CUBO A Mathematical Journal
Vol. 8, № 1, (87 - 96). April 2006.

A covering dimension for C^*-algebras

Takahiro Sudo [1]
Department of Mathematical Sciences, Faculty of Science,
University of the Ryukyus, Nishihara, Okinawa 903-0213, Japan.
sudo@math.u-ryukyu.ac.jp

ABSTRACT

We introduce a notion of dimension for C^*-algebras that is a noncommutative version of the covering dimension for spaces, and study its basic properties and some properties for continuous field C^*-algebras, tensor products and composition series. As an application we estimate the dimension of the group C^*-algebras of some important solvable Lie groups.

RESUMEN

Introducimos la noción de dimensión para C^*-álgebras, las cuales son una versión no conmutativa de la dimensión de recubrimiento por espacios. Estudiamos las propiedades básicas y algunas propiedades para campos continuos de C^*-álgebras, productos tensoriales y series de composición. Como una aplicación, estimamos la dimensión del grupo de C^*-álgebras de algunos importantes grupos de Lie solubles.

Key words and phrases: *C^*-algebra, Covering dimension, Primitive*
Math. Subj. Class.: *Primary 46L05*

Introduction In this paper we introduce a notion of dimension for C^*-algebras, and study its basic properties. Our definition of the dimension for C^*-algebras is

[1] The author is very grateful to the referee for pointing out some kind and useful suggestions and comments for improving the manuscript.

quite parallel to that of the covering dimension for topological spaces. Therefore, our investigations for the dimension given below are quite natural and elementary, but such an attempt has not been done previously in the literature. On the other hand, another covering dimension (completely positive rank) for nuclear C^*-algebras has been introduced by Winter [Wt]. However, our dimension is completely different from Winter's one. Also, the stable rank (a sort of noncommutative complex dimension), the real rank (a sort of noncommutative real dimension) and a topological rank for C^*-algebras have been introduced by Rieffel [Rf], Brown-Pedersen [BP] and the author [Sd3] respectively.

This paper consists of just one section. We first introduce a notion of dimension for C^*-algebras. Then we consider its basic properties and some properties for continuous field C^*-algebras, tensor products and composition series. We also discuss about some connections with the other ranks mentioned above. From this we see that our dimension (as numbers) is certainly the same as the completely positive rank in some cases but not always, and it is rather different from the real rank and much more from the stable rank. Finally, as an application, using [Sd1] and [Sd2] in part we estimate the dimension of the group C^*-algebras of some important solvable Lie groups such as the $ax+b$ group, the real 3-dimensional Heisenberg group, the real 5-dimensional Mautner group and the real 7-dimensional Dixmier group.

1 The covering dimension for C^*-algebras

We first recall the definition of the covering dimension for spaces.

Definition 1.1 Let X be a (non-empty, normal) topological space and $\dim X$ the covering dimension of X. By definition, for $n \geq -1$ an integer, $\dim X \leq n$ if and only if every finite open covering of X has an open refinement (U_j) such that for any $n+2$ distinct U_{j_k} $(1 \leq k \leq n+2)$ of (U_j) we have $\cap_{k=1}^{n+2} U_{j_k} = \emptyset$ (cf. [Wt]). By definition, $\dim \emptyset = -1$.

Passing the above definition directly to C^*-algebras we have

Definition 1.2 Let $\mathfrak{A}$ be a (non-zero) C^*-algebra and $\dim \mathfrak{A}$ denote the covering dimension of $\mathfrak{A}$ defined by: for $n \geq 0$ a non-negative integer, $\dim \mathfrak{A} \leq n$ if and only if every finite covering of closed ideals $(\mathfrak{B}_t)$ of $\mathfrak{A}$, that is, their union is $\mathfrak{A}$, or $\mathfrak{A}$ is generated by the union, has a refinement of closed ideals $(\mathfrak{I}_j)$ of $\mathfrak{A}$, that is, $\mathfrak{I}_j \subset \mathfrak{B}_t$ for some t and its union is $\mathfrak{A}$ such that for any $n+2$ distinct $\mathfrak{I}_{j_k}$ $(1 \leq k \leq n+2)$ of $(\mathfrak{I}_j)$ we have $\cap_{k=1}^{n+2} \mathfrak{I}_{j_k} = \{0\}$.

Remark. It is shown below that this notion as a dimension for C^*-algebras is quite natural and might be some important.

Proposition 1.3 *Let $C(X)$ be the C^*-algebra of continuous complex-valued functions on a compact Hausdorff space X. Then the following are equivalent:*

1. $\dim X \leq n$.

2. $\dim C(X) \le n$.

Furthermore, if X is a non-compact locally compact Hausdorff space, then we can replace $C(X)$ with $C_0(X)$ the C^*-algebra of continuous functions vanishing at infinity on X.

Proof. For a finite open covering (U_j) of X, we assign the finite covering of closed ideals $C_0(U_j)$ of $C(X)$ that consist of continuous functions vanishing at infinity on U_j. Note that $U_j \cap U_k = \emptyset$ if and only if $C_0(U_j) \cap C_0(U_k) = \{0\}$. ■

Remark. We have $\mathrm{cpr}(C_0(X)) = \dim X$ [Wt, Proposition 2.19], where $\mathrm{cpr}(\cdot)$ means the completely positive rank of Winter as a covering dimension for nuclear C^*-algebras. On the other hand, $\mathrm{sr}(C(X)) = [\dim X/2] + 1$ [Rf, Proposition 1.7] and $\mathrm{RR}(C(X)) = \dim X$ [BP, Proposition 1.1], where $\mathrm{sr}(\cdot)$, $\mathrm{RR}(\cdot)$ mean the stable rank of Rieffel [Rf] and the real rank of Brown-Pedersen [BP] respectively, and $[y]$ means the largest interger $\le y$. Furthermore, by definition $\mathrm{sr}(C_0(X)) = \mathrm{sr}(C_0(X)^+)$ and $\mathrm{RR}(C_0(X)) = \mathrm{RR}(C_0(X)^+)$, where $C_0(X)^+$ is the unitization of $C_0(X)$ by $\mathbb{C}$.

Continuous field C^*-algebras

Proposition 1.4 *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}^\wedge$ its spectrum that consists of equivalence classes of irreducible representations of $\mathfrak{A}$. If $\mathfrak{A}^\wedge$ is a locally compact Hausdorff space, then the following are equivalent:*

1. $\dim \mathfrak{A}^\wedge \le n$.

2. $\dim \mathfrak{A} \le n$.

Proof. Note that $\mathfrak{A}^\wedge$ is always locally compact (cf. [Dx, 3.3]). By assumption that $\mathfrak{A}^\wedge$ is Hausdorff, $\mathfrak{A}$ is isomorphic to $\Gamma_0(\mathfrak{A}^\wedge, \{\mathfrak{A}_\pi\}_{\pi\in\mathfrak{A}^\wedge})$ the C^*-algebra of a continuous field (vanishing at infinity) on $\mathfrak{A}^\wedge$ with fibers $\mathfrak{A}_\pi$ given by elementary C^*-algebras, i.e., either matrix algebras $M_n(\mathbb{C})$ or $\mathbb{K}$ the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space, which correspond to elements π of $\mathfrak{A}^\wedge$ (cf. [Dx, Theorem 10.5.4]). For a finite open covering (U_j) of $\mathfrak{A}^\wedge$, we assign the finite covering of closed ideals $\Gamma_0(U_j, \{\mathfrak{A}_\pi\}_{\pi\in U_j})$ of $\mathfrak{A}$ that consist of continuous operator fields vanishing at infinity on U_j. Note that since the fibers $\mathfrak{A}_\pi$ are simple C^*-algebras, any closed ideal of $\mathfrak{A}$ is given by $\Gamma_0(U, \{\mathfrak{A}_\pi\}_{\pi\in U})$ for an open subset U of X. ■

As a generalization of the above proposition, we have

Proposition 1.5 *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field vanishing at infinity on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$ simple C^*-algebras. Then the following are equivalent:*

1. $\dim X \le n$.

2. $\dim \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \le n$.

In particular, we have $\dim C_0(X, \mathfrak{A}) = \dim X$ for a simple C^*-algebra $\mathfrak{A}$, where $C_0(X, \mathfrak{A})$ is the C^*-algebra of continuous $\mathfrak{A}$-valued functions on X.

By definition, we always have

Proposition 1.6 *Let $\mathfrak{A}$ be a simple C^*-algebra. Then* $\dim \mathfrak{A} = 0$. *On the other hand, for C^*-algebras $\mathfrak{A}, \mathfrak{B}$, we have*

$$\dim \mathfrak{A} \oplus \mathfrak{B} = \max\{\dim \mathfrak{A}, \dim \mathfrak{B}\},$$

where $\oplus$ means the direct sum.

In particular, $\dim M_n(\mathbb{C}) = 0$ *and* $\dim \mathbb{K} = 0$, *where $M_n(\mathbb{C})$ is the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$ and $\mathbb{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Furthermore, any finite dimensional C^*-algebra has the covering dimension zero.*

Remark. For $\mathfrak{A}$ any AF-algebra, that is, an inductive limit of finite dimensional C^*-algebras, we have $\mathrm{sr}(\mathfrak{A}) = 1$ [Rf, Proposition 3.5] and $\mathrm{RR}(\mathfrak{A}) = 0$ [BP, Proposition 3.1]. Moreover, $\mathrm{cpr}(\mathfrak{A}) = 0$ if and only if $\mathfrak{A}$ is (separable) AF [Wt, Theorem 3.4].

We in fact have

Theorem 1.7 *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\dim \mathfrak{A} = \dim \mathrm{Prim}(\mathfrak{A}),$$

where $\mathrm{Prim}(\mathfrak{A})$ *is the primitive ideal space of* $\mathfrak{A}$.

Proof. Note that any closed ideal of $\mathfrak{A}$ is an intersection of primitive ideals of $\mathfrak{A}$. Therefore, any finite covering of closed ideals of $\mathfrak{A}$ corresponds to a finite covering of closed subsets of $\mathrm{Prim}(\mathfrak{A})$ (cf. [Dx]). Note also that the covering dimension for a (normal) space may be defined by its coverings of closed subsets. ∎

Remark. This theorem says that the dimension of $\mathfrak{A}$ is determined when and only when the covering dimension of its primitive ideal space is done. Thus, $\mathfrak{A}$ need not to be of continuous trace. By this interpretation we see below that our dimension for C^*-algebras has the same properties as the covering dimension for spaces does. Hence, it is viewed that C^*-algebras (with this intrinsic dimension) are much more like topological spaces in a sense.

Therefore,

Theorem 1.8 *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact Hausdorff space X. Then*

$$\dim \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \leq \dim X + \sup_{t\in X} \dim \mathfrak{A}_t.$$

In particular, for $\mathfrak{A}$ a C^*-algebra, we have

$$\dim C_0(X, \mathfrak{A}) \leq \dim X + \dim \mathfrak{A}.$$

Proof. Note that the primitive ideal space of $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ is regarded as the fiber space over the base space X with fibers the primitive ideal spaces of $\mathfrak{A}_t$. ∎

Tensor products

Proposition 1.9 *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\dim \mathfrak{A} = \dim M_n(\mathfrak{A}), \quad \dim \mathfrak{A} = \dim \mathfrak{A} \otimes \mathbb{K},$$

where $M_n(\mathfrak{A}) \cong \mathfrak{A} \otimes M_n(\mathbb{C})$ is the C^*-algebra of $n \times n$ matrices over $\mathfrak{A}$. Moreover, if $\mathfrak{B}$ is a simple C^*-algebra and if $\mathfrak{A}$ or $\mathfrak{B}$ is nuclear, then

$$\dim \mathfrak{A} = \dim \mathfrak{A} \otimes \mathfrak{B},$$

where $\otimes$ means the minimal (unique) C^*-tensor product.

Proof. Note that $M_n(\mathbb{C})$ and $\mathbb{K}$ are simple C^*-algebras of type I. Therefore, the ideal structures of $M_n(\mathfrak{A})$ and $\mathfrak{A} \otimes \mathbb{K}$ are the same as that of $\mathfrak{A}$ ([RW, Theorem B.45, p. 262]). Thus, the claims of these cases follow. Also note that by [RW, Lemma B.50, p. 264] the ideal structure of $\mathfrak{A} \otimes \mathfrak{B}$ is the same as that of $\mathfrak{A}$ (by taking the spatial representation) (cf. [Mp, 6.3]). ∎

Remark. We have $\mathrm{cpr}(M_n(C_0(X))) = \dim X$ [Wt, Proposition 2.7]. Also, $\mathrm{RR}(M_n(C(X))) = \{\dim X/(2n-1)\}$ [BE, Corollary 3.2] and $\mathrm{sr}(M_n(\mathfrak{A})) = \{(\mathrm{sr}(\mathfrak{A}) - 1)/n\} + 1$ for any C^*-algebra $\mathfrak{A}$ [Rf, Theorem 6.1], where $\{x\}$ means the least integer $\geq x$. Moreover, $\mathrm{RR}(\mathfrak{A} \otimes \mathbb{K}) \leq 1$ [BE, Proposition 3.3] and $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \min\{2, \mathrm{sr}(\mathfrak{A})\}$ [Rf, Theorems 3.6 and 6.4].

Proposition 1.10 *Let $C_0(X \times Y)$ be the C^*-algebra of continuous functions vanishing at infinity on the product space $X \times Y$ of (normal) locally compact Hausdorff spaces X, Y. Then*

$$\dim C_0(X) \otimes C_0(Y) = \dim C_0(X \times Y) \leq \dim X + \dim Y.$$

Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras with their spectrums $\mathfrak{A}^\wedge$, $\mathfrak{B}^\wedge$ (normal) Hausdorff spaces. Then

$$\dim \mathfrak{A} \otimes \mathfrak{B} = \dim \mathfrak{A}^\wedge \times \mathfrak{B}^\wedge \leq \dim \mathfrak{A}^\wedge + \dim \mathfrak{B}^\wedge.$$

Proof. Note that $C_0(X) \otimes C_0(Y) \cong C_0(X \times Y)$. The product theorem of the covering dimension for spaces (cf. [Pc]) implies the first estimate. For the second estimate, note that

$$\begin{aligned}\mathfrak{A} \otimes \mathfrak{B} &\cong \Gamma_0(\mathfrak{A}^\wedge, \{\mathfrak{A}_t\}_{t\in\mathfrak{A}^\wedge}) \otimes \Gamma_0(\mathfrak{B}^\wedge, \{\mathfrak{B}_s\}_{s\in\mathfrak{B}^\wedge}) \\ &\cong \Gamma_0(\mathfrak{A}^\wedge \times \mathfrak{B}^\wedge, \{\mathfrak{A}_t \otimes \mathfrak{B}_s\}_{(s,t)\in\mathfrak{A}^\wedge\times\mathfrak{B}^\wedge}),\end{aligned}$$

and each fiber $\mathfrak{A}_t \otimes \mathfrak{B}_s$ is a simple elementary C^*-algebra ($\mathbb{K}$ or $M_n(\mathbb{C})$ for some n) since $\mathfrak{A}$, $\mathfrak{B}$ are CCR by the assumption on their spectrums. Then use Proposition 1.5 and the product theorem of the covering dimension for spaces. ∎

Composition series

Proposition 1.11 *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{D}$ its quotient C^*-algebra. Then*

$$\dim \mathfrak{A} \geq \dim \mathfrak{D}.$$

Proof. The claim follows from that an open covering of $\mathfrak{D}$ is lifted to that of $\mathfrak{A}$, and its refinement of the covering of $\mathfrak{A}$ passes to that of the first covering of $\mathfrak{D}$. ∎

Remark. For a closed ideal $\mathfrak{I}$ of a C^*-algebra $\mathfrak{A}$, the following is false in general:

$$\dim \mathfrak{I} \leq \dim \mathfrak{A}.$$

For example, there exists a locally compact Hausdorff space X with $\dim X = 1$ but $\dim \beta X = 0$ for βX the Stone-Čech compactification of X (cf. [Pe]). Note that $C_0(X)$ is a closed ideal of $C(\beta X)$ that is isomorphic to the multiplier algebra of $C_0(X)$. Thus, we have $\dim C_0(X) = 1 > 0 = \dim C(\beta X)$.

Proposition 1.12 *Let $\mathfrak{A}$ be a C^*-algebra, $\mathfrak{I}$ its closed ideal and $\mathfrak{A}/\mathfrak{I}$ its quotient C^*-algebra. Then*

$$\dim \mathfrak{A} \leq \max\{\dim \mathfrak{I}, \dim \mathfrak{A}/\mathfrak{I}\}.$$

Proof. This follows from Theorem 1.7, a basic property of primitive ideal spaces (cf. [Dx, 3.2]) and a basic formula for the covering dimension for spaces. ∎

Remark. Let $C_0(X)^+$ be the unitization of $C_0(X)$ by $\mathbb{C}$. Then we have

$$\dim C_0(X)^+ \leq \max\{\dim C_0(X), \dim \mathbb{C}\} = \dim C_0(X).$$

It is known that there exists a locally compact Hausdorff space X with $\dim X = 1$ but $\dim X^+ = 0$, where X^+ is the one point compactification of X (see [Os]). Since $C_0(X)^+ \cong C(X^+)$, in this case we have $\dim C_0(X)^+ = 0 < \dim C_0(X) = 1$. Therefore, similarly we have $\dim \mathfrak{A}^+ \leq \dim \mathfrak{A}$ for $\mathfrak{A}$ a C^*-algebra and $\mathfrak{A}^+$ its unitization by $\mathbb{C}$. But the equality is false in general. In this point our dimension is different from the stable rank of Rieffel [Rf] and the real rank of Brown and Pedersen [BP] since by definition $\mathrm{sr}(\mathfrak{A}^+) = \mathrm{sr}(\mathfrak{A})$ and $\mathrm{RR}(\mathfrak{A}^+) = \mathrm{RR}(\mathfrak{A})$ for any C^*-algebra $\mathfrak{A}$. From [Rf, Theorems 4.3 and 4.4] and [Eh, Theorem 1.4] we always have

$$\mathrm{sr}(\mathfrak{A}) \geq \max\{\mathrm{sr}(\mathfrak{I}), \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\}, \quad \mathrm{RR}(\mathfrak{A}) \geq \max\{\mathrm{RR}(\mathfrak{I}), \mathrm{RR}(\mathfrak{A}/\mathfrak{I})\}.$$

But the reverse inequalities are false in general, and this is often an obstruction in computing the ranks of extensions of C^*-algebras in general. We see below the merit of this proposition.

Example 1.13 Let $\mathbb{B}$ be the C^*-algebra of bounded operators on a Hilbert space. Then we have the following short exact sequence:

$$0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0, \quad \text{and}$$

$$\dim \mathbb{B} = 0 = \max\{\dim \mathbb{K}, \dim \mathbb{B}/\mathbb{K}\}$$

since $\mathbb{K}$ and $\mathbb{B}/\mathbb{K}$ are simple and $\mathbb{K}$ is the unique nontrivial closed ideal of $\mathbb{B}$. Let $\mathfrak{T}$ be the Toeplitz C^*-algebra. Then

$$0 \to \mathbb{K} \to \mathfrak{T} \to C(\mathbb{T}) \to 0, \quad \text{and}$$
$$\dim C(\mathbb{T}) = 1 \le \dim \mathfrak{T} \le \max\{\dim \mathbb{K}, \dim C(\mathbb{T})\} = 1.$$

Let $E = \{f \in C([0,1], M_2(\mathbb{C})) \mid f(1) \in \mathbb{C} \oplus \mathbb{C}\,(\text{diagonal})\}$. Then

$$0 \to C_0([0,1), M_2(\mathbb{C})) \to E \to \mathbb{C} \oplus \mathbb{C} \to 0,$$
$$\dim E \le \max\{\dim C_0([0,1), M_2(\mathbb{C})), \dim \mathbb{C} \oplus \mathbb{C}\} = 1.$$

Since $C([0,1/2], M_2(\mathbb{C}))$ is a quotient of E, we have

$$1 \ge \dim E \ge \dim C([0,1/2], M_2(\mathbb{C})) = 1.$$

Note also that the primitive ideal space of E is not Hausdorff because it is identified with the union of $[0,1)$ and two points attached at 1 so that the two points are not separated.

On the other hand, it is known from [Rf, Proposition 6.5] and [BP, Proposition 1.3] that $\mathrm{sr}(\mathbb{B}) = \infty$ and $\mathrm{RR}(\mathbb{B}) = 0$. Also, $\mathrm{sr}(\mathfrak{T}) = 2$ [Rf, Examples 4.13] and $\mathrm{RR}(\mathfrak{T}) = 1$ [Os, Corollary 1.5].

Theorem 1.14 *Let $\mathfrak{A}$ be a C^*-algebra of type I and $(\mathfrak{I}_j)_j$ its composition series of closed ideals such that subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ have continuous trace. If the series is finite, then*

$$\dim \mathfrak{A} \le \max_j \dim \Gamma_0(X_j, \{\mathfrak{A}_t\}_{t \in X_j}) = \max_j \dim X_j$$
$$= \max_j \dim \mathrm{Prim}(\mathfrak{I}_j/\mathfrak{I}_{j-1}),$$

where $\Gamma_0(X_j, \{\mathfrak{A}_t\}_{t \in X_j}) \cong \mathfrak{I}_j/\mathfrak{I}_{j-1}$ where X_j are the spectrums of $\mathfrak{I}_j/\mathfrak{I}_{j-1}$, $\mathfrak{A}_t$ are elementary C^*-algebras, and $\mathrm{Prim}(\mathfrak{I}_j/\mathfrak{I}_{j-1})$ are the primitive ideal spaces of $\mathfrak{I}_j/\mathfrak{I}_{j-1}$. Furthermore,

$$\dim \mathfrak{A} \ge \sup_n \dim \cup_{k=1}^n Y_k,$$

where Y_k are the subspaces of the spectrum of $\mathfrak{A}$ consisting of equivalence classes of k-dimensional irreducible representations of $\mathfrak{A}$.

Proof. Note that a C^*-algebra is of type I if and only if it has a composition series of closed ideals such that its subquotients have continuous trace. Also, a C^*-algebra of continuous trace is isomorphic to the C^*-algebra of a continuous field on its spectrum with fibers elementary simple C^*-algebras as given in the statement. Hence, using Propositions 1.5 and 1.12 repeatedly we obtain the first estimate. For the second estimate, note that the unions $\cup_{k=1}^n Y_k$ for $n \in \mathbb{N}$ are closed subspaces of the spectrum of $\mathfrak{A}$. Thus, $\mathfrak{A}$ has quotient C^*-algebras $\mathfrak{D}_n$ with $\mathfrak{D}_n^\wedge = \cup_{k=1}^n Y_k$ (cf. [Dx]). Then use Proposition 1.11. ∎

More generally,

Theorem 1.15 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a finite composition series $(\mathfrak{I}_j)_j$ of closed ideals such that $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are isomorphic to $\Gamma_0(X_j, \{\mathfrak{A}_t\}_{t\in X_j})$ with X_j locally compact Hausdorff spaces and $\mathfrak{A}_t$ simple C^*-algebras, then*

$$\dim \mathfrak{A} \le \max_j \dim \Gamma_0(X_j, \{\mathfrak{A}_t\}_{t\in X_j}) = \max_j \dim X_j$$
$$= \max_j \dim \mathrm{Prim}(\mathfrak{I}_j/\mathfrak{I}_{j-1}).$$

Proof. Use Propositions 1.5 and 1.12 repeatedly. ∎

Theorem 1.16 *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras. If they have finite composition series $(\mathfrak{I}_j)_j$, $(\mathfrak{K}_l)_l$ of closed ideals such that subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$, $\mathfrak{K}_l/\mathfrak{K}_{l-1}$ are respectively isomorphic to $\Gamma_0(X_j, \{\mathfrak{A}_t\}_{t\in X_j})$, $\Gamma_0(Y_l, \{\mathfrak{B}_s\}_{s\in Y_l})$ with X_j, Y_l locally compact Hausdorff spaces and $\mathfrak{A}_t$, $\mathfrak{B}_s$ simple C^*-algebras, then*

$$\dim \mathfrak{A}\otimes\mathfrak{B} \le \max_{j,l} \dim \Gamma_0(X_j\times Y_l, \{\mathfrak{A}_t\otimes\mathfrak{B}_s\}_{t\in X_j, s\in Y_l})$$
$$= \max_{j,l} \dim \mathrm{Prim}(\mathfrak{I}_j/\mathfrak{I}_{j-1}\otimes \mathfrak{K}_l/\mathfrak{K}_{l-1}),$$

where $\Gamma_0(X_j\times Y_l, \{\mathfrak{A}_t\otimes\mathfrak{B}_s\}_{t\in X_j, s\in Y_l}) \cong \mathfrak{I}_j/\mathfrak{I}_{j-1}\otimes \mathfrak{K}_l/\mathfrak{K}_{l-1}$.

Proof. Note that $(\mathfrak{I}_j\otimes\mathfrak{K}_l)_{j,l}$ is a finite composition series of $\mathfrak{A}\otimes\mathfrak{B}$, and its subquotients are given by $(\mathfrak{I}_j/\mathfrak{I}_{j-1}\otimes\mathfrak{K}_l/\mathfrak{K}_{l-1})_{j,l}$. Furthermore,

$$\mathfrak{I}_j/\mathfrak{I}_{j-1}\otimes\mathfrak{K}_l/\mathfrak{K}_{l-1} \cong \Gamma_0(X_j, \{\mathfrak{A}_t\}_{t\in X_j})\otimes\Gamma_0(Y_l, \{\mathfrak{B}_s\}_{s\in Y_l})$$
$$\cong \Gamma_0(X_j\times Y_l, \{\mathfrak{A}_t\otimes\mathfrak{B}_s\}_{t\in X_j, s\in Y_l}).$$

∎

Example 1.17 Let $C^*(A)$ be the group C^*-algebra of the real $ax+b$ group $A = \mathbb{R}\rtimes_\alpha\mathbb{R}$ (a semi-direct product) with the action α defined by $\alpha_t(x) = e^t x$ for $t, x\in\mathbb{R}$. Then we have the following short exact sequence:

$$0\to\mathbb{K}\oplus\mathbb{K}\to C^*(A)\to C_0(\mathbb{R})\to 0, \quad \text{and}$$
$$\dim C_0(\mathbb{R}) = 1 \le \dim C^*(A) \le 1 = \max\{\dim\mathbb{K}\oplus\mathbb{K}, \dim C_0(\mathbb{R})\}.$$

Let $C^*(H)$ be the group C^*-algebra of the real 3-dimensional Heisenberg group $H = \mathbb{R}^2\rtimes_\alpha\mathbb{R}$ with the action α defined by $\alpha_t(x,y) = (x+ty, y)$ for $t\in\mathbb{R}$ and $(x,y)\in\mathbb{R}^2$. Then we have the following short exact sequence:

$$0\to C_0(\mathbb{R}\setminus\{0\})\otimes\mathbb{K}\to C^*(H)\to C_0(\mathbb{R}^2)\to 0, \quad \text{and}$$
$$\dim C_0(\mathbb{R}^2) = 2\le \dim C^*(H)\le 2 = \max\{\dim C_0(\mathbb{R}\setminus\{0\})\otimes\mathbb{K}, \dim C_0(\mathbb{R}^2)\}.$$

More generally, let $H_{2n+1} = \mathbb{R}^{n+1}\rtimes_\alpha\mathbb{R}^n$ be the real $(2n+1)$-dimensional Heisenberg group with the action α defined by $\alpha_t(x,y) = (x+\sum_{j=1}^n t_j y_j, y)$ for $x\in\mathbb{R}$ and $t = (t_j), y = (y_j)\in\mathbb{R}^n$. Then we have the following short exact sequence:

$$0\to C_0(\mathbb{R}\setminus\{0\})\otimes\mathbb{K}\to C^*(H_{2n+1})\to C_0(\mathbb{R}^{2n})\to 0, \quad \text{and}$$
$$\dim C_0(\mathbb{R}^{2n}) = 2n\le \dim C^*(H_{2n+1})\le 2n =$$
$$\max\{\dim C_0(\mathbb{R}\setminus\{0\})\otimes\mathbb{K}, \dim C_0(\mathbb{R}^{2n})\}.$$

Example 1.18 Let $C^*(M)$ be the group C^*-algebra of the real 5-dimensional Mautner group $M = \mathbb{C}^2 \rtimes_{\alpha^\theta} \mathbb{R}$ with the action α^θ defined by $\alpha^\theta_t(z,w) = (e^{2\pi it}z, e^{2\pi i\theta t}w)$ for $t \in \mathbb{R}$, $z, w \in \mathbb{C}$ and θ an irrational number. Then we have the following short exact sequence:

$$0 \to \mathfrak{I} \to C^*(M) \to C_0(\mathbb{R}) \to 0,$$
$$0 \to C_0(\mathbb{R}^2) \otimes \mathfrak{A}_\theta \otimes \mathbb{K} \to \mathfrak{I} \to \oplus^2 C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K} \to 0 \quad \text{and}$$
$$\dim C_0(\mathbb{R}) = 1 \leq \dim C^*(M) \leq \max\{\dim \mathfrak{I}, \dim C_0(\mathbb{R})\}$$
$$\leq \max\{\dim C_0(\mathbb{R}^2) \otimes \mathfrak{A}_\theta \otimes \mathbb{K}, \dim C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K}, \dim C_0(\mathbb{R})\} = 2$$

where $\mathfrak{A}_\theta$ is the irrational rotation algebra associated with θ and is simple. See [Sd1] for the composition series given above and $\mathrm{sr}(C^*(M))$.

More generally, in [Sd1] we have constructed finite composition series $(\mathfrak{I}_j)$ of group C^*-algebras $C^*(G)$ of semi-direct products $G = \mathbb{C}^n \rtimes_\alpha \mathbb{R}$ with general actions α such that $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are isomorphic to $C_0(X_j, \mathfrak{A}_j)$ with X_j certain locally compact Hausdorff spaces and $\mathfrak{A}_j$ given by either $\mathbb{C}$, $\mathbb{K}$ or $\mathbb{K} \otimes \mathfrak{A}_\Theta$, where $\mathfrak{A}_\Theta$ are simple or non-simple noncommutative tori. Using this structure and Theorem 1.8 and Proposition 1.12 we can estimate $\dim C^*(G)$ as follows:

$$\dim C^*(G) \leq \max_j \dim \mathfrak{I}_j/\mathfrak{I}_{j-1} = \max_j \dim C_0(X_j, \mathfrak{A}_j)$$
$$\leq \max_j (\dim X_j + \dim \mathfrak{A}_j).$$

By Proposition 1.11 we also have

$$\dim C^*(G) \geq \dim C_0(G_1^\wedge)$$

(for G a locally compact group), where $G_1^\wedge$ is the space of all 1-dimensional representations of G (cf. [Sd1]).

For more applications as above, let $D_{6n+1} = \mathbb{C}^{2n} \rtimes_\beta H_{2n+1}$ be the real $(6n+1)$-dimensional generalized Dixmier groups with the action β defined by $\beta_g(z,w) = ((e^{it_j}z_j), (e^{iy_j}w_j))$ for $z = (z_j), w = (w_j) \in \mathbb{C}^n$ and $g = (x, y, t) \in H_{2n+1}$ the real $(2n+1)$-dimensional Heisenberg group as in Example 1.17. See [Sd2] for $\mathrm{sr}(C^*(D_{6n+1}))$ and the structure of the group C^*-algebras $C^*(D_{6n+1})$ of D_{6n+1} as finite composition series of closed ideals with subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ given by $\Gamma_0(X_j, \{\mathfrak{A}_t\}_{t \in X_j})$ (in general) with X_j certain locally compact Hausdorff spaces and the fibes $\mathfrak{A}_t$ given by either $\mathbb{C}$, $\mathbb{K}$ or $\mathbb{K} \otimes \mathfrak{A}_{\Theta_t}$ (where $\mathfrak{A}_{\Theta_t}$ is a special case of noncommutative tori such as finite tensor products of rotation algebras $\mathfrak{A}_{\theta_t}$ for θ_t varying). In this case we have

$$\dim C^*(D_{6n+1}) \leq \max_j \dim \mathfrak{I}_j/\mathfrak{I}_{j-1} = \max_j \dim \Gamma_0(X_j, \{\mathfrak{A}_t\}_{t \in X_j})$$
$$\leq \max_j (\dim X_j + \sup_{t \in X_j} \dim \mathfrak{A}_t).$$

From this estimate we in fact have

$$2n \leq \dim C^*(D_{6n+1}) \leq 4n + 1.$$

Remark. As the final remark, our results Proposition 1.3, Theorem 1.8, Propositions 1.9, 1.10, 1.11 and 1.12 above just say that our dimension satisfies (not all) the axioms for the topological rank introduced by [Sd3], and only one axiom for inductive limits of C^*-algebras is missing.

Received: April 2005. Revised: May 2005.

References

[BE] E.J. BEGGS and D.E. EVANS, *The real rank of algebras of matrix valued functions*, Internat. J. Math. **2** (1991), 131-138.

[BP] L.G. BROWN and G.K. PEDERSEN, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[Dx] J. DIXMIER, *C^*-algebras*, North-Holland (1962).

[Eh] N. ELHAGE HASSAN, *Rang réel de certaines extensions*, Proc. Amer. Math. Soc. **123** (1995), 3067-3073.

[Mp] G.J. MURPHY, *C^*-algebras and operator theory*, Academic Press (1990).

[Os] H. OSAKA, *Real rank of crossed products by connected compact groups*, Bull. London Math. Soc. **27** (1995), 257-264.

[Pe] A.R. PEARS, *Dimension theory of general spaces*, Cambridge Univ. Press (1975).

[RW] I. RAEBURN and D.P. WILLIAMS, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Amer. Math. Soc. (1998).

[Rf] M.A. RIEFFEL, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333.

[Sd1] T. SUDO, *Structure of group C^*-algebras of Lie semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}$*, J. Operator Theory **46** (2001), 25–38.

[Sd2] T. SUDO, *The structure of group C^*-algebras of the generalized Dixmier groups*, Publ. Res. Inst. Math. Sci. **39** (2003), 205–225.

[Sd3] T. SUDO, *A topological rank for C^*-algebras*, Far East J. Math. Sci. **15** (2004), 71–86.

[Wt] W. WINTER, *Covering dimension for nuclear C^*-algebras*, J. Funct. Anal. **199** (2003), 535–556.