CUBO A Mathematical Journal
Vol. 8, Nº 1, (97 - 108). April 2006.

Erhling's Inequality and Pseudo-Differential Operators on $L^p(\mathbb{R}^n)$

M. W. Wong [1]
Department of Mathematics and Statistics, York University
4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
mwwong@mathstat.yorku.ca

ABSTRACT

We give a version of Erhling's inequality for L^p-Sobolev spaces $H^{s,p}$ on $\mathbb{R}^n$, $-\infty < s < \infty$, $1 \leq p < \infty$, and use it to establish an analogue of the Agmon–Douglis–Nirenberg inequality for pseudo-differential operators perturbed by singular potentials on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Applications to essential spectra of pseudo-differential operators and strongly continuous one-parameter semigroups generated by pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, are given.

RESUMEN

Entregamos una versión de la desigualdad de Erhling para espacios L^p-Sobolev $H^{s,p}$ en $\mathbb{R}^n$, $-\infty < s < \infty$, $1 \leq p < \infty$, y los usamos para establecer una desigualdad análoga a la de Agmon–Douglis–Nirenberg para operadores seudo-diferenciales perturbados por potenciales singulares sobre $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Se muestran aplicaciones al espectro esencial de operadores seudo-diferenciales y semigrupos de un parámetro fuertemente continuos generados por operadores seudo-diferenciales en $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

[1]This research has been supported by the Natural Sciences and Engineering Research Council of Canada. The author is grateful to the referee for the very useful observations and constructive comments on the contents and the presentation of this paper.

Key words and phrases: *Erhling's inequality, Sobolev spaces, pseudo-differential operators, Agmon–Douglis–Nirenberg inequality, essential spectra, dissipative operators, semigroups*

Math. Subj. Class.: *35S05, 47G05*

1 Introduction

Let $m \in \mathbb{R}$. Then we define S^m to be the set of all C^∞ functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β, there exist positive constants $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

We call any function in S^m a symbol of order m. Let $\sigma \in S^m$. Then the pseudo-differential operator T_σ is defined on the Schwartz space $\mathcal{S}$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

It is easy to prove that T_σ maps $\mathcal{S}$ into $\mathcal{S}$ continuously. It can be shown that $T_\sigma : \mathcal{S} \to \mathcal{S}$ can be extended to a continuous linear mapping from $\mathcal{S}'$ into $\mathcal{S}'$, where $\mathcal{S}'$ is the space of all tempered distributions. The well-known L^p-boundedness result states that if $\sigma \in S^m$, then $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a bounded linear operator for $-\infty < s < \infty$ and $1 < p < \infty$, where $H^{s,p}$ is the L^p-Sobolev space of order s defined by

$$H^{s,p} = \{u \in \mathcal{S}' : J_{-s}u \in L^p(\mathbb{R}^n)\},$$

and J_s is the pseudo-differential operator with symbol σ_s given by

$$\sigma_s(\xi) = (1+|\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$

It can be easily shown that $H^{s,p}$ is a Banach space in which the norm $\| \, \|_{s,p}$ is given by

$$\|u\|_{s,p} = \|J_{-s}u\|_p, \quad u \in H^{s,p},$$

where $\| \, \|_p$ is the norm in $L^p(\mathbb{R}^n)$.

Let $\sigma \in S^m$, $m > 0$. Then we say that the symbol σ is elliptic or the pseudo-differential operator T_σ is elliptic if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \ge C(1+|\xi|)^m, \quad |\xi| \ge R.$$

Using parametrices and the L^p-boundedness of pseudo-differential operators, we can prove the following analogue of the celebrated Agmon–Douglis–Nirenberg inequality for pseudo-differential operators. The origin of the inequality dates back to the study of partial differential equations in [1].

Theorem 1.1 *Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol. Then there exist positive constants C_1 and C_2 such that*

$$C_1\|\varphi\|_{m,p} \le \|T_\sigma\varphi\|_{0,p} + \|\varphi\|_{0,p} \le C_2\|\varphi\|_{m,p}, \quad \varphi \in \mathcal{S}.$$

The results hitherto described can be found in the book [18] by Wong. As an easy and interesting corollary to Theorem 1.1, we give the following result.

Corollary 1.2 *Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol and let V be a pseudo-differential operator of order s, where $s < m$. Then there exist positive constants C_1 and C_2 such that*

$$C_1\|\varphi\|_{m,p} \le \|(T_\sigma + V)\varphi\|_{0,p} + \|\varphi\|_{0,p} \le C_2\|\varphi\|_{m,p}, \quad \varphi \in \mathcal{S}.$$

Proof Let $V = T_\tau$, where $\tau \in S^s$. Then the proof is complete if we can show that $\sigma + \tau$ is an elliptic symbol in S^m. Indeed,

$$|\sigma(x,\xi) + \tau(x,\xi)| \ge |\sigma(x,\xi)| - |\tau(x,\xi)|, \quad x,\xi \in \mathbb{R}^n.$$

Since σ is elliptic, there exist positive constants C and R such that

$$|\sigma(x,\xi)| \ge C(1+|\xi|)^m, \quad |\xi| \ge R.$$

Since $\tau \in S^s$, there is a positive constant C_1 such that

$$|\tau(x,\xi)| \le C_1(1+|\xi|)^s, \quad x,\xi \in \mathbb{R}^n.$$

Thus, for $|\xi| \ge R$, we get

$$\begin{aligned} |\sigma(x,\xi) + \tau(x,\xi)| &\ge C(1+|\xi|)^m - C_1(1+|\xi|)^s \\ &= (1+|\xi|)^m(C - C_1(1+|\xi|)^{s-m}). \end{aligned}$$

Since $(1+|\xi|)^{s-m} \to 0$ as $|\xi| \to \infty$, it follows that there exists a positive constant R_1 such that

$$C_1(1+|\xi|)^{s-m} < \frac{C}{2}, \quad |\xi| \ge R_1.$$

Thus, for $|\xi| \ge \max(R, R_1)$, we get

$$|\sigma(x,\xi) + \tau(x,\xi)| \ge \frac{C}{2}(1+|\xi|)^m,$$

which is the same as saying that $\sigma + \tau$ is an elliptic symbol of order m. ∎

Remark 1.3 In view of the L^p-boundedness of pseudo-differential operators, there exists a positive constant C such that

$$\|V\varphi\|_{0,p} \le C\|\varphi\|_{s,p}, \quad \varphi \in \mathcal{S}. \tag{1.1}$$

The simple proof of Corollary 1.2 is due to the fact that V is also a pseudo-differential operator of the same kind as T_σ. It is an interesting problem to seek an analogue of Corollary 1.2 in which the operator V satisfies (1.1) and the corresponding symbol has some singularities in x.

We give a solution to the problem alluded to in Remark 1.3 for the case when the linear operator in (1.1) is identified with the multiplication by a measurable function V on $\mathbb{R}^n$ such that there exists a positive constant C for which

$$\|V\varphi\|_{0,p} \le C\|\varphi\|_{s,p}, \quad \varphi \in \mathcal{S}.$$

To see examples of such functions with singularities, let $M_{\alpha,p}$ be the set of all measurable functions V on $\mathbb{R}^n$ such that

$$M_{\alpha,p}(V) = \sup_{y\in\mathbb{R}^n} \left\{ \int_{|x|<1} |V(x-y)|^p \omega_\alpha(x)\,dx \right\}^{1/p} < \infty, \tag{1.2}$$

where $1 < p < \infty$, $\alpha > 0$ and

$$\omega_\alpha(x) = \begin{cases} |x|^{\alpha-n}, & 0 < \alpha < n, \\ 1 - \ln|x|^2, & \alpha = n, \\ 1, & \alpha > n. \end{cases}$$

Let $s > \alpha/p$. Then, as a special case of Theorem 7.1 in Chapter 6 of the book [10] by Schechter, there exists a positive constant C depending only on α, s, p and n such that for all functions V in $M_{\alpha,p}$,

$$\|V\varphi\|_{0,p} \le C\, M_{\alpha,p}(V)\, \|\varphi\|_{s,p}, \quad \varphi \in \mathcal{S}. \tag{1.3}$$

Moreover, if

$$\lim_{|y|\to\infty} \int_{|x|<1} |V(x-y)|^p \omega_\alpha(x)\,dx = 0, \tag{1.4}$$

then by Lemma 9.1 in Chapter 6 of the book [10] by Schechter, the multiplication by V is a compact operator from $H^{s,p}$ into $L^p(\mathbb{R}^n)$.

Theorem 1.4 *Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol and let V be a measurable function on $\mathbb{R}^n$ such that there exists a positive constant C for which*

$$\|V\varphi\|_{0,p} \le C\|\varphi\|_{s,p}, \quad \varphi \in \mathcal{S},$$

where $s < m$. Then there exist positive constants C_1 and C_2 such that

$$C_1\|\varphi\|_{m,p} \le \|(T_\sigma + V)\varphi\|_{0,p} + \|\varphi\|_{0,p} \le C_2\|\varphi\|_{m,p}, \quad \varphi \in \mathcal{S}.$$

To prove Theorem 1.4, we use a version of Erhling's inequality for $H^{s,p}$, $-\infty < s < \infty, 1 \le p < \infty$, in the Ph.D. dissertation [5] by Iancu. To make the paper self-contained, we state the inequality and give a more streamlined proof in Section 2. Erhling's inequality for $H^{s,2}$, $-\infty < s < \infty$, tells us that if $s < t$, then $H^{t,2} \subset H^{s,2}$ and for every positive number ε, there exists a positive constant C_ε such that

$$\|\varphi\|_{s,2} \le \varepsilon\|\varphi\|_{t,2} + C_\varepsilon\|\varphi\|_{0,2}, \quad \varphi \in \mathcal{S}.$$

The proof is very easy because the Plancherel theorem for the Fourier transform on $L^2(\mathbb{R}^n)$ gives a characterization of $H^{s,2}$ as

$$H^{s,2} = \{u \in \mathcal{S}' : (1 + |\cdot|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\}$$

and

$$\|u\|_{s,2} = \left\{\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right\}^{1/2}, \quad u \in H^{s,2}.$$

The proof of Theorem 1.4 is given in Section 3. The usefulness of Erhling's inequality is amplified by an application to essential spectra of pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, given in Section 4 and another application to strongly continuous one-parameter semigroups generated by pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, in Section 5.

2 Erhling's Inequality for $H^{s,p}, -\infty < s < \infty, 1 \le p < \infty$

Theorem 2.1 *Let $1 \le p < \infty$ and $0 < s < t$. Then for every positive number ε, there exists a positive constant C_ε such that*

$$\|\varphi\|_{s,p} \le \varepsilon\|\varphi\|_{t,p} + C_\varepsilon\|\varphi\|_{0,p}, \quad \varphi \in \mathcal{S}.$$

Proof Let s be a positive number and let $\varphi \in \mathcal{S}$. Then, as has been shown in Chapter 11 of the book [18] by Wong,

$$J_s\varphi = (2\pi)^{-n/2}(G_s * \varphi),$$

where

$$G_s(x) = \frac{1}{2^{s/2}\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-r/2} e^{-|x|^2/2r} r^{-(n-s)/2} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}(J_s\varphi)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} G_s(y)\varphi(x-y)\,dy\\ &= \frac{(2\pi)^{-n/2}}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\int_{\mathbb{R}^n}\left\{\int_0^\infty e^{-r/2}e^{-|y|^2/2r}r^{-(n-s)/2}\frac{dr}{r}\right\}\varphi(x-y)\,dy\\ &= \frac{(2\pi)^{-n/2}}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\int_0^\infty e^{-r/2}r^{s/2}\left\{\int_{\mathbb{R}^n} r^{-n/2}e^{-|y|^2/2r}\varphi(x-y)\,dy\right\}\frac{dr}{r}\\ &= \frac{(2\pi)^{-n/2}}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\int_0^\infty e^{-r/2}r^{s/2}(\psi_r*\varphi)(x)\frac{dr}{r},\quad x\in\mathbb{R}^n,\end{aligned}$$

where

$$\psi_r(x) = r^{-n/2}e^{-|x|^2/2r},\quad x\in\mathbb{R}^n.$$

Let δ be a positive number. Then we can write

$$(J_s\varphi)(x) = \frac{(2\pi)^{-n/2}}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\left\{\left(\int_0^\delta + \int_\delta^\infty\right) e^{-r/2}r^{s/2}(\psi_r*\varphi)(x)\frac{dr}{r}\right\}$$

for all x in $\mathbb{R}^n$. By Minkowski's inequality in integral form, we get

$$\|J_s\varphi\|_p \le \frac{(2\pi)^{-n/2}}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\left\{\left(\int_0^\delta + \int_\delta^\infty\right) e^{-r/2}r^{s/2}\|\psi_r*\varphi\|_p\frac{dr}{r}\right\}. \tag{2.1}$$

Now, using Young's inequality and

$$\|\psi_r\|_1 = (2\pi)^{n/2}, \tag{2.2}$$

we get

$$\begin{aligned}\int_0^\delta e^{-r/2}r^{s/2}\|\psi_r*\varphi\|_p\frac{dr}{r} &= \int_0^\delta e^{-r/2}r^{s/2}\|\psi_r\|_1\|\varphi\|_p\frac{dr}{r}\\ &= (2\pi)^{n/2}\int_0^\delta e^{-r/2}r^{s/2}\frac{dr}{r}\|\varphi\|_p\\ &\le (2\pi)^{n/2}\frac{2}{s}\delta^{s/2}\|\varphi\|_p.\end{aligned} \tag{2.3}$$

On the other hand, we get

$$\begin{aligned}&\int_\delta^\infty e^{-r/2}r^{s/2}\|\psi_r*\varphi\|_p\frac{dr}{r}\\ &= \int_\delta^\infty e^{-r/2}r^{s/2}\|\psi_r*(J_{-t}J_t\varphi)\|_p\frac{dr}{r}\\ &= \int_\delta^\infty e^{-r/2}r^{s/2}\|(J_{-t}\psi_r)*(J_t\varphi)\|_p\frac{dr}{r}\\ &\le \int_\delta^\infty e^{-r/2}r^{s/2}\|J_{-t}\psi_r\|_1\|J_t\varphi\|_p\frac{dr}{r}.\end{aligned} \tag{2.4}$$

Using the relationship between the Bessel potential and the Riesz potential given by Part (ii) in Lemma 2 on Page 133 of the book [12] by Stein, there exists a positive constant K such that

$$\|J_{-t}\psi_r\|_1 \le K(\|I_{-t}\psi_r\|_1 + \|\psi_r\|_1), \quad r > 0, \tag{2.5}$$

where I_{-t} is the Riesz potential defined by

$$(I_{-t}\varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^t \hat{\varphi}(\xi)\,d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in $\mathcal{S}$. Now, we note that for $r > 0$ and all x in $\mathbb{R}^n$,

$$\begin{aligned}(I_{-t}\psi_r)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^t \widehat{\psi_r}(\xi)\,d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^t e^{-\frac{r|\xi|^2}{2}}\,d\xi \\ &= r^{-n/2}r^{-t/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\frac{x}{\sqrt{r}}\cdot\xi}|\xi|^t e^{-\frac{|\xi|^2}{2}}\,d\xi \\ &= r^{-n/2}r^{-t/2}(I_{-t}\psi_1)\left(\frac{x}{\sqrt{r}}\right).\end{aligned}$$

Hence for $r > 0$,

$$\|I_{-t}\psi_r\|_1 = r^{-t/2}\|I_{-t}\psi_1\|_1. \tag{2.6}$$

Therefore, by (2.2), (2.4), (2.5) and (2.6), there exists a positive constant C such that

$$\begin{aligned}&\int_\delta^\infty e^{-r/2}r^{s/2}\|\psi_r * \varphi\|_p \frac{dr}{r} \\ \le\ & C\int_\delta^\infty e^{-r/2}r^{s/2}(1 + r^{-t/2})\frac{dr}{r}\|J_t\varphi\|_p \\ \le\ & C\left(2^{s/2}\Gamma\left(\frac{s}{2}\right) + \frac{2}{t-s}\delta^{(s-t)/2}\right)\|J_t\varphi\|_p.\end{aligned} \tag{2.7}$$

So, by (2.1), (2.3) and (2.7), we get

$$\|J_s\varphi\|_p \le \frac{1}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\left\{\frac{2}{s}\delta^{s/2}\|\varphi\|_p + C\left(2^{s/2}\Gamma\left(\frac{s}{2}\right) + \frac{2}{t-s}\delta^{(s-t)/2}\right)\|J_t\varphi\|_p\right\}.$$

Hence, for every positive number ε, we can choose δ such that

$$\frac{1}{2^{s/2}\Gamma\left(\frac{s}{2}\right)}\frac{s}{2}\delta^{s/2} < \varepsilon.$$

Thus, for this choice of δ, there exists a positive number C_ε such that

$$\|J_s\varphi\|_p \le \varepsilon\|\varphi\|_p + C_\varepsilon\|J_t\varphi\|_p.$$

Therefore for every positive number ε, there exists a positive number C_ε such that

$$\begin{aligned}\|\varphi\|_{s,p} &= \|J_{-s}\varphi\|_p = \|J_{t-s}J_{-t}\varphi\|_p \le \varepsilon\|J_{-t}\varphi\|_p + C_\varepsilon\|J_tJ_{-t}\varphi\|_p \\ &= \varepsilon\|\varphi\|_{t,p} + C_\varepsilon\|\varphi\|_p, \quad \varphi \in \mathcal{S}.\end{aligned}$$

∎

Using a simple density argument, we can extend Erhling's inequality from Schwartz functions to functions in L^p-Sobolev spaces.

Corollary 2.2 *For $1 \le p < \infty$ and $0 < s < t$, we have $H^{t,p} \subset H^{s,p}$. Moreover, for every positive number ε, there exists a positive number C_ε such that*

$$\|u\|_{s,p} \le \varepsilon\|u\|_{t,p} + C_\varepsilon\|u\|_{0,p}, \quad u \in H^{t,p}.$$

3 Proof of Theorem 1.4

To prove Theorem 1.4, let $\varphi \in \mathcal{S}$. Using the inequality in the hypothesis and the Agmon–Douglis–Nirenberg inequality in Theorem 1.1, we get

$$\begin{aligned}\|(T_\sigma + V)\varphi\|_{0,p} + \|\varphi\|_{0,p} &\le \|T_\sigma\varphi\|_{0,p} + \|V\varphi\|_{0,p} + \|\varphi\|_{0,p} \\ &\le \|T_\sigma\varphi\|_{0,p} + C\|\varphi\|_{s,p} + \|\varphi\|_{0,p} \\ &\le (C_2 + C)\|\varphi\|_{m,p}.\end{aligned}$$

On the other hand, for every positive number ε, we can use Erhling's inequality in Theorem 2.1 to get a positive constant C_ε such that

$$\begin{aligned}\|(T_\sigma + V)\varphi\|_{0,p} &\ge \|T_\sigma\varphi\|_{0,p} - \|V\varphi\|_{0,p} \ge \|T_\sigma\varphi\|_{0,p} - C\|\varphi\|_{s,p} \\ &\ge \|T_\sigma\varphi\|_{0,p} - \varepsilon\|\varphi\|_{m,p} - C_\varepsilon\|\varphi\|_{0,p}.\end{aligned}$$

So, using the first half of the Agmon–Douglis–Nirenberg inequality in Theorem 1.1, we get

$$\|(T_\sigma + V)\varphi\|_{0,p} \ge (C_1 - \varepsilon)\|\varphi\|_{m,p} - (C_\varepsilon + 1)\|\varphi\|_{0,p}$$

and the proof is complete if we choose $\varepsilon < C_1$.

Remark 3.1 We observe that the proof of Theorem 1.4 does not depend on the fact that V is a multiplication operator. In fact, Theorem 1.4 is valid for any linear operator V from $\mathcal{S}$ into $L^p(\mathbb{R}^n)$ satisfying (1.1).

4 An Application: Essential Spectra

Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol. Following the approach in Browder [2], Hörmander [4], Kato [6], Schechter [9, 10, 11], Vishik [13] and Wong [18], we look at the pseudo-differential operator T_σ as a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with dense domain $\mathcal{S}$. Then we denote the minimal operator of

$T_\sigma : \mathcal{S} \to L^p(\mathbb{R}^n)$ by $T_{\sigma,0}$. To recall, a function u in $L^p(\mathbb{R}^n)$ is in the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ and $T_{\sigma,0}u = f$ if and only if there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ of functions in $\mathcal{S}$ such that $\varphi_j \to u$ and $T_\sigma \varphi_j \to f$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Then, using the Agmon–Douglis–Nirenberg inequality in Theorem 1.1, we can prove that $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$. Details can be found in the works [14, 18] by Wong.

Let V be a measurable function on $\mathbb{R}^n$. Then we can look at the multiplication operator $\mathcal{D}(V) \ni u \mapsto Vu \in L^p(\mathbb{R}^n)$, where the domain $\mathcal{D}(V)$ is given by

$$\mathcal{D}(V) = \{u \in L^p(\mathbb{R}^n) : Vu \in L^p(\mathbb{R}^n)\}.$$

It is an easy matter to prove that $V : \mathcal{D}(V) \to L^p(\mathbb{R}^n)$ is a closed linear operator.

Theorem 4.1 *Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol. Let V be a measurable function on $\mathbb{R}^n$ such that the multiplication operator $V : H^{s,p} \to L^p(\mathbb{R}^n)$, $s < m$, is compact. Then $T_{\sigma,0} + V : H^{m,p} \to L^p(\mathbb{R}^n)$ is a closed linear operator such that*

$$\Sigma_e(T_{\sigma,0} + V) = \Sigma_e(T_{\sigma,0}),$$

where the notation $\Sigma_e(A)$ is used to denote the essential spectrum of a closed linear operator A from a complex Banach space X into X.

Remark 4.2 Let us recall that $\Sigma_e(A) = \mathbb{C} \setminus \Phi(A)$, where $\Phi(A)$ is the set of all complex numbers λ for which $A - \lambda I$ is Fredholm with zero index. This notion of the essential spectrum is due to Schechter [9] and explained in details in the books [10, 11] by Schechter. Examples of functions V satisfying the hypothesis of Theorem 4.1 are given by (1.2) and (1.4). Information about the essential spectrum $\Sigma_e(T_{\sigma,0})$ can be found in the papers [15, 17] by Wong.

To prove Theorem 4.1, we need an extension of Theorem 1.4 from Schwartz functions to functions in $H^{m,p}$.

Theorem 4.3 *Under the hypotheses of Theorem 1.4, there exist positive constants C_1 and C_2 such that*

$$C_1\|u\|_{m,p} \le \|(T_{\sigma,0} + V)u\|_{0,p} + \|u\|_{0,p} \le C_2\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Proof Let $u \in H^{m,p}$. Then there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ of functions in $\mathcal{S}$ such that $\varphi_j \to u$ in $H^{m,p}$ as $j \to \infty$. Using the second half of the Agmon–Douglis–Nirenberg inequality in Theorem 1.1, $T_\sigma \varphi_j \to T_{\sigma,0}u$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Now,

$$\|V\varphi_j - V\varphi_k\|_{0,p} \le C\|\varphi_j - \varphi_k\|_{s,p} \le C\|\varphi_j - \varphi_k\|_{m,p} \to 0$$

as $j, k \to \infty$. So, $V\varphi_j \to v$ for some v in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Since $V : \mathcal{D}(V) \to L^p(\mathbb{R}^n)$ is closed, we get $Vu = v$. By the Agmon–Douglis–Nirenberg inequality in Theorem 1.4, we have for $j = 1, 2, \ldots$,

$$C_1\|\varphi_j\|_{m,p} \le \|(T_\sigma + V)\varphi_j\|_{0,p} \le C_2\|\varphi_j\|_{m,p},$$

and the proof is complete if we let $j \to \infty$. ∎

Proof of Theorem 4.1 To prove that $T_{\sigma,0} + V : H^{m,p} \to L^p(\mathbb{R}^n)$ is a closed linear operator, let $\{u_j\}_{j=1}^{\infty}$ be a sequence of functions in $\mathcal{D}(T_{\sigma,0} + V) = H^{m,p}$ such that $u_j \to u$ and $(T_{\sigma,0} + V)u_j \to v$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. By the L^p-boundedness result for pseudo-differential operators, there exists a positive constant C' such that

$$\|T_{\sigma,0}u_j\|_{0,p} \le C'\|u_j\|_{m,p}, \quad j = 1, 2, \ldots.$$

Hence, by the first half of the Agmon–Douglis–Nirenberg inequality in Theorem 4.3,

$$\|T_{\sigma,0}u_j\|_{0,p} \le \frac{C'}{C_1}(\|(T_{\sigma,0} + V)u_j\|_{0,p} + \|u_j\|_{0,p}), \quad j = 1, 2, \ldots.$$

So, $T_{\sigma,0}u_j \to w$ for some w in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Since $T_{\sigma,0}$ is closed, $u \in \mathcal{D}(T_{\sigma,0}) \subset \mathcal{D}(V)$ and $T_{\sigma,0}u = w$. Thus, $Vu_j \to v - w$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Since V is closed, we get $Vu = v - w$ and consequently,

$$(T_{\sigma,0} + V)u = w + (v - w) = v.$$

Therefore $T_{\sigma,0} + V : H^{m,p} \to L^p(\mathbb{R}^n)$ is closed. Since $V : H^{s,p} \to L^p(\mathbb{R}^n)$ is compact, it follows that $V : H^{m,p} \to L^p(\mathbb{R}^n)$ is compact. Since the essential spectrum is invariant with respect to relatively compact perturbations, the proof is complete.

5 Another Application: One-Parameter Semigroups

Let us begin with an explicit semi-inner-product $(\,,\,)$ in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which is compatible with the norm $\|\;\|_p$ in $L^p(\mathbb{R}^n)$.

Theorem 5.1 *The Banach space* $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *has a semi-inner-product* $(\,,\,)$ *compatible with the norm* $\|\;\|_p$ *in* $L^p(\mathbb{R}^n)$ *given by*

$$(f, g) = \int_{\mathbb{R}^n} f(x)\overline{g^*(x)}\,dx,$$

where

$$g^*(x) = \begin{cases} g(x)|g(x)|^{p-2}/\|g\|_p^{p-2}, & g(x) \ne 0, \\ 0, & g(x) = 0. \end{cases}$$

See the paper [7] by Lumer for the notion and properties of a semi-inner-product. Dissipative operators on Banach spaces defined in terms of semi-inner-products can be found in the paper [8] by Lumer and Phillips. To see an example of a dissipative operator, let $V \in L^p_{loc}(\mathbb{R}^n)$, $1 < p < \infty$. Then, by Theorem 5.3 in Wong [16], the multiplication operator $V : \mathcal{D}(V) \to L^p(\mathbb{R}^n)$ is dissipative if and only if $\mathrm{Re}\, V(x) \le 0$ for almost all x in $\mathbb{R}^n$.

The following result is the same as Corollary 3.8 in the book [3] by Davies.

Theorem 5.2 *Let X be a complex Banach space in which the norm is denoted by $\| \ \|$. Let A be the infinitesimal generator of a one-parameter strongly continuous semigroup of contractions on a complex Banach space X. Let B be a dissipative operator such that there exist positive numbers a and C for which $a < 1$ and*

$$\|Bx\| \leq a\,\|Ax\| + C\|x\|, \quad x \in \mathcal{D}(A).$$

Then $A + B$ is the infinitesimal generator of a one-parameter strongly continuous semigroup of contractions on X.

Theorem 5.3 *Let $\sigma \in S^m$, $m > n/p$, be an elliptic symbol such that $T_{\sigma,0}$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Let V be a measurable function on $\mathbb{R}^n$ such that $\mathrm{Re}\, V(x) \leq 0$ for almost all x in $\mathbb{R}^n$ and $M_{n,p}(V) < \infty$, where $M_{n,p}(V)$ is defined by (1.2). Then $T_{\sigma,0} + V$ is the infinitesimal generator of a one-parameter strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

Proof Let $s \in (n/p, m)$. Then, by (1.3),

$$\|Vu\|_{0,p} \leq M_{n,p}(V)\,\|u\|_{s,p}, \quad u \in H^{s,p}.$$

Let $\varepsilon \in (0,1)$. Then, by Erhling's inequality in Corollary 2.2, we can get a positive constant C_ε such that

$$\|Vu\|_{0,p} \leq \varepsilon(\|T_{\sigma,0}u\|_{0,p} + \|u\|_{0,p}) + C_\varepsilon\|u\|_{0,p}, \quad u \in H^{m,p}.$$

Hence, by Theorem 5.2, the proof is complete. ∎

Received: January 2005. Revised: March 2005.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, I, Comm. Pure Appl. Math. 12 (1959), 623–727.

[2] F. E. Browder, *On the spectral theory of elliptic differential operators*, I, Math. Ann. 142 (1960/1961), 22–130.

[3] E. B. Davies, *One-Parameter Semigroups*, Academic Press, 1980.

[4] L. Hörmander, On the theory of general partial differential operators, Acta Math. 94 (1955), 161–248.

[5] G. M. IANCU, *Dynamical Systems Generated by Pseudo-Differential Operators*, Ph.D. Dissertation, York University, 1999.

[6] T. Kato, Perturbation Theory for Linear Operators, Second Edition, Springer-Verlag, 1976.

[7] G. LUMER, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29–43.

[8] G. LUMER and R. S. PHILLIPS, *Dissipative operators in a Banach space*, Pacific J. Math. 11 (1961), 679–698.

[9] M. SCHECHTER, *On the essential spectrum of an arbitrary operator*, I, J. Math. Anal. Appl. 13 (1966), 205–215.

[10] M. SCHECHTER, *Spectra of Partial Differential Operators*, Second Edition, North-Holland, 1986.

[11] M. SCHECHTER, *Principles of Functional Analysis*, Second Edition, American Mathematical Society, 2002.

[12] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[13] M. I. VISHIK, *On general boundary problems for elliptic differential equations (Russian)*, Trudy Moskov. Mat. Obshch. 1 (1952), 187–246.

[14] M. W. WONG, *On some properties of elliptic pseudo-differential operators*, Proc. Amer. Math. Soc. 99 (1987), 683–689.

[15] M. W. WONG, *Essential spectra of elliptic pseudo-differential operators*, Comm. Partial Differential Equations 13 (1988), 1209–1221.

[16] M. W. WONG, *A contraction semigroup generated by a pseudo-differential operator*, Differential Integral Equations 5 (1992), 193–200.

[17] M. W. WONG, *Spectral theory of pseudo-differential operators*, Adv. Appl. Math. 15 (1994), 437–451.

[18] M. W. WONG, *An Introduction to Pseudo-Differential Operators*, Second Edition, World Scientific, 1999.