CUBO A Mathematical Journal
Vol. 7, Nº 1, (1 - 21). April 2005.

Concentration of solutions of non-linear elliptic equations involving critical Sobolev exponent

Man Chun Leung [1]
Department of Mathematics, National University of Singapore
2 Science Drive 2, Singapore 117543. Republic of Singapore
matlmc@math.nus.edu.sg

ABSTRACT

In $\mathbb{R}^n$ $(n \ge 3)$, an interesting property of the semi-linear equation

$$\Delta u_o + c_n K_o u_o^{\frac{n+2}{n-2}} = 0$$

is that, when K_o is a positive constant, solutions can concentrate at any point. When K_o is not a constant, we show that concentration of solutions requires strong conditions on K_o. Through the stereographic projection, the discussion can be extended to S^n, and is related to bubbling, or the blow-up phenomenon.

RESUMEN

En $\mathbb{R}^n$ $(n \ge 3)$, una propiedad interesante de la ecuación semi lineal

$$\Delta u_o + c_n K_o u_o^{\frac{n+2}{n-2}} = 0$$

es que, cuando K_o es una constante positiva, las soluciones pueden concentrase en cualquier punto. Cuando K_o no es constante, mostramos que concentraciones de soluciones requiere condiciones fuertes en K_o. A través de la proyección estereográfica la discusión puede ser extendida a S^n, y relacionada con "bubbling" o el fenómeno "blow-up".

[1] The author is grateful to the anonymous referee for the sensible and helpful comments.

Key words and phrases: *Concentration, blow-up, scalar curvature equation.*
Math. Subj. Class.: *Primary 35J60; Secondary 53C21.*

1 Introduction

In this article we consider positive smooth solutions u of the equation

$$\Delta_{g_1} u - c_n\, n(n-1)u + c_n\, K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } S^n. \tag{1.1}$$

Here Δ_{g_1} is the Laplacian for the standard metric g_1 on the unit sphere S^n $(n \ge 3)$, and $c_n = (n-2)/[4(n-1)]$. Equation (1.1) describes the scalar curvature K of the conformal metric $u^{\frac{4}{n-2}} g_1$ [4]. Under the stereographic projection $\mathcal{P} : S^n \to \mathbb{R}^n$ (cf. [15] and § 7), with

$$K_o(y) := K(\mathcal{P}^{-1}(y)), \qquad u_o(y) = u\,(\mathcal{P}^{-1}(y)) \left(\frac{2}{1 + \|y\|^2} \right)^{\frac{n-2}{2}} \quad \text{for } y \in \mathbb{R}^n, \tag{1.2}$$

equation (1.1) can be expressed as

$$\Delta\, u_o + c_n\, K_o u_o^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

The geometric accent of the equations is reflected analytically in the critical Sobolev exponent. Together with conformal invariance, they may cause bubbles to appear [26]. Active studies are conducted on existence of solutions and fine asymptotic properties, employing powerful ideas in partial differential equations and global geometry (see, for instances, recent publications [1], [3], [7], [8], [9], [10], [11], [18], [23], and the references within). However, key questions like the Nirenberg problem and the Kazdan-Warner problem remain unresolved.

An exquisite result of Gidas, Ni and Nirenberg ([13], [14]; cf. [5], [24]) shows that when K_o is a positive constant, say (after rescaling), $K_o = 4\,n(n-1)$, any positive smooth solution of equation (1.3) is of the form

$$u_{\lambda,\,p}(y) := \left(\frac{\lambda}{\lambda^2 + \|y - p\|^2} \right)^{\frac{n-2}{2}} \quad \text{for } y \in \mathbb{R}^n. \tag{1.4}$$

Here p is a fixed point in $\mathbb{R}^n$, and λ a positive number. Thus the rigidity and flexibility of the equation are captured. As an interesting consequence, solutions can concentrate near p when $\lambda \to 0^+$. Indeed, direct calculation reveals that

$$\int_{\mathbb{R}^n} u_{\lambda,\,p}^{\frac{2n}{n-2}}(y)\, dy \quad \text{is independent on } \lambda \text{ and } p,$$

$$\int_{\mathbb{R}^n \setminus B_p(\rho)} u_{\lambda,\,p}^{\frac{2n}{n-2}}(y)\, dy \le C \left(\frac{\lambda}{\rho} \right)^n \longrightarrow 0 \quad \text{as} \quad \lambda \to 0^+.$$

In the above, C is a positive constant, $B_p(\rho)$ the open ball in $\mathbb{R}^n$ with center at p and radius $\rho > 0$. Observe that $u_{\lambda,\, p}(p) \to \infty$ as $\lambda \to 0^+$. We use the term concentration to denote the general phenomenon when the solution is large in a tiny neighborhood of a point, and small outside. The precise meaning is made evident in each of the following theorems.

When K_o is *not* a constant, we show that positive smooth solutions of equation (1.3) in the form of (1.2) can only concentrate on particular places. The first observation is that concentration *cannot* take place at a point p with $K_o(p) \le 0$ (see propositions 2.2 and 2.4 for the precise statements). Here we make use of the fact that conformal deformations on $(S^n,\ g_1)$ tend to increase the $L^{\frac{n}{2}}$-norm of the scalar curvatures [16].

The second obstruction for concentration is $\|\nabla K_o(p)\| \ne 0$, a consequence of the famed Kazdan-Warner *balance* formula:

$$\int_{S^n} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,. \tag{1.5}$$

Here X is an arbitrary conformal Killing vector field on $(S^n,\ g_1)$ (cf. [10] [15]).

Formula (1.5), when projected onto $\mathbb{R}^n$ via $\mathcal{P}$, and when X is generated by rescaling, gives rise to the Pohozaev identity

$$\int_{\mathbb{R}^n} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy = 0\,. \tag{1.6}$$

From (1.6), we derive the third obstruction, namely, *high concentration cannot take place at a point* p *with* $\Delta K_o(p) \ne 0$ (the precise statement is found in theorem 4.1). Earlier, Chang-Gursky-Yang [6] and Schoen-Zhang [28] consider similar situation. ((1.6) is satisfied by solutions u_o and K_o related to u and K through (1.2), which guarantees the convergence of the integral and non-existence of boundary terms. This requirement can be relaxed by imposing suitable decay condition on u_o. cf.[6].)

Observe that equation (1.3) is invariant under translations. Using this, we also discover that $\nabla(\Delta K_o)(p) \ne 0$ is an obstruction (theorem 5.1). Further exploration on the Pohozaev identity shows that

$$3\left(\sum_{i=1}^{n} \frac{\partial^4 K_o}{\partial y_i^4}(p) \right) + \Delta^2 K_o(p) \ne 0 \tag{1.7}$$

places an additional restriction on strong concentration, see theorem 6.1. Here, $\Delta^2 K_o = \Delta(\Delta K_o)$.

A natural link with the kind of concentrations discussed in this article is shown in blow-up or bubbling. Let $\{u_i\} \subset C_+^\infty(S^n)$ be a sequence of solutions of equation (1.1). A point x_b is called a blow-up point of $\{u_i\}$ if there exists a sequence $\{x_i\} \subset S^n$ such that $\lim_{i\to\infty} u_i(x_i) = \infty$ and $\lim_{i\to\infty} x_i = x_b$. Point singularity of this type is studied in detail by R. Schoen, Y.Y. Li (cf. [23]), Chen and Lin (cf. [7] [8] [25]), and others. Under suitable conditions [10] [21], u_i can be approximated near x_b by a standard solution as in (1.4).

In order to obtain *a priori* bounds and existence results, methods are developed to eliminate the possibility of blow-up (see the elegant works of Aubin [2], Chang-Gursky-Yang [6], Chen-Li [10], Y.-Y. Li [21] [22], Chen-Lin (op. cit.), Schoen [27], Schoen-Escobar [12], and Schoen-Zhang [28]). Conditions allow, uniform upper bound also implies uniform lower bound, thanks to the Harnack inequality. This becomes crucial as certain blow-up tends to pull down the solution to zero outside a small neighborhood of the blow-up point (see [8]). The conditions discussed here help to avoid this specific type of bubbling (unboundedness).

Conventions. Throughout this article, $n \geq 3$ is an integer; the functions $u_o \in C_+^\infty(\mathbb{R}^n)$ and $K_o \in C^\infty(\mathbb{R}^n)$ descend from the corresponding functions on S^n via (1.2). We observe the practice of summation over repeated indices, and use C, possibly with sub-indices, to denote various positive constants, which may be rendered differently according to the contents.

2 Zeroth order condition

Let

$$\begin{aligned} V &:= \int_{S^n} u^{\frac{2n}{n-2}}\, dV_{g_1} = \int_{\mathbb{R}^n} u_o^{\frac{2n}{n-2}}\, dy\,, \\ T &:= \int_{S^n} K\, u^{\frac{2n}{n-2}}\, dV_1 = \int_{\mathbb{R}^n} K_o\, u_o^{\frac{2n}{n-2}}\, dy\,. \end{aligned}$$

On account of (1.1), we have

$$T = c_n^{-1} \int_{S^n} \| \nabla_1\, u\|^2\, dV_{g_1} + n(n-1) \int_{S^n} |u|^2\, dV_{g_1} > 0\,. \tag{2.1}$$

Proposition 2.2. *Let K_o be as in (1.2). Assume that $K_o(p) < 0$ for a point $p \in \mathbb{R}^n$. There exist positive constants ρ_o and ε_o such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequality*

$$\int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \;\le\; \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \tag{2.3}$$

cannot hold for $\rho \le \rho_o$ and $\varepsilon \le \varepsilon_o$.

Proof. Take ρ_o to be small enough so that $\sup_{B_p(\rho_o)} K_o < 0$. Set

$$\sigma := -\left(\sup_{B_p(\rho_o)} K_o\right) > 0\,, \quad \text{and} \quad \varepsilon_o = \left(\sup_{\mathbb{R}^n} K_o\right)^{-1} \sigma\,.$$

Suppose that (2.3) holds for $\rho \le \rho_o$ and $\varepsilon \le \varepsilon_o$. We have

$$\begin{aligned} T &= \int_{B_p(\rho)} K_o\, u_o^{\frac{2n}{n-2}}\, dy + \int_{\mathbb{R}^n \setminus B_p(\rho)} K_o\, u_o^{\frac{2n}{n-2}}\, dy \\ &\le -\sigma \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o\right) \int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \\ &\le -\sigma \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o\right) \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le 0\,, \end{aligned}$$

which contradicts (2.1). ■

From above, it is not immediately clear that concentration cannot take place at a point p with $K_o(p) = 0$. This can be shown with the help of a result in [16].

Proposition 2.4. *Let K_o be as in (1.2). Assume that $K_o(p) = 0$ for a point $p \in \mathbb{R}^n$. Given any positive number C, there exist positive constants ρ_1 and ε_1 such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities*

$$\int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \quad \text{and} \quad V \le C \tag{2.5}$$

do not hold for $\rho \le \rho_1$ and $\varepsilon \le \varepsilon_1$.

Proof. By applying lemma 4.5 in [16] on S^n with the standard metric, and then transferring to $\mathbb{R}^n$ by the stereographic projection as in (1.2), we obtain

$$\int_{\mathbb{R}^n} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \ge [\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n\,. \tag{2.6}$$

Here ω_n is the volume of the standard n-sphere. Take ρ_1 to be small enough so that

$$|K_o(y)|^{\frac{n}{2}} \le \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2\,C}$$

for $y \in B_p(\rho)$. Let

$$\varepsilon_1 = \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2\,C} \left(\sup_{\mathbb{R}^n} K_o\right)^{-\frac{n}{2}}\,.$$

Here C is the positive constant in (2.5). Suppose that (2.5) holds for $\rho \le \rho_1$ and $\varepsilon \le \varepsilon_1$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \\
= & \int_{B_p(\rho)} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy + \int_{\mathbb{R}^n\setminus B_p(\rho)} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \\
\le & \frac{[n(n-1)]^{\frac{n}{2}}\,\omega_n}{2C} \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o\right)^{\frac{n}{2}} \int_{\mathbb{R}^n\setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \\
< & \frac{[n(n-1)]^{\frac{n}{2}}\,\omega_n}{2} + \varepsilon V \left(\sup_{\mathbb{R}^n} K_o\right)^{\frac{n}{2}} \\
= & [n(n-1)]^{\frac{n}{2}}\,\omega_n\,.
\end{aligned}$$

The strict inequality above provides a contradiction with (2.6). ■

Remark 2.7. *In proposition 2.4, whether the bound on V can be removed is not known. Interestingly, there are examples which show that V can become very large due to strong concentration at a point, even though K_o is very close to a positive constant (see [19] and [30]; cf. also [17]). However, under mild conditions on K_o (see [18]), it can be shown that if x_b is a blow-up point as defined in the introduction, then $K(x_b) > 0$.*

3 First order property

The stereographic projection $\mathcal{P}$ enables us to bring the discussion from $\mathbb{R}^n$ to S^n, or vice versa. For first order obstruction, it is more convenient to consider S^n. Denote by $\mathcal{B}_q(r)$ the open (metric) ball in the standard sphere S^n, where q is the center and $r \in (0,\ \pi)$ the radius of the ball.

Proposition 3.1. *Let $K \in C^\infty(S^n)$. Assume that ($K(q) > 0$ and) $\nabla_1 K(q) \ne 0$ for a point $q \in S^n$. There exist positive constants ρ_2 and ε_2 such that for any positive smooth solution u of equation (1.1), the concentration inequality*

$$\int_{S^n\setminus\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \le \varepsilon \int_{\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \tag{3.2}$$

cannot hold for $r \le \rho_2$ and $\varepsilon \le \varepsilon_2$.

Proof. Let $\|\nabla_1 K(q)\| = \delta^2 > 0$. In the Kazdan-Warner formula (1.5), we can choose the coordinate system so that the conformal Killing vector field X has the property that $\|X(q)\| = 1$ and $X(q)$ is in the direction of $\nabla_1 K(q)$. This is possible because of the innate symmetry of S^n. Furthermore, we may take $\|X\| \le 1$ in S^n.

Fix a positive constant ρ_2 such that

$$\langle X\,,\ \nabla_1 K(x)\rangle_{g_1} \ge \frac{\delta^2}{2} \qquad \text{for} \quad x \in \mathcal{B}_q(\rho_2)\,. \tag{3.3}$$

Let D be a positive constant such that

$$\|\nabla_1 K\| \le D \quad \text{in } S^n\,.$$

It follows that $X(K) = \langle X\,,\ \nabla_1 K\rangle_{g_1} \le \|X\| \cdot \|\nabla_1 K\| \le D$. Take

$$\varepsilon_2 = \frac{\delta^2}{3D}\,.$$

For $r \le \rho_2$, we have

$$\begin{aligned}
&\int_{S^n} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \\
\Longrightarrow\ &\int_{\mathcal{B}_q(r)} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = -\int_{S^n\setminus\mathcal{B}_q(r)} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} \\
\Longrightarrow\ &\frac{\delta^2}{2}\int_{\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \le D\int_{S^n\setminus\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \\
\Longrightarrow\ &\int_{S^n\setminus\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \ge \frac{\delta^2}{2D}\int_{\mathcal{B}_q(r)} u^{\frac{2n}{n-2}}\, dV_{g_1}\,.
\end{aligned}$$

The imbalance renders (3.2) invalid for $\varepsilon \le \varepsilon_2$ and $r \le \rho_2$. ■

Related to the above, we refer to [29] and [31] for first order conditions on concentration for certain singularly perturbed elliptic equations in $\mathbb{R}^n$.

4 Second order property

Because equation (1.3) is invariant under translations, for the moment, we focus the discussion on the origin.

Theorem 4.1. *Let K_o be as in (1.2). Assume that $\nabla K_o(0) = 0$ and $\Delta K_o(0) \ne 0$. Given any positive numbers C and ρ, there exist positive numbers c_1 and c_2, such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration*

$$\left\| \left(\frac{u_o}{u_{\lambda,0}}\right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o(B_o(\rho))} \le \delta\,, \qquad \int_{\mathbb{R}^n\setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le C\,\lambda^3 \tag{4.2}$$

cannot take place for $\lambda \le c_1$ and $\delta \le c_2$. Here $u_{\lambda,0}$ is the standard spherical solution defined in (1.4).

Proof. Consider the case $\Delta K_o(0) = \Lambda > 0$ first. As

$$
\begin{aligned}
(4.3) \qquad & \int_{\mathbb{R}^n} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy &=& \; 0 \qquad \text{(the Pohozaev identity)} \\
\Longrightarrow & \int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy &=& \; - \int_{\mathbb{R}^n \setminus B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy\,,
\end{aligned}
$$

we intend to show that, under concentration as expressed in (4.2), the left hand side of the above is $O(\lambda^2)$, and the other side $O(\lambda^3)$. Thus (4.3) cannot be balanced for small λ. To this end we apply Taylor's expansion and the fact that $\nabla K_o(0) = 0$, obtaining

$$
K_o(y) = K_o(0) + \frac{1}{2} \sum_{i,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) + R(y)\,.
$$

Here R is a smooth function with vanishing first and second order derivatives at the origin. It follows that

$$
y \cdot \nabla K_o(y) = \sum_{i,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) + y \cdot \nabla R(y)\,. \tag{4.4}
$$

By the remainder theorem for Taylor's expansions, we have

$$
|y \cdot \nabla R(y)| \le |y| \cdot \| \nabla R(y) \| \le C_1 \|y\|^3 \quad \text{for } \|y\| \le \rho\,. \tag{4.5}
$$

It follows from (4.4) that

$$
|y \cdot \nabla K_o(y)| \le C_2 |y|^2 \quad \text{for } \|y\| \le \rho\,. \tag{4.6}
$$

Assuming that (4.2) holds. We have

$$
\begin{aligned}
(4.7) \qquad & \int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\
= & \int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy + \int_{B_o(\rho)} (y \cdot \nabla K_o) \left[u_o^{\frac{2n}{n-2}} - u_{\lambda,0}^{\frac{2n}{n-2}} \right] dy \\
\ge & \int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy - \int_{B_o(\rho)} |y \cdot \nabla K_o| \left| u_o^{\frac{2n}{n-2}} - u_{\lambda,0}^{\frac{2n}{n-2}} \right| dy \\
\ge & \sum_{i,j} \left[\frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) \right] \int_{B_o(\rho)} y_i\, y_j\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy - C_2\, \delta \int_{B_o(\rho)} \|y\|^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \\
& - C_1 \int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \qquad \text{(using (4.2), (4.4) -- (4.6))}\,.
\end{aligned}
$$

As $u_{\lambda,0}$ depends only on $r = \|y\|$, by symmetry, one obtains

$$\begin{aligned}
\int_{B_o(\rho)} y_i\, y_j\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= 0 \quad \text{for } i \neq j\,, \\
\int_{B_o(\rho)} y_i^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \int_{B_o(\rho)} y_j^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\,, \\
\Longrightarrow \int_{B_o(\rho)} y_i^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \frac{1}{n} \int_{B_o(\rho)} r^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\,.
\end{aligned}$$

We compute

$$\begin{aligned}
(4.8) \qquad & \int_{B_o(\rho)} r^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \\
&= \omega_{n-1} \int_0^\rho \left(\frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n+1}\, dr \\
&= \omega_{n-1} \int_0^{\arctan(\rho/\lambda)} \left(\frac{\lambda}{\lambda^2 \sec^2\phi} \right)^n \lambda^{n+2} \tan^{n+1}\phi\, \sec^2\phi\, d\phi \qquad (r = \lambda \tan\phi) \\
&= \lambda^2\, \omega_{n-1} \int_0^{\arctan(\rho/\lambda)} \cos^{2(n-1)}\phi \left(\frac{\sin^{n+1}\phi}{\cos^{n+1}\phi} \right) d\phi \\
&= \lambda^2\, \omega_{n-1} \int_0^{\arctan(\rho/\lambda)} \sin^{n+1}\phi\, \cos^{n-3}\phi\, d\phi \qquad (n \geq 3) \\
&= \lambda^2\, I_{\rho/\lambda}\,.
\end{aligned}$$

Here

$$I_{\rho/\lambda} := \omega_{n-1} \int_0^{\arctan(\rho/\lambda)} \sin^{n+1}\phi\, \cos^{n-3}\phi\, d\phi\,, \tag{4.9}$$

and ω_{n-1} is the volume of the standard sphere S^{n-1}. As it can be seen in (4.9), $I_{\rho/\lambda}$ is bounded from above, and its value is larger for smaller λ, assuming that ρ is fixed. Similarly,

$$\int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = \lambda^3\, \omega_{n-1} \int_0^{\arctan(\rho/\lambda)} \sin^{n+2}\phi\, \cos^{n-4}\phi\, d\phi\,. \tag{4.10}$$

When $n \geq 4$,

$$\int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \leq C_4\, \lambda^3.$$

When $n = 3$,

$$
\begin{aligned}
& \int_0^{\arctan(\rho/\lambda)} \sin^{n+2}\phi \,\sec\phi \, d\phi \\
\le\ & \int_0^{\arctan(\rho/\lambda)} \sec\phi \, d\phi \\
=\ & \ln\, |\sec y + \tan y|_{\,y=\arctan(\rho/\lambda)} = \ln \left| \sqrt{1+\tan^2 y} + \tan y \right|_{\,y=\arctan(\rho/\lambda)} \\
=\ & \ln \left(\sqrt{1 + \frac{\rho^2}{\lambda^2}} + \frac{\rho}{\lambda} \right) \le \ln\left(\frac{3\rho}{\lambda} \right) \le \sqrt{\frac{3\rho}{\lambda}} \qquad \text{(provided } \rho/\lambda \ge 1).
\end{aligned}
$$

Thus

$$
\int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \le C_5\, \lambda^{\frac{5}{2}} \qquad \text{for } \lambda \le \rho \quad (n \ge 3),
$$

where C_5 is a positive constant that depends on ρ and n only. It follows from the symmetry and (4.7)–(4.10) that

$$
\int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy \ge \left[\frac{\Delta K_o(0)}{n} - C\,\delta \right] I_{\rho/\lambda}\, \lambda^2 - C_6\, \lambda^{\frac{5}{2}}. \tag{4.11}
$$

We choose

$$
c_2 = \frac{\Lambda}{2n\, C}
$$

so that when $\delta \le \Lambda_2$, we have

$$
\left[\frac{\Delta K_o(0)}{n} - C\,\delta \right] \ge \frac{\Lambda}{n} - \frac{\Lambda}{2n} = \frac{\Lambda}{2n}.
$$

By the decay property of K_o as in (1.2), there exists a positive constant C_7 such that $|y \cdot \nabla K_o(y)| \le C_7$ for all $y \in \mathbb{R}^n$. From (4.3) we have

$$
\begin{aligned}
\text{(4.12)} \qquad & \left(I_{\rho/\lambda}\, \frac{\Lambda}{2n} \right) \lambda^2 - C_6\, \lambda^{\frac{5}{2}} \le \int_{\mathbb{R}^n \setminus B_o(\rho)} |y \cdot \nabla K_o|\, u_o^{\frac{2n}{n-2}}\, dy \\
\Longrightarrow \quad & \left(I_{\rho/\lambda}\, \frac{\Lambda}{2n} \right) \lambda^2 \le C_6\, \lambda^{\frac{5}{2}} + C_8\, \lambda^3 \\
\Longrightarrow \quad & \frac{\Lambda\, I_{20}}{2n} \le C_9\, \lambda^{\frac{1}{2}} \qquad \left(\text{provided that } \lambda \le \min\left\{ \frac{\rho}{20},\, 1 \right\} \right) \\
\Longrightarrow \quad & \lambda > \left[\frac{\Lambda\, I_{20}}{2n\, C_9} \right]^2.
\end{aligned}
$$

Hence we may choose

$$
c_1 = \left[\frac{\Lambda\, I_{20}}{2n\, C_9} \right]^2.
$$

From (4.12), we conclude that (4.2) cannot hold for $\lambda \le c_1$ and $\delta \le c_2$. The case $\Delta K_o(0) < 0$ is similar. ■

Remark 4.13. *Fixing ρ in (4.2), we observe that when λ is small enough, (4.2) guarantees that*

$$\int_{\mathbb{R}^n \setminus B_0(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le \lambda^2 \int_{B_0(\rho)} u_o^{\frac{2n}{n-2}}\, dy .$$

(Compare with the calculation in the introduction following (1.4).) It follows that (4.2), when projected back to S^n, also implies inequality of the form (3.2).

5 Third order restriction

Theorem 5.1. *Let K_o be as in (1.2). Assume that $\|\nabla K_o(0)\| = \Delta K_o(0) = 0$ and $\nabla(\Delta K_o)(0) \ne 0$. Given any positive numbers C and ρ, there exist positive constants c_3 and c_4, such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities*

$$\left\| \left(\frac{u_o}{u_{\lambda,0}} \right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o(B_o(\rho))} \le \delta\lambda, \qquad \int_{\mathbb{R}^n \setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le C\lambda^3 \tag{5.2}$$

cannot take place for $\lambda \le c_3$ and $\delta \le c_4$.

Proof. We proceed as in the proof of theorem 4.1, and observe the effect of translations. Take a point $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ such that

$$\gamma := p \cdot \nabla(\Delta K_o)(0) > 0. \tag{5.3}$$

Consider the translation

$$K_p(y) := K_o(y-p) \quad \text{and} \quad u_p := u_o(y-p) \quad \text{for } y \in \mathbb{R}^n. \tag{5.4}$$

It follows that u_p satisfies the equation

$$\Delta u_p + c_n K_p u_p^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{5.5}$$

In addition, u_p has similar decay property as expressed in (1.2). We also have

$$\left\| \left(\frac{u_p(y)}{u_{\lambda,0}(y-p)} \right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o(B_p(\rho))} \le \delta\lambda, \qquad \int_{\mathbb{R}^n \setminus B_p(\rho)} u_p^{\frac{2n}{n-2}}\, dy \le C\lambda^3. \tag{5.6}$$

Let

$$u_{\lambda,p}(y) := u_{\lambda,0}(y-p) = \left(\frac{\lambda}{\lambda^2 + \|y-p\|^2} \right)^{\frac{n-2}{2}} \qquad \text{for } y \in \mathbb{R}^n.$$

One obtains

$$
\begin{aligned}
&\int_{B_p(\rho)} (y\cdot \nabla K_p)\, u_p^{\frac{2n}{n-2}}\, dy \\
= &\int_{B_p(\rho)} [(y-p)\cdot \nabla K_p]\, u_p^{\frac{2n}{n-2}}\, dy + \int_{B_p(\rho)} (p\cdot \nabla K_p)\, u_p^{\frac{2n}{n-2}}\, dy \\
\ge &\int_{B_p(\rho)} [(y-p)\cdot \nabla K_p]\, u_{\lambda,p}^{\frac{2n}{n-2}}\, dy - \int_{B_p(\rho)} |(y-p)\cdot \nabla K_p|\, \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,p}^{\frac{2n}{n-2}} \right| dy \\
&+ \int_{B_p(\rho)} (p\cdot \nabla K_p)\, u_{\lambda,p}^{\frac{2n}{n-2}}\, dy - \int_{B_p(\rho)} |p\cdot \nabla K_p|\, \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,p}^{\frac{2n}{n-2}} \right| dy\,.
\end{aligned}
$$

We apply the Taylor expansion

$$
\begin{aligned}
K_p(y) &= K_p(p) + \frac{1}{2}\sum_{i,j} (y_i-p_i)(y_j-p_j)\, \frac{\partial^2 K_p}{\partial y_i \partial y_j}(p) \\
&\quad + \frac{1}{3!}\sum_{i,j,k} (y_i-p_i)(y_j-p_j)(y_k-p_k)\, \frac{\partial^3 K_p}{\partial y_i \partial y_j \partial y_k}(p) + R(y)\,, \\
&= K_o(0) + \frac{1}{2}\sum_{i,j} (y_i-p_i)(y_j-p_j)\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) \\
&\quad + \frac{1}{3!}\sum_{i,j,k} (y_i-p_i)(y_j-p_j)(y_k-p_k)\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0) + R(y)\,.
\end{aligned}
$$

Here R is a smooth function with vanishing derivatives (up to at least third order) at p. As in (4.4),

$$
\begin{aligned}
(y-p)\cdot \nabla K_p(y) = &\sum_{i,j} (y_i-p_i)(y_j-p_j)\, \tfrac{\partial^2 K_o}{\partial y_i \partial y_j}(0) \\
&+\tfrac{1}{2}\sum_{i,j,k} (y_i-p_i)(y_j-p_j)(y_k-p_k)\, \tfrac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0) \\
&+(y-p)\cdot \nabla R(y)\,.
\end{aligned} \tag{5.7}
$$

By the remainder theorem for Taylor's expansions, we have

$$
\begin{aligned}
|p\cdot \nabla R(y)| &\le C_1 \|y-p\|^3, \\
|(y-p)\cdot \nabla R(y)| &\le C_2 \|y-p\|^4 \qquad \text{for } \|y-p\| < \rho\,.
\end{aligned} \tag{5.8}
$$

From (5.7), we also have

$$|(y-p)\cdot\nabla K_p(y)| \le C_3\|y-p\|^2 \quad \text{for } \|y-p\|<\rho\,. \tag{5.9}$$

Likewise,

$$\begin{aligned} p\cdot\nabla K_p(y) = \ & \tfrac{1}{2}\textstyle\sum_{i,j}\,[p_i\,(y_j-p_j)+p_j\,(y_i-p_i)]\,\frac{\partial^2 K_o}{\partial y_i\partial y_j}(0)\\ & +\tfrac{1}{3!}\textstyle\sum_{i,j,k}\,[p_i\,(y_j-p_j)\,(y_k-p_k)+(y_i-p_i)\,p_j\,(y_k-p_k)\\ & +(y_i-p_i)\,(y_j-p_j)\,p_k\,]\,\frac{\partial^3 K_o}{\partial y_i\partial y_j\partial y_k}(0)+p\cdot\nabla R(y)\,. \end{aligned} \tag{5.10}$$

By symmetry,

$$\int_{B_p(\rho)}(y_j-p_j)\,u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy=0\,,$$

$$\int_{B_p(\rho)}(y_i-p_i)\,(y_j-p_j)\,u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy=0 \quad \text{for } i\ne j\,,$$

$$\int_{B_p(\rho)}(y_i-p_i)^2\,u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy=\frac{1}{n}\int_{B_p(\rho)}r^2\,u_\lambda^{\frac{2n}{n-2}}(y)\,dy \qquad (\text{here } r^2=\|y-p\|^2)\,,$$

$$\int_{B_p(\rho)}(y_i-p_i)\,(y_j-p_j)\,(y_k-p_k)\,u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy=0 \quad \text{for } 1\le i,\ j,\ k\le n\,.$$

Similar to the proof of theorem 4.1, $\Delta K_p(p)=\Delta K_o(0)=0$ and (5.8) imply that

$$\begin{aligned}\int_{B_p(\rho)}[(y-p)\cdot\nabla K_p]\,u_{\lambda,p}^{\frac{2n}{n-2}}\,dy &= \int_{B_p(\rho)}[(y-p)\cdot\nabla R]\,u_{\lambda,p}^{\frac{2n}{n-2}}\,dy\\ &\le C\,\lambda^4\,\omega_{n-1}\int_0^{\arctan(\rho/\lambda)}\sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\,.\end{aligned}$$

In the above, if $n=3$, we have

$$\int_0^{\arctan(\rho/\lambda)}\sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\le\int_0^{\arctan(\rho/\lambda)}\sec^2\phi\,d\phi=\rho/\lambda\,.$$

Whiles $n\ge 4$, the situation is akin to (4.9). Hence

$$\int_{B_p(\rho)}[(y-p)\cdot\nabla K_p]\,u_{\lambda,p}^{\frac{2n}{n-2}}\,dy\le C\,\lambda^3\,. \tag{5.11}$$

Assuming that (5.2) holds, it follows from (5.10) that

$$\int_{B_p(\rho)} [p \cdot \nabla K_p(y)]\, u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy$$
$$\geq \frac{1}{3!} \sum_{i,j,k} \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0) \times \int_{B_p(\rho)} [p_i (y_j - p_j)(y_k - p_k) + (y_i - p_i) p_j (y_k - p_k)$$
$$+ (y_i - p_i)(y_j - p_j) p_k]\, u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy - C \int_{B_p(\rho)} \|y - p\|^3 u_{\lambda,p}^{\frac{2n}{n-2}}(y - p)\, dy$$
$$= \frac{1}{3!} \Bigg[\sum_i \frac{\partial^3 K_o}{\partial y_i^3}(0) \int_{B_p(\rho)} 3 p_i (y_i - p_i)^2 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy$$

(all three indices equal)

$$+ \sum_{i \neq j} \frac{\partial^3 K_o}{\partial y_i \partial y_j^2}(0) \int_{B_p(\rho)} 3 p_i (y_j - p_j)^2 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy \Bigg]$$

(exactly two indices equal) (5.12)

$$- C \int_{B_p(\rho)} \|y - p\|^3 u_{\lambda,p}^{\frac{2n}{n-2}}(y - p)\, dy$$
$$= \frac{1}{2} \left[\sum_i p_i \frac{\partial^3 K_o}{\partial y_i^3}(0) + \sum_{i \neq j} p_i \frac{\partial^3 K_o}{\partial y_i \partial y_j^2}(0) \right] \int_{B_p(\rho)} (y_j - p_j)^2 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy$$
$$- C \int_{B_p(\rho)} \|y - p\|^3 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy$$
$$= \frac{1}{2} [p \cdot \nabla(\Delta K_o)(0)] \int_{B_p(\rho)} (y_i - p_i)^2 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy - C \int_{B_p(\rho)} \|y - p\|^3 u_{\lambda,p}^{\frac{2n}{n-2}}(p)\, dy$$
$$\geq \frac{\gamma}{2n} I_{\rho/\lambda}\left(\tfrac{\rho}{\lambda}\right) \lambda^2 - C_4 \lambda^{\frac{5}{2}}.$$

Here $I_{\rho/\lambda}$ is defined as in (4.9). Similarly,

$$\int_{B_p(\rho)} |(y - p) \cdot \nabla K_p| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,p}^{\frac{2n}{n-2}} \right| dy \leq C_5\, \delta\, \lambda^3$$
$$\int_{B_p(\rho)} |p \cdot \nabla K_p| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,p}^{\frac{2n}{n-2}} \right| dy \leq C_6\, \delta\, \lambda^2, \qquad (5.13)$$

where we use (5.6) and the estimate $|p \cdot \nabla K_p| \leq C_7 \|y - p\|$ for $y \in B_p(\rho)$. Using (4.3), (5.10), (5.12) and (5.13) (compare also with the proof of theorem 4.1), we obtain a contradiction when δ and λ are small enough. ■

6 Fourth order

One may ask what is likely to happen when $\Delta K(0) = \| \nabla K(0) \| = 0$? (Interesting examples include *homogeneous harmonic polynomials* of higher degrees, see the next section.) The method expounded in theorem 4.1 can be used to search for algebraic relations on higher order derivatives of K, cf. [21]. Here we continue with the fourth order condition.

Theorem 6.1. *For $n \geq 5$, Let K_o be as in (1.2). Assume that* $\| \nabla K_o(0)\| = \Delta K(0) = 0$, *and*

$$\Upsilon := 3\left(\sum_{i=1}^{n} \frac{\partial^4 K_o}{\partial y_i^4}(0)\right) + \Delta^2 K_o(0) \neq 0\,. \tag{6.2}$$

Given positive constants C and ρ, there exist positive numbers c_5 and c_6 such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities

$$\left\| \left(\frac{u_o}{u_{\lambda,0}}\right)^{\frac{2n}{n-2}} - 1 \right\|_{B_o(\rho)} \leq \delta\,\lambda^2\,, \qquad \int_{I\!R^n \setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \leq C\,\lambda^5 \tag{6.3}$$

cannot hold for $\lambda \leq c_5$ and $\delta \leq c_6$.

Proof. We explore the main ideas in the proof of theorem 4. Starting with the case that $\Upsilon > 0$, consider the following Taylor expansion

$$\begin{aligned} K_o(y) &= K_o(0) + \frac{1}{2}\sum_{i,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) + \frac{1}{3!}\sum_{i,j,k} y_i\, y_j\, y_k\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0) \\ &\quad + \frac{1}{4!}\sum_{i,j,k,l} y_i\, y_j\, y_k\, y_l\, \frac{\partial^4 K_o}{\partial y_i \partial y_j \partial y_k \partial y_l}(0) + R_5(y)\,, \end{aligned}$$

which implies that

$$\begin{aligned} y \cdot \nabla K_o(y) &= \sum_{i,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) + \frac{1}{2!}\sum_{i,j,k} y_i\, y_j\, y_k\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0) \\ &\quad + \frac{1}{3!}\sum_{i,j,k,l} y_i\, y_j\, y_k\, y_l\, \frac{\partial^4 K_o}{\partial y_i \partial y_j \partial y_k \partial y_l}(0) + y \cdot \nabla R_5(y)\,. \end{aligned}$$

Here

$$|y \cdot \nabla R_5(y)| \leq C_1\, |y|^5 \quad \text{for} \quad |y| < \rho\,. \tag{6.4}$$

As in the proof of theorem 4.1, by symmetry and the fact that $\Delta K_o(0) = 0$, we have

$$\sum_{i,j}\left(\frac{\partial^2 K_o}{\partial y_i \partial y_j}(0) \int_{B_o(\rho)} y_i y_j\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\right) = 0\,,$$

$$\int_{B_o(\rho)} y_i y_j y_k\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for } 1 \leq i,\ j,\ k \leq n\,,$$

$$\int_{B_o(\rho)} y_i y_j y_k y_l\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for } i,\ j,\ k,\ l \text{ being all distinct}\,,$$

$$\int_{B_o(\rho)} y_i^2 y_j y_k\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for } j \neq k\,,$$

$$\int_{B_o(\rho)} y_i^3 y_j\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for } i \neq j\,.$$

Assuming that (6.3) holds, it follows as in (4.7) that

$$\begin{aligned}
&\int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\
&\geq \frac{1}{3!}\left[\sum_i \frac{\partial^4 K_o}{\partial y_i^4}(0)\int_{B_o(\rho)} y_i^4\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy + \sum_{i\neq j}\frac{\partial^4 K_o}{\partial y_i^2 \partial y_j^2}(0)\int_{B_o(\rho)} y_i^2\, y_j^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\right] \qquad (6.5)\\
&\quad - C_2\,\delta \int_{B_o(\rho)} \lambda^2\, |y|^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy - C_3 \int_{B_o(\rho)} |y|^5\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\,.
\end{aligned}$$

To compute the first two integrals in (6.5), let θ be the angle to the y_n-axis. That is, $y_n = |y|\,\cos\theta$. Set

$$\begin{aligned}
I_n &:= \omega_{n-2}\int_0^\pi \cos^4\theta\,\sin^{n-2}\theta\, d\theta = \frac{3\,\omega_{n-1}}{n(n+2)}\,,\\
J_n &:= \frac{\omega_{n-2}}{n-1}\int_0^\pi \cos^2\theta\,\sin^n\theta\, d\theta = \frac{\omega_{n-1}}{n(n+2)} \qquad \Longrightarrow \quad I_n = 3\,J_n\,.
\end{aligned}$$

Here we use the formulas

$$\begin{aligned}
\int_0^\pi \sin^l\theta\,\cos^m\theta\, d\theta &= \frac{m-1}{m+l}\int_0^\pi \sin^l\theta\,\cos^{m-2}\theta\, d\theta\,,\\
\omega_{n-1} &= \omega_{n-2}\int_0^\pi \sin^{n-2}\theta\, d\theta\,,
\end{aligned}$$

where $l \geq 1$, $m \geq 2$, $n \geq 3$, and ω_{n-2} is the volume of the standard sphere S^{n-2}. It follows that

$$\begin{aligned}
\int_{B_o(\rho)} y_n^4\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \int_0^\rho \left[\omega_{n-2}\int_0^\pi (r\cos\theta)^4\,(r\sin\theta)^{n-2}\, r\, d\theta\right]\left(\frac{\lambda}{\lambda^2+r^2}\right)^n dr\\
&= \omega_{n-2}\int_0^\rho \left[\int_0^\pi \cos^4\theta\,\sin^{n-2}\theta\, d\theta\right]\left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\, dr\\
&= I_n \int_0^\rho \left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\, dr\\
&= I_n\,\lambda^4 \int_0^{\arctan(\rho/\lambda)} \sin^{n+3}\phi\,\cos^{n-5}\phi\, d\phi
\end{aligned}$$

(using the substitution $r = \lambda\tan\phi$).

On the other hand,

$$\int_{B_o(\rho)} y_n^2 (y_1^2 + \cdots + y_{n-1}^2)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy$$
$$= \int_{B_o(\rho)} y_n^2 (r^2 - y_n^2)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy$$
$$= \int_0^\rho \left[\omega_{n-2} \int_0^\pi (r\cos\theta)^2 (r\sin\theta)^2 (r\sin\theta)^{n-2}\, r\, d\theta \right] \left(\frac{\lambda}{\lambda^2 + r^2} \right)^n dr$$
$$\Longrightarrow \quad (n-1) \int_{B_o(\rho)} y_i^2 y_j^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \quad (i \neq j)$$
$$= \omega_{n-2} \int_0^\rho \left(\int_0^\pi \cos^2\theta \sin^2\theta \sin^{n-2}\theta\, d\theta \right) \left(\frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n+3}\, dr$$
$$\Longrightarrow \quad \int_{B_o(\rho)} y_i^2 y_j^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = J_n \int_0^\rho \left(\frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n+3}\, dr$$
$$\Longrightarrow \quad \int_{B_o(\rho)} y_i^2 y_j^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy = J_n\, \lambda^4 \int_0^{\arctan(\rho/\lambda)} \sin^{n+3}\phi \cos^{n-5}\phi\, d\phi, \quad i \neq j.$$

Setting $\sigma = \Upsilon\, J_n$, we obtain

$$\begin{aligned} &\textstyle\int_{B_o(\rho)} (y \cdot \nabla K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\ &\geq \left(\tfrac{\sigma}{6}\right) \lambda^4 \textstyle\int_0^{\arctan(\rho/\lambda)} \sin^{n+3}\phi \cos^{n-5}\phi\, d\phi - C_2\,\delta \int_{B_o(\rho)} \lambda^2 \|y\|^2 u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \\ &\quad - C_3 \textstyle\int_{B_o(\rho)} \|y\|^5 u_{\lambda,0}^{\frac{2n}{n-2}}\, dy. \end{aligned} \tag{6.7}$$

As in (4.8) and (4.9),

$$\begin{aligned} \textstyle\int_{B_o(\rho)} \lambda^2 \|y\|^2 u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \lambda^4 \textstyle\int_0^{\arctan(\rho/\lambda)} \sin^{n+1}\phi \cos^{n-3}\phi \leq C_4\, \lambda^4, \\ \textstyle\int_{B_o(\rho)} \|y\|^5 u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \lambda^5 \textstyle\int_0^{\arctan(\rho/\lambda)} \sin^{n+4}\phi \cos^{n-6}\phi \leq C_5\, \lambda^{5-\varepsilon}. \end{aligned} \tag{6.8}$$

Here $n \geq 5$ and $\varepsilon \in (0,\, 1)$ is a positive constant. With (6.3), (6.7) and (6.8), we conclude as in the proof of theorem 4.1 that contradiction arises when λ and δ are small enough. The case $\Upsilon < 0$ is similar. ■

7 Homogeneous harmonic polynomials

Here we present some simple functions which satisfy the conditions in theorem 6.1. Let

$$Q_k(x) = \sum C_{i_1,\cdots,i_k}\, x_{i_1} \cdots x_{i_k} \tag{7.1}$$

be a *homogeneous harmonic polynomial* of degree $k \geq 2$ in $\mathbb{R}^{n+1}$. It is shown in [20] that Q_k satisfies the Kazdan-Warner type identity, namely,

$$\int_{S^n} X(Q_k)\, dV_{g_1} = 0$$

for any conformal Killing vector field X on S^n.

Assuming that the indices $i_1, \cdots, i_k$ *in* (7.1) *are all smaller than* $n+1$. Consider the stereographic projection $\mathcal{P}$ from

$$S^n := \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$$

to $\mathbb{R}^n$, with Cartesian coordinates $(y_1, \cdots, y_n)$. It is given by

$$\begin{aligned} y_i &= \tfrac{x_i}{1-x_{n+1}}, \quad 1 \le i \le n, \\ x_i &= \tfrac{2y_i}{1+\|y\|^2}, \quad 1 \le i \le n, \quad \text{and} \quad x_{n+1} = \tfrac{\|y\|^2-1}{\|y\|^2+1}. \end{aligned} \tag{7.2}$$

Using $\mathcal{P}$, we transfer Q_k into $\mathbb{R}^n$ and obtain

$$Q_k(y) = \left(\frac{2}{1+\|y\|^2}\right)^k \sum C_{i_1,\cdots,i_k}\, y_{i_1}\cdots y_{i_k}. \tag{7.3}$$

It follows that $\nabla Q_k(0) = 0$ (as $k \ge 2$). Moreover,

$$\begin{aligned} \Delta_y Q_k(y) &= \left(\frac{2}{1+\|y\|^2}\right)^k \Delta_y\left(\sum C_{i_1,\cdots,i_k}\, y_{i_1}\cdots y_{i_k}\right) \\ &\quad - \frac{2^{k+1}k}{(1+\|y\|^2)^{k+1}} \sum C_{i_1,\cdots,i_k}\,[y\cdot\nabla_y(y_{i_1}\cdots y_{i_k})] \\ &\quad + \left[\frac{2^{k+2}k(k+1)\|y\|^2}{(1+\|y\|^2)^{k+2}} - \frac{2^{k+1}k\,n}{(1+\|y\|^2)^{k+1}}\right]\sum C_{i_1,\cdots,i_k}\, y_{i_1}\cdots y_{i_k} \\ &= 0 - \frac{2^{k+1}k^2}{(1+\|y\|^2)^{k+1}}\sum C_{i_1,\cdots,i_k}\, y_{i_1}\cdots y_{i_k} \\ &\quad + \left[\frac{2^{k+2}k(k+1)\|y\|^2}{(1+\|y\|^2)^{k+2}} - \frac{2^{k+1}k\,n}{(1+\|y\|^2)^{k+1}}\right]\sum C_{i_1,\cdots,i_k}\, y_{i_1}\cdots y_{i_k} \\ &= 2^{k+1}k\left[\frac{(k+2-n)\|y\|^2-(n+k)}{(1+\|y\|^2)^{k+2}}\right] Q_k(y) \\ \Longrightarrow \quad & \Delta_y Q_k(0) = 0. \end{aligned}$$

Likewise, $\nabla_y(\Delta_y Q_k)(0) = 0$ and $\Delta_y(\Delta_y Q_k)(0) = 0$. Here we make use of the fact that Q_k is a harmonic polynomial and x_{n+1} is not present in $Q_k(x)$.

Using above, one can construct the desired functions. For instance, let Q_4 be the homogeneous harmonic polynomial defined by

$$Q_4(x) := x_1^4 + x_2^4 - 6x_1^2x_2^2 \quad \text{for } x \in S^n \subset \mathbb{R}^{n+1}.$$

Using the stereographic projection, we obtain

$$Q_4(y) = \left(\frac{2}{1+\|y\|^2}\right)^4 (y_1^4 + y_2^4 - 6y_1^2y_2^2).$$

Let $K_o := 1 + Q_4$ in $\mathbb{R}^n$. We have

$$K_o(0) = 1 > 0\,, \quad \nabla K_o(0) \; = \; 0\,, \quad \Delta K_o(0) = 0\,, \quad \nabla(\Delta K_o)(0) = 0\,,$$

but $$3\left(\sum_{i=1}^{n} \frac{\partial^4 Q_4}{\partial y_i^4}(0)\right) + \Delta_y^2 Q_4(0) \; = \; 3\left(\sum_{i=1}^{n} \frac{\partial^4 Q_4}{\partial y_i^4}(0)\right) \neq 0\,.$$

Received: July 2003. Revised: October 2003.

References

[1] A. Ambrosetti, J. Garcia Azorero and I. Peral, *Elliptic variational problems in R^N with critical growth,* J. Differential Equations **168** (2000), 10–32.

[2] T. Aubin, *Équations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire,* J. Math. Pures Appl. 55 (1976), 269–296.

[3] T. Aubin and A. Bahri, *Une hypothése topologique pour le probléme de la courbure scalaire prescrite*, J. Math. Pures Appl. **76** (1997), 843–850.

[4] A. L. Besse, *Einstein Manifolds,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol 10, Springer-Verlag, Berlin, 1987.

[5] L. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth,* Comm. Pure Appl. Math. **42** (1989), 271–297.

[6] S.Y.A. Chang, M. Gursky and P. Yang, *The scalar curvature equation on 2- and 3-spheres,* Calc. Var. Partial Differential Equations **1** (1993), 215–257.

[7] C.-C. Chen and C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes,* Comm. Pure Appl. Math. **50** (1997), 971–1019.

[8] C.-C. Chen and C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes. II,* J. Differential Geom. **49** (1998), 115–178.

[9] C.-C. Chen and C.-S. Lin, *Prescribing scalar curvature on S^N. I. A priori estimates,* J. Differential Geom. **57** (2001), 67–171.

[10] W.-X. Chen and C.-M. Li, *A priori estimates for prescribing scalar curvature equations*, Ann. of Math. **145** (1997), 547–564.

[11] K.-L. Cheung and M.-C. Leung, *Asymptotic behavior of positive solutions of the equation $\Delta u + K u^{\frac{n+2}{n-2}} = 0$ in $I\!R^n$ and positive scalar curvature*, Discrete Contin. Dynam. Systems, Added Volume (Proceedings of the International Conference on Dynamical Systems and Differential Equations, Edited by Joshua Du and Shouchuan Hu) (2001), 109–120.

[12] J. Escobar and R. Schoen, *Conformal metrics with prescribed scalar curvature*, Invent. Math. **86** (1986), 243–254.

[13] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[14] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in $I\!R^n$*, Mathematical Analysis and Applications, Part A, p. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[15] Z.-C. Han & Y.-Y. Li, *A note on the Kazdan-Warner type condition*, Ann. Inst. H. Poincarè Anal. Non Linèaire **13** (1996), 283–292.

[16] M.-C. Leung, *On the $L^{\frac{n}{2}}$-norm of scalar curvature*, Illinois J. Math. **40** (1996), 606–631.

[17] M.-C. Leung, *Exotic solutions of the conformal scalar curvature equation in R^n*, Ann. Inst. H. Poincarè Anal. Non Linèaire **18** (2001), 297–307.

[18] M.-C. Leung, *Conformal scalar curvature equations in open spaces*, Cubo Matemática Educacional, **3** (2001), 415–443. Also available via the Differential Geometry Server at math.DG/0110105, or the URL http://www.math.nus.edu.sg/~matlmc/survey.pdf.

[19] M.-C. Leung, *Blow-up solutions of nonlinear elliptic equations in R^n with critical exponent*, Math. Ann. **327** (2003), 723–744.

[20] M.-C. Leung, *Local properties of the Kazdan-Warner problem on prescribing scalar curvature on S^n*, preprint.

[21] Y.-Y. Li, *Prescribing scalar curvature on S^n and related problems. I*, J. Differential Equations **120** (1995), 319–410.

[22] Y.-Y. Li, *Prescribing scalar curvature on S^n and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. **49** (1996), 541–597.

[23] Y.-Y. Li, *Fine analysis of blow up and applications*, First International Congress of Chinese Mathematicians (Beijing, 1998), 411–421, AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., Providence, RI, 2001.

[24] Y.-Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, to appear in Journal of the European Mathematical Society.

[25] C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes. III,* Comm. Pure Appl. Math. **53** (2000), 611–646.

[26] T. Parker, *What is a bubble tree?* Notices A.M.S. **50** (2003), 666–667.

[27] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature,* J. Differential Geom. **20** (1984), 479–495.

[28] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere,* Calc. Var. Partial Differential Equations **4** (1996), 1–25.

[29] S. Secchi and M. Squassina, *On the location of concentration points for singularly perturbed elliptic equations,* preprint.

[30] S. Taliaferro and L. Zhang, *Arbitrarily large solutions of the conformal scalar curvature problem at an isolated singularity*, Proc. Amer. Math. Soc. **131** (2003), 2895–2902.

[31] X. Wang and B. Zeng, *On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions,* SIAM J. Math. Anal. **28** (1997), 633–655.