D)$ be the weak solution of (2.1) with $f = v|_{\partial\Omega}$. The function $w = u - v \in X_0(\Omega \setminus \Sigma; D)$ does not depend on the choice of D satisfying $(*)$. w is called the *reflected solution* of v by Σ .

For the proof of Theorems A and B we start with describing the following elementary lemma which can be easily proved (see also [15]).

Lemma 4.1. *Let $v \in H^1(\Omega)$ be a weak solution of the Laplace equation in Ω and $u \in X(\Omega \setminus \Sigma; D)$ be the weak solution of (2.1) for $f = v|_{\partial\Omega}$. Then $w = u - v \in X_0(\Omega \setminus \Sigma; D)$ satisfies, for all $\Psi \in X_0(\Omega \setminus \Sigma; D)$*

$$\int_{\Sigma} \frac{\partial v}{\partial \nu} (\Psi_+ - \Psi_-) dS = \int_{\Omega \setminus \Sigma} \nabla w \cdot \nabla \Psi dy. \tag{4.1}$$

Moreover the formula

$$\langle (\Lambda_0 - \Lambda_{\Sigma})f, \bar{f} \rangle = \int_{\Omega \setminus \Sigma} |\nabla w|^2 dy, \tag{4.2}$$

is valid.

4.2 A Key Lemma

We called w in Lemma 4.1 the reflected solution of v by Σ . The following lemma is the key of this paper and gives an estimate of $\|\nabla w\|_{L^2(\Omega \setminus \Sigma)}$ from below by using v only. The proof is quite elementary and everything has been done in the context of the weak solution.

Lemma 4.2. *Let $\eta \in C_0^\infty(\Omega)$ and $M > 0$ satisfy*

$$\|\eta\|_{L^\infty(\Omega_-)} + \|\nabla \eta\|_{L^\infty(\Omega_-)} \leq M \tag{4.3}$$

and

$$\text{supp}(\eta|_{\Gamma}) \subset \Sigma. \tag{4.4}$$

Set $(\Omega_-)_\eta = \Omega_- \cap \text{supp} \eta$. Let $v \in H^1(\Omega)$ be a weak solution of the Laplace equation in Ω and $w \in X_0(\Omega \setminus \Sigma; D)$ be the reflected solution of v by Σ . If

$$\int_{\Omega} |\nabla v|^2 dy - \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS \geq 0,$$

then we have two estimates:

$$\frac{(\int_{\Omega_-} |\nabla v|^2 dy - \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS)^2}{M^2 (\int_{(\Omega_-)_\eta} |\nabla v|^2 dy + \int_{(\Omega_-)_\eta} |v|^2 dy)} \leq \|\nabla w\|_{L^2((\Omega_-)_\eta)}^2; \tag{4.5}$$

$$\frac{(\int_{\Omega_-} |\nabla v|^2 dy - \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS)^2}{C_1^2 C_2^2 M^2 (\int_{(\Omega_-)_\eta} |\nabla v|^2 dy + \int_{(\Omega_-)_\eta} |v|^2 dy)} \leq \|\nabla w\|_{L^2(\Omega_+)}^2 \tag{4.6}$$

where $C_1 > 0$ and $C_2 > 0$ are independent of v (see (4.9) and (4.10) below).

Proof. Define

$$\Psi(y) = \begin{cases} 0, & \text{if } y \in \Omega_+, \\ -\eta(y)\bar{v}(y), & \text{if } y \in \Omega_-. \end{cases}$$

The trace of Ψ onto $\partial\Omega$ vanishes and we see $\Psi_+ - \Psi_- = \eta\bar{v}$ on Γ . (4.4) ensures that $\Psi \in X_0(\Omega \setminus \Sigma; D)$. From (4.3) we have

$$\|\nabla\Psi\|_{L^2(\Omega_-)}^2 = \|\nabla(\eta\bar{v})\|_{L^2(\Omega_-)}^2 \leq M^2 \left(\int_{(\Omega_-)_n} |\nabla v|^2 dy + \int_{(\Omega_-)_n} |v|^2 dy \right). \quad (4.7)$$

Integration by parts gives

$$\begin{aligned} \int_{\Omega_-} |\nabla v|^2 dy &= \int_{\Gamma} \frac{\partial v}{\partial \nu} \bar{v} dS = \int_{\Gamma} \frac{\partial v}{\partial \nu} \eta \bar{v} dS + \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS \\ &= \int_{\Sigma} \frac{\partial v}{\partial \nu} (\Psi_+ - \Psi_-) dS + \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS. \end{aligned}$$

From (4.1) we have

$$\begin{aligned} \int_{\Omega_-} |\nabla v|^2 dy &\leq \left| \int_{\Omega \setminus \Sigma} \nabla w \cdot \nabla \Psi dy \right| + \left| \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS \right| \\ &\leq \|\nabla w\|_{L^2((\Omega_-)_n)} \|\nabla \Psi\|_{L^2((\Omega_-)_n)} + \left| \int_{\Gamma} \frac{\partial v}{\partial \nu} (1 - \eta) \bar{v} dS \right|. \end{aligned} \quad (4.8)$$

A combination of (4.7) and (4.8) gives (4.5).

Next from the trace theorem ([8]) one can choose $p \in H^1(\Omega_+)$ in such a way that

$$p = \eta\bar{v} \text{ on } \Gamma,$$

$$p = 0 \text{ on } \partial\Omega$$

and satisfies

$$\|p\|_{H^1(\Omega_+)} \leq C_1 \|\eta\bar{v}\|_{H^{1/2}(\Gamma)} \quad (4.9)$$

where $C_1 = C_1(\Omega_+)$ is a positive constant and independent of η and v .

Define

$$\Psi'(y) = \begin{cases} p(y), & \text{if } y \in \Omega_+, \\ 0, & \text{if } y \in \Omega_-. \end{cases}$$

The trace of Ψ' onto $\partial\Omega$ vanishes and we see $\Psi'_+ - \Psi'_- = \eta\bar{v}$ on Γ . (4.4) ensures that $\Psi' \in X(\Omega \setminus \Sigma; D)$. Let $C_2 = C_2(\Omega_-) > 0$ satisfy, for all $\varphi \in H^1(\Omega_-)$

$$\|\varphi\|_{\partial\Omega} \|H^{1/2}(\Gamma)\| \leq C_2 \|\varphi\|_{H^1(\Omega_-)}. \quad (4.10)$$

From (4.3), (4.9) and (4.10) we have

$$\|\nabla\Psi'\|_{L^2(\Omega_+)}^2 \leq C_1^2\|\eta\bar{v}\|_{H^{1/2}(\Gamma)}^2 \leq C_1^2C_2^2M^2\left(\int_{(\Omega_-)_n}|\nabla v|^2dy + \int_{(\Omega_-)_n}|v|^2dy\right). \quad (4.11)$$

Hereafter the completely parallel argument to the previous one yields (4.6). ■

4.3 Dominance of Gradient

We prove

Lemma 4.3. *Let $x \in \Omega$ and σ be a needle with tip at x . Let $\xi = \{v_n\}$ be a needle sequence for (x, σ) . If*

$$\lim_{n \rightarrow \infty} \int_{\Omega_-} |\nabla v_n|^2 dy = \infty,$$

then there exists a natural number n_0 such that the sequence

$$\left\{ \frac{\int_{\Omega_-} |v_n|^2 dy}{\int_{\Omega_-} |\nabla v_n|^2 dy} \right\}_{n \geq n_0},$$

is bounded.

Proof. We describe only the case when $\Omega_- = D$ consists of a single domain for simplicity of description. Choose a sequence $\{K_l\}$ of compact sets of \mathbb{R}^m in such a way that $K_l \subset \Omega \setminus \sigma([0, 1])$; $\bar{K}_l \subset K_{l+1}$ for $l = 1, \dots$; $\Omega \setminus \sigma([0, 1]) = \cup_{l=1}^{\infty} K_l$. Then $|K_l \cap \Omega_-| \rightarrow |\Omega_- \setminus \sigma([0, 1])| = |\Omega_-|$ as $l \rightarrow \infty$. Thus one can take a large l_0 in such a way that the set $A \equiv K_{l_0} \cap \Omega_-$ satisfies $|A| > 0$. Then, from the Poincaré inequality (e.g., [14, 19, 20]) we have

$$\begin{aligned} \int_{\Omega_-} |v_n|^2 dy &\leq 2 \int_{\Omega_-} |v_n - (v_n)_A|^2 dy + 2 \int_{\Omega_-} |(v_n)_A|^2 dy \\ &\leq 2C(\Omega_-, A)^2 \int_{\Omega_-} |\nabla v_n|^2 dy + 2|\Omega_-| |(v_n)_A|^2 \end{aligned}$$

where $C(\Omega_-, A)$ is a positive constant independent of v_n and

$$(v_n)_A = \frac{1}{|A|} \int_A v_n dy.$$

We know that the sequence $\{(v_n)_A\}$ is always convergent since $\bar{A} \subset \Omega \setminus \sigma([0, 1])$. Now we have the desired conclusion. ■

4.4 Blowup of Indicator Sequence

We are ready to prove Theorem B. Let $x \in \Omega \setminus \partial\Sigma$ and $\sigma \in N_x$ satisfy $\emptyset \neq \sigma([0, 1]) \cap \Sigma \subset \text{Int } \Sigma$. The point is the choice of a suitable modification of the original D .

Define $t(\sigma; \Sigma) = \sup\{0 < t < 1 \mid \forall s \in]0, t[\sigma(s) \in \Omega \setminus \Sigma\}$. The number $t(\sigma; \Sigma)$ is nothing but the first hitting parameter of σ with respect to Σ . If $\sigma([0, t(\sigma; \Sigma)]) \subset \Omega_+$, then choose a modification D' satisfying (\star) of the original D in such a way that $\sigma([0, 1]) \cap \Gamma' \subset \text{Int } \Sigma$ where $\Gamma' = \partial D'$. If not so, choose another $D' \subset \Omega_+$ satisfying (\star) in such a way that $\sigma([0, t(\sigma; \Sigma)]) \subset \Omega'_+$ where $\Omega'_+ = \Omega \setminus \overline{D'}$ and apply again the former argument.

Thus we can assume, in advance, that D satisfies that $\sigma([0, 1]) \cap \Gamma \subset \text{Int } \Sigma$. Since $\sigma([0, 1]) \cap \Gamma$ is compact and $\sigma([0, 1]) \cap \Gamma \subset \text{Int } \Sigma$, there exists $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in a neighbourhood of $\sigma([0, 1]) \cap \Gamma$ and η satisfies (4.4). Let $\xi = \{v_n\}$ be an arbitrary needle sequence for (x, σ) . We see that

$$v_n \longrightarrow G(\cdot - x) \text{ in } H_{\text{loc}}^2(\Omega \setminus \sigma([0, 1])).$$

Since the set $\{y \in \Gamma \mid \eta(y) \neq 1\}$ is contained in $\Omega \setminus \sigma([0, 1])$, we conclude that the sequence

$$\left\{ \int_{\Gamma} \frac{\partial v_n}{\partial \nu} (1 - \eta) \bar{v}_n dS \right\}$$

is bounded. On the other hand, from Lemmas 3.1 and 3.2 we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_-} |\nabla v_n|^2 dy = \infty. \quad (4.12)$$

Thus we have

$$\lim_{n \rightarrow \infty} \frac{\int_{\Gamma} \frac{\partial v_n}{\partial \nu} (1 - \eta) \bar{v}_n dS}{\int_{\Omega_-} |\nabla v_n|^2 dy} = 0. \quad (4.13)$$

On the other hand, from Lemma 4.3 we know that there exists a positive constant K such that, for all n

$$\int_{\Omega_-} |\nabla v_n|^2 dy + \int_{\Omega_-} |v_n|^2 dy \leq K \int_{\Omega_-} |\nabla v_n|^2 dy. \quad (4.14)$$

Now from (4.12), (4.13), (4.14) and (4.5) we see that, as $n \rightarrow \infty$

$$M^2 K \|\nabla w_n\|_{L^2((\Omega_-)_n)}^2 \geq \left(1 - \frac{\left| \int_{\Gamma} \frac{\partial v_n}{\partial \nu} (1 - \eta) \bar{v}_n dS \right|}{\int_{\Omega_-} |\nabla v_n|^2 dy}\right)^2 \int_{\Omega_-} |\nabla v_n|^2 dy \longrightarrow \infty.$$

This completes the proof of Theorem B. ■

Remark. Using a similar argument and Lemma 4.2, one can easily conclude the validity of (A.3). The validity of (A.1) and (A.2) are almost trivial. Thus finally we got a solid understanding of the probe method.

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