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Triple solutions of constant sign for a system of fredholm integral equations

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ABSTRACT

We consider the following system of Fredholm integral equations

$$u_i(t) = \int_0^1 g_i(t,s)P_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \quad t \in [0,1], \ 1 \le i \le n.$$

Criteria for the existence of three constant-sign solutions of the system will be presented. The generality of the results obtained is illustrated through applications to several well known boundary value problems. We also consider a similar problem on the half-line $[0,\infty)$

$$u_i(t) = \int_0^\infty g_i(t,s)P_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \quad t \in [0,\infty), \ 1 \le i \le n.$$

RESUMEN

Consideramos el siguiente sistema de ecuaciones integrales de Fredholm.

$$u_i(t) = \int_0^1 g_i(t,s)P_i(s,u_1(s),u_2(s),\cdots,u_n(s))ds, \quad t \in [0,1], \ 1 \le i \le n.$$

Se presentarán criterios para la existencia de tres soluciones de signo constante del sistema. La generalidad de los resultados obtenidos es ilustrada a través de la aplicación de varios problemas de límite bien conocidos. Además, consideramos un problema similar en la recta real positiva

$$u_i(t) = \int_0^\infty g_i(t,s)P_i(s,u_1(s),u_2(s),\cdots,u_n(s))ds, \quad t \in [0,\infty), \ 1 \le i \le n.$$

Key words and phrases: *Constant-sign solutions, system of Fredholm integral equations, boundary value problems.*

Math. Subj. Class.: *45B05, 45G15, 45M20*

1 Introduction

In this paper we shall consider two systems of Fredholm integral equations, one is on a finite interval

$$u_i(t) = \int_0^1 g_i(t,s)P_i(s,u_1(s),u_2(s),\cdots,u_n(s))ds, \quad t \in [0,1], \ 1 \le i \le n \tag{1.1}$$

while the other is on the half-line $[0,\infty)$

$$u_i(t) = \int_0^\infty g_i(t,s)P_i(s,u_1(s),u_2(s),\cdots,u_n(s))ds, \quad t \in [0,\infty), \ 1 \le i \le n. \tag{1.2}$$

A solution $u = (u_1, u_2, \cdots, u_n)$ of (1.1) will be sought in $(C[0,1])^n = C[0,1] \times \cdots \times C[0,1]$ (n times), whereas a solution $u = (u_1, u_2, \cdots, u_n)$ of (1.2) will be sought in a subset of $(BC[0,\infty))^n$ where $\lim_{t\to\infty} u_i(t)$ exists for each $1 \le i \le n$. Here $BC[0,\infty)$ denotes the space of fucntions that are bounded and continuous on $[0,\infty)$. In both cases, we say that u is a solution of *constant sign* if for each $1 \le i \le n$, we have $\theta_i u_i \ge 0$ on $[0,1]$ for (1.1), or on $[0,\infty)$ for (1.2), where $\theta_i \in \{1,-1\}$ is fixed.

For each of (1.1) and (1.2), we shall establish criteria so that the system has at least *three* constant-sign solutions.

Recently, Agarwal et al [1, 2] have investigated the existence of positive solutions of the nonlinear Fredholm integral equation

$$y(t) = \int_0^1 g(t,s)f(y(s))ds + h(t), \quad t \in [0,1]. \tag{1.3}$$

Particular cases of this equation are also considered in [15, 16, 20]. We remark that a generalization of (1.3) to a system with existence criteria for single and multiple constant-sign solutions has recently been presented in [4, 5]. The main tool employed has been Krasnosel'skii's fixed point theorem. In the present work, besides extending (1.3) to a system, we will be using *other* fixed point theorems, namely, that of Leggett and Williams [19] as well as Avery [9], to derive criteria for the existence of *triple* constant-sign solutions. Note that the term $h(t)$ in (1.3) has been excluded as we wish to apply the results to homogeneous boundary value problems (in which case $h(t) \equiv 0$), which have received almost all the attention in the recent literature (see the monographs [3, 6] and the references cited therein). However, it is not difficult to develop parallel results with the inclusion of $h(t)$ or even $h_i(t)$, $1 \le i \le n$. Many papers have discussed triple solutions of boundary value problems [7,8,10–14,23,25,27,28,30–32]. Our problems (1.1), (1.2) generalize almost all the work in the literature to date as we are considering *systems* as well as *more general nonlinear terms*. Moreover, our present approach is not only generic, but also improves, corrects and completes the arguments in many papers in the literature.

The outline of the paper is as follows. In Section 2, we shall state the relevant fixed point theorems of Leggett and Williams [19] and Avery [9]. Our main results for (1.1) are presented in Section 3, whose usefulness is illustrated in Section 4 when we apply them to several well known boundary value problems. Finally, parallel results are established for system (1.2) in Section 5.

2 Preliminaries

Definition 2.1. Let C ($\subset B$) be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

(a) If $u \in C$ and $\alpha \ge 0$, then $\alpha u \in C$;

(b) If $u \in C$ and $-u \in C$, then $u = 0$.

The cone C induces an ordering $\le$ on B. For $y, z \in B$, we write $y \le z$ if and only if $z - y \in C$. If $y, z \in B$ with $y \le z$, we let $\langle y, z\rangle$ denote the closed order interval given by

$$\langle y, z\rangle = \{u \in B \mid y \le u \le z\}.$$

Definition 2.2. Let C ($\subset B$) be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:

(a) $\psi : C \to [0, \infty)$ is continuous;

(b) $\psi(ty + (1-t)z) \ge t\psi(y) + (1-t)\psi(z)$ for all $y, z \in C$ and $0 \le t \le 1$.

Definition 2.3. Let C ($\subset B$) be a cone. A map β is a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

(a) $\beta : C \to [0, \infty)$ is continuous;

(b) $\beta(ty + (1-t)z) \le t\beta(y) + (1-t)\beta(z)$ for all $y, z \in C$ and $0 \le t \le 1$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C. For nonnegative numbers w_i, $1 \le i \le 3$, we shall introduce the following notations:

$$C(w_1) = \{u \in C \mid \|u\| < w_1\},$$
$$C(\psi, w_1, w_2) = \{u \in C \mid \psi(u) \ge w_1 \text{ and } \|u\| \le w_2\},$$
$$P(\gamma, w_1) = \{u \in C \mid \gamma(u) < w_1\},$$
$$P(\gamma, \alpha, w_1, w_2) = \{u \in C \mid \alpha(u) \ge w_1 \text{ and } \gamma(u) \le w_2\},$$
$$Q(\gamma, \beta, w_1, w_2) = \{u \in C \mid \beta(u) \le w_1 \text{ and } \gamma(u) \le w_2\},$$
$$P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{u \in C \mid \alpha(u) \ge w_1,\ \Theta(u) \le w_2 \text{ and } \gamma(u) \le w_3\},$$
$$Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{u \in C \mid \psi(u) \ge w_1,\ \beta(u) \le w_2 \text{ and } \gamma(u) \le w_3\}.$$

The following fixed point theorems are needed later. The first is usually called *Leggett-Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem.*

Theorem 2.1. [19] *Let C ($\subset B$) be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(u) \le \|u\|$ for all $u \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \to \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 where $0 < w_1 < w_2 < w_3 \le w_4$ such that*

(a) *$\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \ne \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$;*

(b) *$\|Su\| < w_1$ for all $u \in \overline{C}(w_1)$;*

(c) *$\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.*

Then, S has (at least) three fixed points u^1, u^2 and u^3 in $\overline{C}(w_4)$. Furthermore, we have

$$u^1 \in C(w_1), \quad u^2 \in \left\{ u \in C(\psi, w_2, w_4) \;\middle|\; \psi(u) > w_2 \right\} \quad \text{and}$$
$$u^3 \in \overline{C}(w_4) \setminus \left(C(\psi, w_2, w_4) \cup \overline{C}(w_1) \right). \tag{2.1}$$

Theorem 2.2. [9] *Let C ($\subset B$) be a cone. Assume that there exist positive numbers w_5, M, nonnegative continuous convex functionals γ, β, Θ on C, and nonnegative continuous concave functionals α, ψ on C, with*

$$\alpha(u) \le \beta(u) \quad \text{and} \quad \|u\| \le M\gamma(u)$$

for all $u \in \overline{P}(\gamma, w_5)$. *Let* $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$ *be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers* w_i, $1 \le i \le 4$ *with* $0 < w_2 < w_3$ *such that*

(a) $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \neq \emptyset$, *and* $\alpha(Su) > w_3$ *for all* $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;

(b) $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \neq \emptyset$, *and* $\beta(Su) < w_2$ *for all* $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;

(c) $\alpha(Su) > w_3$ *for all* $u \in P(\gamma, \alpha, w_3, w_5)$ *with* $\Theta(Su) > w_4$;

(d) $\beta(Su) < w_2$ *for all* $u \in Q(\gamma, \beta, w_2, w_5)$ *with* $\psi(Su) < w_1$.

Then, S has (at least) three fixed points u^1, u^2 *and* u^3 *in* $\overline{P}(\gamma, w_5)$. *Furthermore, we have*

$$\beta(u^1) < w_2, \quad \alpha(u^2) > w_3, \quad \text{and} \quad \beta(u^3) > w_2 \quad \text{with} \quad \alpha(u^3) < w_3. \tag{2.2}$$

Remark 2.1. We note that the five-functional fixed point theorem is more general than Leggett-Williams' fixed point theorem. Indeed, in Theorem 2.2 if we replace w_i by w_{i-1}, $2 \le i \le 5$, and choose the functionals $\gamma = \Theta = \beta = \|\cdot\|$ and $\alpha = \psi$, then we obtain Theorem 2.1.

We also require the definition of a *L^q-Carathéodory function.*

Definition 2.4. [22] A function $P : [0,1] \times \mathbf{R}^n \to \mathbf{R}$ is a *L^q-Carathéodory function* if the following conditions hold:

(a) The map $t \to P(t, u)$ is measurable for all $u \in \mathbf{R}^n$.

(b) The map $u \to P(t, u)$ is continuous for almost all $t \in [0,1]$.

(c) For any $r > 0$, there exists $\mu_r \in L^q[0,1]$ such that $|u| \le r$ implies that $|P(t,u)| \le \mu_r(t)$ for almost all $t \in [0,1]$.

3 Triple solutions of (1.1)

Throughout we shall denote $u = (u_1, u_2, \cdots, u_n)$. Let the Banach space

$$B = \left\{ u \;\middle|\; u \in (C[0,1])^n \right\} \tag{3.1}$$

be equipped with norm

$$\|u\| = \max_{1 \le i \le n} \sup_{t \in [0,1]} |u_i(t)| = \max_{1 \le i \le n} |u_i|_0 \tag{3.2}$$

where we let $|u_i|_0 = \sup_{t\in[0,1]} |u_i(t)|$, $1 \le i \le n$. Moreover, for fixed $\theta_i \in \{1, -1\}$, $1 \le i \le n$, define

$$\tilde{K} = \left\{ u \in B \;\middle|\; \theta_i u_i \ge 0,\ 1 \le i \le n \right\}$$

and

$$K = \left\{ u \in \tilde{K} \;\middle|\; \theta_j u_j > 0 \text{ for some } j \in \{1, 2, \cdots, n\} \right\} = \tilde{K} \backslash \{0\}.$$

For the purpose of clarity, we shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1, -1\}$, $1 \le i \le n$ are fixed.

(C1) Let integers p, q be such that $1 \le p \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $1 \le i \le n$, assume that $P_i : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^q-Carathéodory function, and

$$g_i^t(s) \equiv g_i(t, s) \ge 0,\ t \in [0,1],\ a.e.\ s \in [0, 1],$$
$$g_i^t(s) \in L^p[0,1],\ t \in [0,1],$$
$$\text{the map } t \to g_i^t \text{ is continuous from } [0,1] \text{ to } L^p[0,1].$$

(C2) For each $1 \le i \le n$, there exists a constant $0 < M_i < 1$, a function $H \in L^p[0,1]$, and an interval $[a, b] \subseteq [0, 1]$ such that

$$g_i(t, s) \ge M_i H(s) \ge 0,\quad t \in [a, b],\ a.e.\ s \in [0, 1].$$

(C3) For each $1 \le i \le n$,

$$g_i(t, s) \le H(s),\quad t \in [0, 1],\ a.e.\ s \in [0, 1].$$

(C4) For each $1 \le i \le n$, assume that

$$\theta_i P_i(t, u) \ge 0,\ \ u \in \tilde{K},\ a.e.\ t \in (0, 1) \text{ and } \theta_i P_i(t, u) > 0,\ \ u \in K,\ a.e.\ t \in (0, 1).$$

(C5) There exist continuous functions f, b and a_i, $1 \le i \le n$ with $f : \mathbb{R}^n \to [0, \infty)$ and $b, a_i : (0, 1) \to [0, \infty)$ such that for each $1 \le i \le n$,

$$a_i(t) \le \frac{\theta_i P_i(t, u)}{f(u)} \le b(t),\quad u \in \tilde{K},\ a.e.\ t \in (0, 1).$$

(C6) For each $1 \le i \le n$, there exists a number $0 < \rho_i \le 1$ such that

$$a_i(t) \ge \rho_i b(t),\ a.e.\ t \in (0, 1).$$

To begin the discussion, let the operator $S : B \to B$ be defined by

$$Su(t) = (Su_1(t), Su_2(t), \cdots, Su_n(t)),\ t \in [0, 1] \tag{3.3}$$

where

$$Su_i(t) = \int_0^1 g_i(t,s)P_i(s,u(s))ds, \quad t \in [0,1], \ 1 \le i \le n. \tag{3.4}$$

Clearly, a fixed point of the operator S is a solution of the system (1.1).

Next, we define a cone in B as

$$C = \left\{ u \in B \;\middle|\; \text{for each } 1 \le i \le n, \ \theta_i u_i(t) \ge 0 \text{ for } t \in [0,1], \right.$$
$$\left. \text{and } \min_{t\in[a,b]} \theta_i u_i(t) \ge M_i \rho_i |u_i|_0 \right\} \tag{3.5}$$

where M_i and ρ_i are defined in (C2) and (C6) respectively. Note that $C \subseteq \tilde{K}$. A fixed point of S obtained in C or $\tilde{K}$ will be a *constant-sign solution* of the system (1.1).

Remark 3.1. Instead of the cone C defined in (3.5), we can also use the cone C' ($\subset C$) given by

$$C' = \left\{ u \in B \;\middle|\; \text{for each } 1 \le i \le n, \ \theta_i u_i(t) \ge 0 \text{ for } t \in [0,1], \right.$$
$$\left. \text{and } \min_{t\in[a,b]} \theta_i u_i(t) \ge M_i \rho_i \|u\| \right\}.$$

The arguments that follow will be similar.

If (C1), (C4) and (C5) hold, then it is clear from (3.4) that for $u \in \tilde{K}$,

$$\int_0^1 g_i(t,s)a_i(s)f(u(s))ds \le \theta_i Su_i(t) \le \int_0^1 g_i(t,s)b(s)f(u(s))ds, \quad t \in [0,1], \ 1 \le i \le n. \tag{3.6}$$

Lemma 3.1. *Let (C1) hold. Then, the operator S is continuous and completely continuous.*

Proof. As in [22, Theorem 4.2.2], (C1) ensures that S is continuous and completely continuous. ■

Lemma 3.2. *Let (C1)-(C6) hold. Then, the operator S maps C into itself.*

Proof. Let $u \in C$. From (3.6) we have for $t \in [0,1]$ and $1 \le i \le n$,

$$\theta_i Su_i(t) \ge \int_0^1 g_i(t,s)a_i(s)f(u(s))ds \ge 0. \tag{3.7}$$

Next, using (3.6) and (C3) gives for $t \in [0,1]$ and $1 \le i \le n$,

$$|Su_i(t)| = \theta_i Su_i(t) \le \int_0^1 g_i(t,s)b(s)f(u(s))ds \le \int_0^1 H(s)b(s)f(u(s))ds.$$

Hence, we have

$$|Su_i|_0 \le \int_0^1 H(s)b(s)f(u(s))ds, \ 1 \le i \le n. \tag{3.8}$$

Indeed, this immediately gives

$$\|Su\| = \max_{1\le i\le n} |Su_i|_0 \le \int_0^1 H(s)b(s)f(u(s))ds. \tag{3.9}$$

Now, employing (3.6), (C2), (C6) and (3.8) we find for $t \in [a,b]$ and $1 \le i \le n$,

$$\begin{aligned} \theta_i Su_i(t) &\ge \int_0^1 g_i(t,s)a_i(s)f(u(s))ds \\ &\ge \int_0^1 M_i H(s)a_i(s)f(u(s))ds \\ &\ge \int_0^1 M_i H(s)\rho_i b(s)f(u(s))ds \\ &\ge M_i\rho_i|Su_i|_0. \end{aligned}$$

This leads to

$$\min_{t\in[a,b]} \theta_i Su_i(t) \ge M_i\rho_i|Su_i|_0, \ 1 \le i \le n. \tag{3.10}$$

Inequalities (3.7) and (3.10) imply that $Su \in C$. ∎

For subsequent results, we define the following constants for each $1 \le i \le n$ and fixed numbers $\tau_j \in [0,1]$, $1 \le j \le 4$:

$$\begin{aligned} q_i &= \sup_{t\in[0,1]} \int_0^1 g_i(t,s)b(s)ds, \\ r_i &= \min_{t\in[a,b]} \int_a^b g_i(t,s)a_i(s)ds, \\ d_{1,i} &= \min_{t\in[\tau_2,\tau_3]} \int_{\tau_2}^{\tau_3} g_i(t,s)a_i(s)ds, \\ d_{2,i} &= \max_{t\in[\tau_1,\tau_4]} \int_{\tau_1}^{\tau_4} g_i(t,s)b(s)ds, \\ d_{3,i} &= \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\tau_1} g_i(t,s)b(s)ds + \int_{\tau_4}^1 g_i(t,s)b(s)ds\right]. \end{aligned} \tag{3.11}$$

In view of (C3) and (C2), it is clear that for each $1 \le i \le n$,

$$q_i \le \int_0^1 H(s)b(s)ds, \qquad r_i \ge \int_a^b M_i H(s)a_i(s)ds \qquad \text{and} \qquad d_{2,i} \le \int_{\tau_1}^{\tau_4} H(s)b(s)ds. \tag{3.12}$$

Lemma 3.3. *Let (C1)-(C6) hold, and assume*

(C7) *for each* $1 \le i \le n$ *and each* $t \in [0,1]$, *the function* $g_i(t,s)b(s)$ *is nonzero on a subset of* $[0,1]$ *of positive measure.*

Suppose that there exists a number $d > 0$ *such that for* $\theta_j u_j \in [0,d]$, $1 \le j \le n$,

$$f(u_1, u_2, \cdots, u_n) < \frac{d}{q_i}, \ 1 \le i \le n. \tag{3.13}$$

Then,

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d). \tag{3.14}$$

Proof. Let $u \in \overline{C}(d)$. Clearly, we have $\theta_j u_j \in [0,d]$, $1 \le j \le n$. Applying (3.6), (3.13) and (3.11), for each $t \in [0,1]$ and $1 \le i \le n$ we find

$$\begin{aligned} |Su_i(t)| &= \theta_i Su_i(t) \le \int_0^1 g_i(t,s)b(s)f(u(s))ds \\ &< \sup_{t\in[0,1]} \int_0^1 g_i(t,s)b(s)\,\frac{d}{q_i}\,ds \\ &= \frac{q_i}{q_i}\,d = d. \end{aligned}$$

This implies $|Su_i|_0 < d$, $1 \le i \le n$ and so $\|Su\| < d$. From Lemma 3.2, we already have $Su \in C$, thus it follows that $Su \in C(d)$. The conclusion (3.14) is now immediate. ∎

The next lemma is similar to Lemma 3.3 and its proof is omitted.

Lemma 3.4. *Let (C1)-(C6) hold. Suppose that there exists a number* $d > 0$ *such that for* $\theta_j u_j \in [0,d]$, $1 \le j \le n$,

$$f(u_1, u_2, \cdots, u_n) \le \frac{d}{q_i}, \ 1 \le i \le n.$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

Our first result makes use of Theorem 2.1.

Theorem 3.1. *Let (C1)-(C7) hold, and assume*

(C8) *for each* $1 \le i \le n$ *and each* $t \in [a,b]$, *the function* $g_i(t,s)a_i(s)$ *is nonzero on a subset of* $[a,b]$ *of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 *with*

$$0 < w_1 < w_2 < \frac{w_2}{\min_{1\le i\le n} M_i \rho_i} \le w_3$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1,u_2,\cdots,u_n) < \frac{w_1}{q_i}$ *for* $\theta_j u_j \in [0,w_1],\ 1 \le j \le n$;

(Q) *one of the following holds:*

(Q1) $\limsup_{|u_1|,|u_2|,\cdots,|u_n|\to\infty} \frac{f(u_1,u_2,\cdots,u_n)}{|u_j|} < \frac{1}{q_i}$ *for some* $j \in \{1,2,\cdots,n\}$ *(j depends on i);*

(Q2) *there exists a number* η $(\ge w_3)$ *such that* $f(u_1,u_2,\cdots,u_n) \le \frac{\eta}{q_i}$ *for* $\theta_j u_j \in [0,\eta],\ 1 \le j \le n$;

(R) $f(u_1,u_2,\cdots,u_n) > \frac{w_2}{r_i}$ *for* $\theta_j u_j \in [w_2,w_3],\ 1 \le j \le n$.

Then, the system (1.1) has (at least) three constant-sign solutions $u^1,u^2,u^3 \in C$ *such that*

$$\begin{aligned} &\|u^1\| < w_1; && |u_i^2(t)| > w_2,\ \ t\in[a,b],\ 1\le i\le n;\\ &\|u^3\| > w_1 && \text{and} \quad \min_{1\le i\le n}\ \min_{t\in[a,b]} |u_i^3(t)| < w_2. \end{aligned} \tag{3.15}$$

Proof. We shall employ Theorem 2.1. First, we shall prove that condition (Q) implies the existence of a number w_4 where $w_4 \ge w_3$ such that

$$S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \tag{3.16}$$

Suppose that (Q2) holds. Then, by Lemma 3.4 we immediately have (3.16) where we pick $w_4 = \eta$. Suppose now that (Q1) is satisfied. Then, for each $1 \le i \le n$, there exist $N_i > 0$ and $\epsilon_i < \frac{1}{q_i}$ such that

$$\frac{f(u_1,u_2,\cdots,u_n)}{|u_j|} < \epsilon_i, \qquad |u_1|,|u_2|,\cdots,|u_n| > N_i. \tag{3.17}$$

Define

$$M_i = \max_{|u_m|\in[0,N_i],\ 1\le m\le n} f(u_1,u_2,\cdots,u_n),\ \ 1\le i\le n.$$

In view of (3.17), it is clear that for each $1 \le i \le n$ and some $j(i) \in \{1,2,\cdots,n\}$, the following holds for all $(u_1,u_2,\cdots,u_n) \in \mathbb{R}^n$,

$$f(u_1,u_2,\cdots,u_n) \le M_i + \epsilon_i\,|u_j|. \tag{3.18}$$

Now, pick the number w_4 so that

$$w_4 > \max\left\{ w_3,\ \max_{1\le i\le n} M_i \left(\frac{1}{q_i} - \epsilon_i\right)^{-1} \right\}. \tag{3.19}$$

Let $u \in \overline{C}(w_4)$. For $t \in [0,1]$ and $1 \le i \le n$, using (3.6), (3.18) and (3.19) gives

$$\begin{aligned} |Su_i(t)| = \theta_i Su_i(t) &\le \int_0^1 g_i(t,s)b(s)f(u(s))ds \\ &\le \sup_{t\in[0,1]} \int_0^1 g_i(t,s)b(s)(M_i + \epsilon_i\ |u_j(s)|)ds \\ &\le \sup_{t\in[0,1]} \int_0^1 g_i(t,s)b(s)(M_i + \epsilon_i\ w_4)ds \\ &= q_i(M_i + \epsilon_i\ w_4) \\ &< q_i\left[w_4\left(\frac{1}{q_i} - \epsilon_i\right) + \epsilon_i\ w_4\right] = w_4. \end{aligned}$$

This leads to $|Su_i|_0 < w_4$, $1 \le i \le n$. Hence, $\|Su\| < w_4$ and so $Su \in C(w_4) \subset \overline{C}(w_4)$. Thus, (3.16) follows immediately.

Let $\psi : C \to [0,\infty)$ be defined by

$$\psi(u) = \min_{1\le i\le n}\ \min_{t\in[a,b]} \theta_i u_i(t).$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(u) \le \|u\|$ for all $u \in C$.

We shall verify that condition (a) of Theorem 2.1 is satisfied. In fact, it is obvious that

$$\begin{aligned} u(t) &= \left(\tfrac{\theta_1}{2}(w_2+w_3), \tfrac{\theta_2}{2}(w_2+w_3), \cdots, \tfrac{\theta_n}{2}(w_2+w_3)\right) \\ &\in \left\{ u \in C(\psi, w_2, w_3)\ \middle|\ \psi(u) > w_2 \right\} \end{aligned}$$

and so $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \ne \emptyset$. Next, let $u \in C(\psi, w_2, w_3)$. Then, $w_2 \le \psi(u) \le \|u\| \le w_3$ and hence for $s \in [a,b]$, we have

$$\theta_j u_j(s) \in [w_2, w_3],\ 1 \le j \le n. \tag{3.20}$$

In view of (3.6), (3.20), (R) and (3.11), it follows that

$$\begin{aligned} \psi(Su) = \min_{1\le i\le n}\ \min_{t\in[a,b]} \theta_i(Su_i)(t) &\ge \min_{1\le i\le n}\ \min_{t\in[a,b]} \int_0^1 g_i(t,s)a_i(s)f(u(s))ds \\ &\ge \min_{1\le i\le n}\ \min_{t\in[a,b]} \int_a^b g_i(t,s)a_i(s)f(u(s))ds \\ &> \min_{1\le i\le n}\ \min_{t\in[a,b]} \int_a^b g_i(t,s)a_i(s)\ \frac{w_2}{r_i}\ ds \\ &= \min_{1\le i\le n} \frac{r_i}{r_i}\ w_2 = w_2. \end{aligned}$$

Therefore, we have shown that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$.

Next, by Lemma 3.3 and condition (P), we have $S(\overline{C}(w_1)) \subseteq C(w_1)$. Hence, condition (b) of Theorem 2.1 is fulfilled.

Finally, we shall show that condition (c) of Theorem 2.1 holds. Recall that w_3 satisfies

$$w_3 \geq \frac{w_2}{\min_{1\leq i\leq n} M_i\rho_i}. \tag{3.21}$$

Let $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$. Using (3.6), (C2), (C6), (3.9) and (3.21), we find

$$\begin{aligned}
\psi(Su) = \min_{1\leq i\leq n} \min_{t\in[a,b]} \theta_i(Su_i)(t) &\geq \min_{1\leq i\leq n} \min_{t\in[a,b]} \int_0^1 g_i(t,s)a_i(s)f(u(s))ds \\
&\geq \min_{1\leq i\leq n} \int_0^1 M_iH(s)a_i(s)f(u(s))ds \\
&\geq \min_{1\leq i\leq n} \int_0^1 M_iH(s)\rho_i b(s)f(u(s))ds \\
&\geq \min_{1\leq i\leq n} M_i\rho_i\|Su\| \\
&> \min_{1\leq i\leq n} M_i\rho_i w_3 \geq w_2.
\end{aligned}$$

Hence, we have proved that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

It now follows from Theorem 2.1 that the system (1.1) has (at least) three *constant-sign* solutions $u^1, u^2, u^3 \in \overline{C}(w_4)$ satisfying (2.1). It is easy to see that here (2.1) reduces to (3.15). ∎

We shall now employ Theorem 2.2 to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

Theorem 3.2. *Let (C1)–(C6) hold. Assume there exist numbers τ_j, $1 \leq j \leq 4$ with*

$$0 \leq \tau_1 \leq a \leq \tau_2 < \tau_3 \leq b \leq \tau_4 \leq 1$$

such that

(C9) *for each $1 \leq i \leq n$ and each $t \in [\tau_2, \tau_3]$, the function $g_i(t,s)a_i(s)$ is nonzero on a subset of $[\tau_2, \tau_3]$ of positive measure;*

(C10) *for each $1 \leq i \leq n$ and each $t \in [\tau_1, \tau_4]$, the function $g_i(t,s)b(s)$ is nonzero on a subset of $[\tau_1, \tau_4]$ of positive measure.*

Suppose that there exist numbers w_i, $2 \leq i \leq 5$ with

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\leq i\leq n} M_i\rho_i} \leq w_4 \leq w_5$$

such that the following hold for each $1 \leq i \leq n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right)$ *for* $\theta_j u_j \in [0, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i}$ *for* $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (1.1) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2, \quad t \in [\tau_1, \tau_4], \ 1 \le i \le n; \qquad |u_i^2(t)| > w_3, \quad t \in [\tau_2, \tau_3], \ 1 \le i \le n;$$

$$\max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \text{and} \quad \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3. \tag{3.22}$$

Proof. To apply Theorem 2.2, we shall define the following functionals on C :

$$\begin{aligned} \gamma(u) &= \|u\|, \\ \psi(u) &= \min_{1\le i\le n} \min_{t\in[a,b]} \theta_i u_i(t), \\ \beta(u) &= \Theta(u) = \max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} \theta_i u_i(t), \\ \alpha(u) &= \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} \theta_i u_i(t). \end{aligned} \tag{3.23}$$

First, we shall show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Let $u \in \overline{P}(\gamma, w_5)$. Then, we have $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$. Using (3.6), (Q) and (3.11), for each $t \in [0,1]$ and $1 \le i \le n$ we find

$$\begin{aligned} |Su_i(t)| = \theta_i Su_i(t) &\le \int_0^1 g_i(t,s)b(s)f(u(s))ds \\ &\le \sup_{t\in[0,1]} \int_0^1 g_i(t,s)b(s)\,\frac{w_5}{q_i}\,ds \\ &= \frac{q_i}{q_i}\,w_5 = w_5. \end{aligned}$$

This implies $|Su_i|_0 \le w_5$, $1 \le i \le n$ and so $\gamma(Su) = \|Su\| \le w_5$. From Lemma 3.2, we already have $Su \in C$, thus it follows that $Su \in \overline{P}(\gamma, w_5)$. Hence, we have shown that $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we note that

$$\begin{aligned} u(t) &= \left(\tfrac{\theta_1}{2}(w_3+w_4), \tfrac{\theta_2}{2}(w_3+w_4), \cdots, \tfrac{\theta_n}{2}(w_3+w_4)\right) \\ &\in \left\{ u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \ \middle|\ \alpha(u) > w_3 \right\} \end{aligned}$$

and so $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \ne \emptyset$. Let $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, by definition we have $\alpha(u) \ge w_3$ and $\Theta(u) \le w_4$ which imply

$$\theta_i u_i(s) \in [w_3, w_4], \quad s \in [\tau_2, \tau_3], \ 1 \le i \le n. \tag{3.24}$$

Noting (3.6), (3.24), (R) and (3.11), we obtain

$$\begin{aligned}
\alpha(Su) &= \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \theta_i(Su_i)(t) \ \geq\ \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \int_0^1 g_i(t,s)a_i(s)f(u(s))ds \\
&\geq \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \int_{\tau_2}^{\tau_3} g_i(t,s)a_i(s)f(u(s))ds \\
&> \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \int_{\tau_2}^{\tau_3} g_i(t,s)a_i(s)\ \frac{w_3}{d_{1,i}}\ ds \\
&= \min_{1\le i\le n} \frac{d_{1,i}}{d_{1,i}}\, w_3 \ =\ w_3.
\end{aligned}$$

Hence, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. Note that

$$\begin{aligned}
u(t) &= \left(\tfrac{\theta_1}{2}(w_1+w_2), \tfrac{\theta_2}{2}(w_1+w_2), \cdots, \tfrac{\theta_n}{2}(w_1+w_2)\right) \\
&\in \left\{ u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \ \middle|\ \beta(u) < w_2 \right\}
\end{aligned}$$

and so $\{u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\} \neq \emptyset$. Let $u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$. Then, we have $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which imply

$$\theta_i u_i(s) \in [0,w_2],\ s\in[\tau_1,\tau_4] \quad \text{and} \quad \theta_i u_i(s) \in [0,w_5],\ s\in[0,1], \quad 1\le i\le n. \tag{3.25}$$

In view of (3.6), (3.25), (P), (Q) and (3.11), we find

$$\begin{aligned}
\beta(Su) &= \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \theta_i(Su_i)(t) \\
&\le \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_0^1 g_i(t,s)b(s)f(u(s))ds \\
&= \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \left[\int_{\tau_1}^{\tau_4} g_i(t,s)b(s)f(u(s))ds \right. \\
&\quad \left. + \int_0^{\tau_1} g_i(t,s)b(s)f(u(s))ds + \int_{\tau_4}^1 g_i(t,s)b(s)f(u(s))ds \right] \\
&< \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_{\tau_1}^{\tau_4} g_i(t,s)b(s)\ \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right) ds \\
&\quad + \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\tau_1} g_i(t,s)b(s)ds + \int_{\tau_4}^1 g_i(t,s)b(s)ds \right] \frac{w_5}{q_i} \\
&= \max_{1\le i\le n} \left[d_{2,i}\, \frac{1}{d_{2,i}} \left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right) + d_{3,i}\, \frac{w_5}{q_i} \right] \ =\ w_2.
\end{aligned}$$

Therefore, $\beta(Su) < w_2$ for all $u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.6) and (C3), for $u \in C$,

$$\begin{aligned}\Theta(Su) &= \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \theta_i(Su_i)(t)\\ &\le \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_0^1 g_i(t,s)b(s)f(u(s))ds\\ &\le \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_0^1 H(s)b(s)f(u(s))ds\\ &= \int_0^1 H(s)b(s)f(u(s))ds. \qquad (3.26)\end{aligned}$$

Moreover, using (3.6), (C6) and (C2), we get for $u \in C$,

$$\begin{aligned}\alpha(Su) &= \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \theta_i(Su_i)(t)\\ &\ge \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \int_0^1 g_i(t,s)a_i(s)f(u(s))ds\\ &\ge \min_{1\le i\le n}\ \min_{t\in[a,b]} \int_0^1 g_i(t,s)\rho_i b(s)f(u(s))ds\\ &\ge \min_{1\le i\le n} M_i\rho_i \int_0^1 H(s)b(s)f(u(s))ds. \qquad (3.27)\end{aligned}$$

Combining (3.26) and (3.27) yields

$$\alpha(Su) \ge \min_{1\le i\le n} M_i\rho_i\ \Theta(Su), \quad u\in C. \qquad (3.28)$$

Let $u \in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Su) > w_4$. Then, it follows from (3.28) that

$$\alpha(Su) \ge \min_{1\le i\le n} M_i\rho_i\ \Theta(Su) > \min_{1\le i\le n} M_i\rho_i w_4 \ge \min_{1\le i\le n} M_i\rho_i\ \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} = w_3. \qquad (3.29)$$

Thus, $\alpha(Su) > w_3$ for all $u \in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Su) > w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$. Then, we have $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which give (3.25). Using (3.6), (3.25), (P), (Q) and (3.11), we get as in an earlier part

$$\begin{aligned}\beta(Su) &\le \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_0^1 g_i(t,s)b(s)f(u(s))ds\\ &< \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \int_{\tau_1}^{\tau_4} g_i(t,s)b(s)\ \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right) ds\\ &\quad + \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\tau_1} g_i(t,s)b(s)ds + \int_{\tau_4}^1 g_i(t,s)b(s)ds\right] \frac{w_5}{q_i}\\ &= \max_{1\le i\le n}\left[d_{2,i}\ \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right) + d_{3,i}\ \frac{w_5}{q_i}\right] = w_2.\end{aligned}$$

Thus, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.

It now follows from Theorem 2.2 that the system (1.1) has (at least) three *constant-sign* solutions $u^1, u^2, u^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ satisfying (2.2). Furthermore, (2.2) reduces to (3.22) immediately. ∎

When $\tau_1 = 0,\ \tau_4 = 1,\ \tau_2 = a$ and $\tau_3 = b$, then

$$d_{1,i} = r_i, \qquad d_{2,i} = q_i \qquad \text{and} \qquad d_{3,i} = 0. \tag{3.30}$$

In this case Theorem 3.2 yields the following corollary.

Corollary 3.1. *Let (C1)–(C6) hold, and assume*

(C9)* *for each $1 \le i \le n$ and each $t \in [a,b]$, the function $g_i(t,s)a_i(s)$ is nonzero on a subset of $[a,b]$ of positive measure;*

(C10)* *for each $1 \le i \le n$ and each $t \in [0,1]$, the function $g_i(t,s)b(s)$ is nonzero on a subset of $[0,1]$ of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$ with

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{w_2}{q_i}$ *for* $\theta_j u_j \in [0, w_2],\ 1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i}$ *for* $\theta_j u_j \in [0, w_5],\ 1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{r_i}$ *for* $\theta_j u_j \in [w_3, w_4],\ 1 \le j \le n$.

Then, the system (1.1) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} &\|u^1\| < w_2; \qquad |u_i^2(t)| > w_3, \quad t \in [a,b],\ 1 \le i \le n; \\ &\|u^3\| > w_2 \quad \text{and} \quad \min_{1\le i\le n}\ \min_{t\in[a,b]} |u_i^3(t)| < w_3. \end{aligned} \tag{3.31}$$

Remark 3.2. Corollary 3.1 is actually Theorem 3.1.

The next result is another application of Theorem 2.2.

Theorem 3.3. *Let (C1)–(C6) hold. Assume there exist numbers τ_j, $1 \le j \le 4$ with*

$$a \le \tau_1 < \tau_2 < \tau_3 < \tau_4 \le b$$

such that (C9) and (C10) hold. Suppose that there exist numbers w_i, $1 \le i \le 5$ with

$$0 < w_1 \le w_2 \cdot \min_{1\le i\le n} M_i\rho_i < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1,u_2,\cdots,u_n) < \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right)$ *for* $\theta_j u_j \in [w_1,w_2],\ 1 \le j \le n$;

(Q) $f(u_1,u_2,\cdots,u_n) \le \frac{w_5}{q_i}$ *for* $\theta_j u_j \in [0,w_5],\ 1 \le j \le n$;

(R) $f(u_1,u_2,\cdots,u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in [w_3,w_4],\ 1 \le j \le n$.

Then, the system (1.1) has (at least) three constant-sign solutions $u^1,u^2,u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2,\ \ t\in[\tau_1,\tau_4],\ 1\le i\le n; \qquad |u_i^2(t)| > w_3,\ \ t\in[\tau_2,\tau_3],\ 1\le i\le n;$$
$$\max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \textit{and} \quad \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3. \tag{3.32}$$

Proof. To apply Theorem 2.2, we shall define the following functionals on C :

$$\begin{aligned}
\gamma(u) &= \|u\|,\\
\psi(u) &= \min_{1\le i\le n}\ \min_{t\in[\tau_1,\tau_4]} \theta_i u_i(t),\\
\beta(u) &= \max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} \theta_i u_i(t),\\
\alpha(u) &= \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} \theta_i u_i(t),\\
\Theta(u) &= \max_{1\le i\le n}\ \max_{t\in[\tau_2,\tau_3]} \theta_i u_i(t).
\end{aligned} \tag{3.33}$$

First, using (Q), as in the proof of Theorem 3.2, we can show that $S:\overline{P}(\gamma,w_5)\to\overline{P}(\gamma,w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we use (R) and a similar argument as in the proof of Theorem 3.2.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Note that

$$\begin{aligned}
u(t) &= \left(\tfrac{\theta_1}{2}(w_1+w_2), \tfrac{\theta_2}{2}(w_1+w_2),\cdots,\tfrac{\theta_n}{2}(w_1+w_2)\right)\\
&\in \left\{ u\in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \;\middle|\; \beta(u) < w_2 \right\}
\end{aligned}$$

and so $\{u\in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\} \ne \emptyset$. Let $u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$. Then, we have $\psi(u)\ge w_1$, $\beta(u)\le w_2$ and $\gamma(u)\le w_5$ which imply

$$\theta_i u_i(s)\in[w_1,w_2],\ s\in[\tau_1,\tau_4] \quad \text{and} \quad \theta_i u_i(s)\in[0,w_5],\ s\in[0,1], \quad 1\le i\le n. \tag{3.34}$$

In view of (3.6), (3.34), (P), (Q) and (3.11), we find

$$\begin{aligned}\beta(Su) &= \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\theta_i(Su_i)(t)\\ &\le \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\int_0^1 g_i(t,s)b(s)f(u(s))ds\\ &= \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\left[\int_{\tau_1}^{\tau_4} g_i(t,s)b(s)f(u(s))ds\right.\\ &\quad\left.+\int_0^{\tau_1} g_i(t,s)b(s)f(u(s))ds+\int_{\tau_4}^1 g_i(t,s)b(s)f(u(s))ds\right]\\ &< \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\int_{\tau_1}^{\tau_4} g_i(t,s)b(s)\,\frac{1}{d_{2,i}}\left(w_2-\frac{w_5d_{3,i}}{q_i}\right)ds\\ &\quad+\max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\left[\int_0^{\tau_1} g_i(t,s)b(s)ds+\int_{\tau_4}^1 g_i(t,s)b(s)ds\right]\frac{w_5}{q_i}\\ &= \max_{1\le i\le n}\left[d_{2,i}\,\frac{1}{d_{2,i}}\left(w_2-\frac{w_5d_{3,i}}{q_i}\right)+d_{3,i}\,\frac{w_5}{q_i}\right] = w_2.\end{aligned}$$

Therefore, $\beta(Su)<w_2$ for all $u\in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.6) and (C3), for $u\in C$,

$$\begin{aligned}\Theta(Su) &= \max_{1\le i\le n}\max_{t\in[\tau_2,\tau_3]}\theta_i(Su_i)(t)\\ &\le \max_{1\le i\le n}\max_{t\in[\tau_2,\tau_3]}\int_0^1 g_i(t,s)b(s)f(u(s))ds\\ &\le \max_{1\le i\le n}\max_{t\in[\tau_2,\tau_3]}\int_0^1 H(s)b(s)f(u(s))ds\\ &= \int_0^1 H(s)b(s)f(u(s))ds.\end{aligned}\tag{3.35}$$

Moreover, using (3.6), (C2) and (C6), we get (3.27) for $u\in C$. Combining (3.27) and (3.35) yields (3.28). The rest then follows as in the proof of Theorem 3.2.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. As in (3.35), by (3.6) and (C3), we see that for $u\in C$,

$$\beta(Su)=\max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]}\theta_i(Su_i)(t)\le\int_0^1 H(s)b(s)f(u(s))ds.\tag{3.36}$$

On the other hand, similar to (3.27) it follows from (3.6), (C2) and (C6) that for $u\in C$,

$$\psi(Su)=\min_{1\le i\le n}\min_{t\in[\tau_1,\tau_4]}\theta_i(Su_i)(t)\ge\min_{1\le i\le n}M_i\rho_i\int_0^1 H(s)b(s)f(u(s))ds.\tag{3.37}$$

A combination of (3.36) and (3.37) gives

$$\psi(Su) \geq \min_{1\leq i\leq n} M_i\rho_i\, \beta(Su), \quad u \in C. \tag{3.38}$$

Let $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$. Then, (3.38) leads to

$$\begin{aligned} \beta(Su) &\leq \frac{1}{\min_{1\leq i\leq n} M_i\rho_i}\,\psi(Su) < \frac{1}{\min_{1\leq i\leq n} M_i\rho_i}\, w_1 \\ &\leq \frac{1}{\min_{1\leq i\leq n} M_i\rho_i}\, w_2 \cdot \min_{1\leq i\leq n} M_i\rho_i = w_2. \end{aligned}$$

Thus, $\beta(Su) < w_2$ for all $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$.

It now follows from Theorem 2.2 that the system (1.1) has (at least) three *constant-sign* solutions $u^1, u^2, u^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ satisfying (2.2). Furthermore, (2.2) reduces to (3.32) immediately. ∎

4 Applications to boundary value problems

In this section we shall illustrate the generality of the results obtained in Section 3 by considering various well known boundary value problems in the literature. Indeed, we shall apply our results to systems of boundary value problems of the following types: (m,p), Lidstone, focal, conjugate, Hermite, Neumann, Sturm-Liouville and periodic.

Case 4.1. (m,p) boundary value problem

Consider the system of (m,p) boundary value problems

$$\begin{aligned} &u_i^{(m)}(t) + P_i(t,u(t)) = 0,\ t \in [0,1] \\ &u_i^{(j)}(0) = 0,\ 0 \leq j \leq m-2; \qquad u_i^{(p)}(1) = 0 \end{aligned} \tag{4.1}$$

where $i = 1,2,\cdots,n$. For each $1 \leq i \leq n$, assume that $m \geq 2$ is fixed, $1 \leq p \leq m-1$ is fixed, and $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} &-y^{(m)}(t) = 0,\ t \in [0,1] \\ &y^{(j)}(0) = 0,\ 0 \leq j \leq m-2; \qquad y^{(p)}(1) = 0. \end{aligned}$$

It is known that [3, p.191]

(a) $G(t,s) = \dfrac{1}{(m-1)!}\begin{cases} t^{m-1}(1-s)^{m-p-1} - (t-s)^{m-1}, & 0 \leq s \leq t \leq 1 \\ t^{m-1}(1-s)^{m-p-1}, & 0 \leq t \leq s \leq 1; \end{cases}$

(b) $\frac{\partial^j}{\partial t^j} G(t,s) \geq 0,\ \ 0 \leq j \leq p,\ (t,s) \in [0,1]\times[0,1]$;

(c) $G(t,s) > 0,\ \ (t,s) \in (0,1)\times(0,1)$;

(d) $G(t,s) \geq \left(\frac{1}{4}\right)^{m-1} \frac{1}{(m-1)!}(1-s)^{m-p-1}[1-(1-s)^p], \quad (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [0,1]$;

(e) $G(t,s) \leq \frac{1}{(m-1)!}(1-s)^{m-p-1}[1-(1-s)^p], \quad (t,s) \in [0,1] \times [0,1]$.

Now, $u = (u_1, u_2, \cdots, u_n)$ is a solution of the system (4.1) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 G(t,s)P_i(s,u(s))ds, \quad t \in [0,1], \ 1 \leq i \leq n. \tag{4.2}$$

In the context of Section 3, let

$$g_i(t,s) = G(t,s), \ 1 \leq i \leq n, \qquad a = \frac{1}{4}, \qquad b = \frac{3}{4},$$

$$M_i = \left(\frac{1}{4}\right)^{m-1} \quad \text{and} \quad H(s) = \frac{1}{(m-1)!}(1-s)^{m-p-1}[1-(1-s)^p], \ 1 \leq i \leq n. \tag{4.3}$$

Then, noting (a)–(e), we have $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$ and the conditions (C1)–(C3) are fulfilled.

The results in Sections 3 reduce to the following theorem, which is *new* in the literature to date.

Theorem 4.1. *With g_i, a, b, M_i and H given in (4.3), and the various constants given in (3.11), we have the following:*

(i) *(Theorem 3.1) Let (C4)–(C8) hold. Suppose that there exist numbers w_1, w_2, w_3 with*

$$0 < w_1 < w_2 < \frac{w_2}{\min_{1\leq i\leq n} M_i\rho_i} \leq w_3$$

such that the following hold for each $1 \leq i \leq n$:

(P) $f(u_1,u_2,\cdots,u_n) < \frac{w_1}{q_i}$ *for* $\theta_j u_j \in [0,w_1], \ 1 \leq j \leq n$;

(Q) *one of the following holds:*

(Q1) $\limsup_{|u_1|,|u_2|,\cdots,|u_n|\to\infty} \frac{f(u_1,u_2,\cdots,u_n)}{|u_j|} < \frac{1}{q_i}$ *for some* $j \in \{1,2,\cdots,n\}$ *(j depends on i)*;

(Q2) *there exists a number η ($\geq w_3$) such that* $f(u_1,u_2,\cdots,u_n) \leq \frac{\eta}{q_i}$ *for* $\theta_j u_j \in [0,\eta], \ 1 \leq j \leq n$;

(R) $f(u_1,u_2,\cdots,u_n) > \frac{w_2}{r_i}$ *for* $\theta_j u_j \in [w_2,w_3], \ 1 \leq j \leq n$.

Then, the system (4.1) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in C$ such that

$$\|u^1\| < w_1; \qquad |u_i^2(t)| > w_2, \ t \in [a,b], \ 1 \leq i \leq n;$$
$$\|u^3\| > w_1 \quad \text{and} \quad \min_{1\leq i\leq n} \min_{t\in[a,b]} |u_i^3(t)| < w_2.$$

(ii) (*Theorem 3.2*) *Let (C4)–(C6) hold. Assume there exist numbers* τ_j, $1 \le j \le 4$ *with*

$$0 \le \tau_1 \le a \le \tau_2 < \tau_3 \le b \le \tau_4 \le 1$$

such that (C9) and (C10) hold. Suppose that there exist numbers w_i, $2 \le i \le 5$ *with*

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right)$ *for* $\theta_j u_j \in [0, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i}$ *for* $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (4.1) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2, \quad t \in [\tau_1, \tau_4],\ 1 \le i \le n; \qquad |u_i^2(t)| > w_3, \quad t \in [\tau_2, \tau_3],\ 1 \le i \le n;$$

$$\max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \text{and} \quad \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3.$$

(iii) (*Theorem 3.3*) *Let (C4)–(C6) hold. Assume there exist numbers* τ_j, $1 \le j \le 4$ *with*

$$a \le \tau_1 < \tau_2 < \tau_3 < \tau_4 \le b$$

such that (C9) and (C10) hold. Suppose that there exist numbers w_i, $1 \le i \le 5$ *with*

$$0 < w_1 \le w_2 \cdot \min_{1\le i\le n} M_i\rho_i < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_{2,i}}\left(w_2 - \frac{w_5 d_{3,i}}{q_i}\right)$ *for* $\theta_j u_j \in [w_1, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i}$ *for* $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (4.1) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2, \quad t \in [\tau_1, \tau_4],\ 1 \le i \le n; \qquad |u_i^2(t)| > w_3, \quad t \in [\tau_2, \tau_3],\ 1 \le i \le n;$$

$$\max_{1\le i\le n}\ \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \text{and} \quad \min_{1\le i\le n}\ \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3.$$

Case 4.2. Lidstone boundary value problem

Consider the system of Lidstone boundary value problems

$$\begin{aligned} (-1)^m\, u_i^{(2m)}(t) &= P_i(t,u(t)),\ t\in[0,1] \\ u_i^{(2j)}(0) &= u_i^{(2j)}(1) = 0,\ 0\le j\le m-1 \end{aligned} \tag{4.4}$$

where $i = 1,2,\cdots,n$. For each $1 \le i \le n$, assume that $m \ge 1$ is fixed and $P_i : [0,1]\times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G_m(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} y^{(2m)}(t) &= 0,\ t\in[0,1] \\ y^{(2j)}(0) &= y^{(2j)}(1) = 0,\ 0\le j\le m-1. \end{aligned}$$

It is known that [31]

(a) $G_m(t,s) = \int_0^1 G(t,u)G_{m-1}(u,s)du$ where

$$G_1(t,s) = G(t,s) = \begin{cases} t(s-1), & 0\le t\le s\le 1 \\ s(t-1), & 0\le s\le t\le 1; \end{cases}$$

(b) $(-1)^m G_m(t,s) \ge 0,\quad (t,s)\in[0,1]\times[0,1]$;

(c) $(-1)^m G_m(t,s) > 0,\quad (t,s)\in(0,1)\times(0,1)$;

(d) $(-1)^m G_m(t,s) \ge 4^{-m}\left(\frac{3}{32}\right)^{m-1} s(1-s),\quad (t,s)\in\left[\frac14,\frac34\right]\times[0,1]$;

(e) $(-1)^m G_m(t,s) \le 6^{-(m-1)}s(1-s),\quad (t,s)\in[0,1]\times[0,1]$.

Clearly, $u = (u_1,u_2,\cdots,u_n)$ is a solution of the system (4.4) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 (-1)^m G_m(t,s)P_i(s,u(s))ds,\quad t\in[0,1],\ 1\le i\le n. \tag{4.5}$$

In the context of Section 3, let

$$\begin{aligned} &g_i(t,s) = (-1)^m G_m(t,s),\ 1\le i\le n,\qquad a=\frac14,\qquad b=\frac34, \\ &M_i = \frac{6^{m-1}}{4^m}\left(\frac{3}{32}\right)^{m-1} = 4^{-m}\left(\frac{9}{16}\right)^{m-1}\ \text{and} \\ &H(s) = 6^{-(m-1)}s(1-s),\quad 1\le i\le n. \end{aligned} \tag{4.6}$$

Then, the conditions (C1)–(C3) are satisfied in view of (a)–(e) (note that $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$).

Applying the results in Section 3, we obtain the following theorem which improves and extends the earlier work of [14, 31] (for $n = 1$). Note that the P_i considered in (4.4) as well as the methodology used are both more general.

Theorem 4.2. *With g_i, a, b, M_i and H given in (4.6), the statements (i)–(iii) of Theorem 4.1 hold for system (4.4).*

Case 4.3. Focal boundary value problem

Consider the system of focal boundary value problems

$$\begin{aligned} &(-1)^{m-p}\, u^{(m)}(t) = P_i(t, u(t)),\ t \in [0,1] \\ &u_i^{(j)}(0) = 0,\ 0 \le j \le p-1; \qquad u_i^{(j)}(1) = 0,\ p \le j \le m-1 \end{aligned} \tag{4.7}$$

where $i = 1, 2, \cdots, n$. For each $1 \le i \le n$, assume that $m \ge 2$ is fixed, $1 \le p \le m-1$ is fixed, and $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} &y^{(m)}(t) = 0,\ t \in [0,1] \\ &y^{(j)}(0) = 0,\ 0 \le j \le p-1; \qquad y^{(j)}(1) = 0,\ p \le j \le m-1. \end{aligned}$$

In [3, p.211] it is documented that

(a) $$G(t,s) = \frac{1}{(m-1)!}\begin{cases} \displaystyle\sum_{j=0}^{p-1} \binom{m-1}{j} t^j (-s)^{m-j-1}, & 0 \le s \le t \le 1 \\ \displaystyle -\sum_{j=p}^{m-1} \binom{m-1}{j} t^j (-s)^{m-j-1}, & 0 \le t \le s \le 1; \end{cases}$$

(b) for $(t,s) \in [0,1] \times [0,1]$,

$$\begin{cases} (-1)^{m-p} \dfrac{\partial}{\partial t^j}\, G(t,s) \ge 0, & 0 \le j \le p-1 \\ (-1)^{m-j} \dfrac{\partial}{\partial t^j}\, G(t,s) \ge 0, & p \le j \le m-1; \end{cases}$$

(c) $(-1)^{m-p} G(t,s) > 0,\quad (t,s) \in (0,1) \times (0,1)$;

(d) for a given $\delta \in \left(0, \frac{1}{2}\right)$,

$$(-1)^{m-p} G(t,s) \ge (-1)^{m-p} G(1,s) \inf_{z \in [0,1]} \frac{G(\delta, z)}{G(1,z)},\quad (t,s) \in [\delta, 1-\delta] \times [0,1];$$

(e) $(-1)^{m-p} G(t,s) \le (-1)^{m-p} G(1,s),\quad (t,s) \in [0,1] \times [0,1]$.

Obviously, $u = (u_1, u_2, \cdots, u_n)$ is a solution of the system (4.7) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 (-1)^{m-p} G(t,s) P_i(s, u(s))ds, \quad t \in [0,1], \ 1 \le i \le n. \tag{4.8}$$

Let $\delta \in \left(0, \frac{1}{2}\right)$ be fixed. In the context of Section 3, let

$$\begin{aligned} g_i(t,s) &= (-1)^{m-p} G(t,s), \ 1 \le i \le n, \qquad a = \delta, \qquad b = 1 - \delta, \\ M_i &= \inf_{z \in [0,1]} \frac{G(\delta, z)}{G(1,z)} \qquad \text{and} \qquad H(s) = (-1)^{m-p} G(1,s), \ 1 \le i \le n. \end{aligned} \tag{4.9}$$

Then, from (a)–(e) we see that the conditions (C1)–(C3) are satisfied (note that $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$).

The results in Section 3 reduce to the following theorem which improves and extends the earlier work of [7, 8, 10, 28] (for $n = 1$). We remark that the P_i considered in (4.7) as well as the methodology used are both more general.

Theorem 4.3. *With g_i, a, b, M_i and H given in (4.9), the statements (i)–(iii) of Theorem 4.1 hold for system (4.7).*

Case 4.4. Conjugate boundary value problem

Consider the system of conjugate boundary value problems

$$\begin{aligned} &(-1)^{m-p}\, u^{(m)}(t) = P_i(t, u(t)), \ t \in [0,1] \\ &u_i^{(j)}(0) = 0, \ 0 \le j \le p-1; \qquad u_i^{(j)}(1) = 0, \ 0 \le j \le m-p-1 \end{aligned} \tag{4.10}$$

where $i = 1, 2, \cdots, n$. For each $1 \le i \le n$, assume that $m \ge 2$ is fixed, $1 \le p \le m-1$ is fixed, and $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} &y^{(m)}(t) = 0, \ t \in [0,1] \\ &y^{(j)}(0) = 0, \ 0 \le j \le p-1; \qquad y^{(j)}(1) = 0, \ 0 \le j \le m-p-1. \end{aligned}$$

It is known that [23, 25, 29]

(a) $$G(t,s) = \begin{cases} \displaystyle\sum_{j=0}^{p-1} \left[\sum_{\tau=0}^{p-1-j} \binom{m-p+\tau-1}{\tau} t^\tau \right] \frac{t^j(-s)^{m-j-1}}{j!(m-j-1)!} (1-t)^{m-p}, \\ \qquad\qquad 0 \le s \le t \le 1 \\ \displaystyle -\sum_{j=0}^{m-p-1} \left[\sum_{\tau=0}^{m-p-1-j} \binom{p+\tau-1}{\tau} (1-t)^\tau \right] \frac{(t-1)^j(1-s)^{m-j-1}}{j!(m-j-1)!} t^p, \\ \qquad\qquad 0 \le t \le s \le 1; \end{cases}$$

(b) $(-1)^{m-p} G(t,s) \ge 0, \quad (t,s) \in [0,1] \times [0,1];$

(c) $(-1)^{m-p}G(t,s) > 0$, $(t,s) \in (0,1) \times (0,1)$;

(d) for a fixed $\delta \in (0,1/2)$,

$$(-1)^{m-p}G(t,s) \geq \nu_\delta \|G(\cdot,s)\|, \quad (t,s) \in [\delta, 1-\delta] \times [0,1]$$

where

$$\|G(\cdot,s)\| = \sup_{t\in[0,1]} |G(t,s)| = \sup_{t\in[0,1]} (-1)^{m-p}G(t,s),$$

the constant $0 < \nu_\delta < 1$ is given by

$$\nu_\delta = \min\left\{ b(p)\cdot\min\{c(p), c(m-p-1)\},\ b(p-1)\cdot\min\{c(p-1), c(m-p)\}\right\},$$

and the functions b and c are defined as

$$b(t) = \frac{(m-1)^{m-1}}{t^t(m-t-1)^{m-t-1}} \quad \text{and} \quad c(t) = \delta^t(1-\delta)^{m-t-1};$$

(e) $(-1)^{m-p}G(t,s) \leq \|G(\cdot,s)\|$, $(t,s) \in [0,1]\times[0,1]$.

Now, $u = (u_1, u_2, \cdots, u_n)$ is a solution of the system (4.10) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 (-1)^{m-p}G(t,s)P_i(s,u(s))ds, \quad t \in [0,1],\ 1 \leq i \leq n. \tag{4.11}$$

Let $\delta \in \left(0, \frac{1}{2}\right)$ be fixed. In the context of Section 3, let

$$\begin{aligned} g_i(t,s) &= (-1)^{m-p}G(t,s),\ 1 \leq i \leq n, \quad a = \delta, \quad b = 1-\delta, \\ M_i &= \nu_\delta \quad \text{and} \quad H(s) = \|G(\cdot,s)\|,\ 1 \leq i \leq n. \end{aligned} \tag{4.12}$$

Then, (a)–(e) ensures that the conditions (C1)–(C3) are fulfilled (note that $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$).

Applying the results in Section 3, we obtain the following theorem which improves and extends the earlier work of [11–13,23] (for $n=1$) and [25] (on systems). Note that the P_i considered in (4.10) as well as the methodology used are both more general.

Theorem 4.4. *With g_i, a, b, M_i and H given in (4.12), the statements (i)–(iii) of Theorem 4.1 hold for system (4.10).*

Case 4.5. Hermite boundary value problem

Let $r \geq 2$ and $0 = t_1 < t_2 < \cdots < t_r = 1$ be given. Consider the system of Hermite boundary value problems

$$\begin{aligned} u_i^{(m)}(t) &= P_i(t, u(t)),\ t \in [0,1] \\ u_i^{(j)}(t_k) &= 0,\ j = 0,\cdots,m_k - 1,\ k = 1,\cdots,r \end{aligned} \tag{4.13}$$

where $i = 1, 2, \cdots, n$. For each $1 \le k \le r$, assume $m_k \ge 1$ is fixed with $\sum_{k=1}^{r} m_k = m$, and for each $1 \le i \le n$, let $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a L^1-Carathéodory function.

For each $k = 1, \cdots, r-1$, define the constant γ_k and the interval I_k as

$$\gamma_k = \sum_{j=k+1}^{r} m_j \quad \text{and} \quad I_k = \left[\frac{3t_k + t_{k+1}}{4}, \frac{t_k + 3t_{k+1}}{4}\right].$$

Let $G(t,s)$ be the Green's function of the boundary value problem

$$y^{(m)}(t) = 0, \ t \in [0,1]$$
$$y^{(j)}(t_k) = 0, \ j = 0, \cdots, m_k - 1, \ k = 1, \cdots, r.$$

It is well known that [26, 30]

(a) $G(t,s) \in C[0,1] \subseteq L^\infty[0,1]$, $t \in [0,1]$ and the map $t \to G(t,s)$ is continuous from $[0,1]$ to $C[0,1]$;

(b) $(-1)^{\gamma_k} G(t,s) \ge 0$, $(t,s) \in [t_k, t_{k+1}] \times [0,1]$, $k = 1, \cdots, r-1$;

(c) $(-1)^{\gamma_k} G(t,s) > 0$, $(t,s) \in (t_k, t_{k+1}) \times (0,1)$, $k = 1, \cdots, r-1$;

(d) for each $k = 1, \cdots, r-1$,

$$(-1)^{\gamma_k} G(t,s) \ge L_k \|G(\cdot,s)\|, \quad (t,s) \in I_k \times [0,1]$$

where

$$\|G(\cdot,s)\| = \sup_{t\in[0,1]} |G(t,s)| = \max_{1\le j\le r-1} \sup_{t\in[t_j,t_{j+1}]} (-1)^{\gamma_j} G(t,s),$$

the constant $0 < L_k < 1$ is given by

$$L_k = \min\left\{ \min\left\{ R\left(\frac{3t_k + t_{k+1}}{4}\right), R\left(\frac{t_k + 3t_{k+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} R(t), \right.$$
$$\left. \min\left\{ Q\left(\frac{3t_k + t_{k+1}}{4}\right), Q\left(\frac{t_k + 3t_{k+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} Q(t) \right\},$$

and the functions R and Q are defined as

$$R(t) = \prod_{j=1}^{r-1} |t - t_j|^{m_j} (1-t)^{m_r - 1} \quad \text{and} \quad Q(t) = t^{m_1 - 1} \prod_{j=2}^{r} |t - t_j|^{m_j};$$

(e) $(-1)^{\gamma_k} G(t,s) \le \|G(\cdot,s)\|$, $(t,s) \in [t_k, t_{k+1}] \times [0,1]$, $k = 1, \cdots, r-1$.

We say that $u = (u_1, u_2, \cdots, u_n)$ is a solution of *constant sign* if for each $1 \le i \le n$, we have $(-1)^{\gamma_k}\theta_i u_i \ge 0$ on $[t_k, t_{k+1}]$, $1 \le k \le r-1$ where $\theta_i \in \{1, -1\}$ is fixed.

In the context of Section 3, let the Banach space $B = (C[0,1])^n$ be equipped with norm $\|u\| = \max_{1\le i\le n} \sup_{t\in[0,1]} |u_i(t)| = \max_{1\le i\le n} |u_i|_0$ where we let

$$|u_i|_0 = \sup_{t\in[0,1]} |u_i(t)| = \max_{1\le k\le r-1} \max_{t\in[t_k,t_{k+1}]} |u_i(t)|,\ 1 \le i \le n.$$

Define the cone C in B as

$$C = \left\{ u \in B \,\middle|\, \text{for each } 1 \le i \le n, (-1)^{\gamma_k}\theta_i u_i(t) \ge 0 \text{ for } t \in [t_k, t_{k+1}], k = 1, \cdots, r-1 \right.$$
$$\left. \text{and } \min_{t\in I_k} (-1)^{\gamma_k}\theta_i u_i(t) \ge L_k \rho_i |u_i|_0,\ k = 1, \cdots, r-1 \right\}. \tag{4.14}$$

Clearly, if $u \in C$, then u is of constant-sign.

Now, $u = (u_1, u_2, \cdots, u_n)$ is a solution of the system (4.13) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 G(t,s)P_i(s,u(s))ds,\quad t \in [0,1],\ 1 \le i \le n. \tag{4.15}$$

It can be verified that S maps C into C.

In the context of Section 3, let

$$g_i(t,s) = (-1)^{\gamma_k}G(t,s),\ 1 \le i \le n,\quad a = \frac{3t_k + t_{k+1}}{4},\quad b = \frac{t_k + 3t_{k+1}}{4},$$
$$M_i = L_k \text{ and } H(s) = \|G(\cdot,s)\|,\quad 1 \le i \le n. \tag{4.16}$$

Then, noting (a)–(e), we have $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$, the conditions (C1), (C3) and (C2) (for $k = 1, 2, \cdots, r-1$) are fulfilled. Moreover, the constants defined earlier in (3.11) are now modified appropriately. We define the following constants for each $1 \le i \le n$ and fixed numbers $\tau_{j,k} \in [0,1]$, $1 \le j \le 4$, $1 \le k \le r-1$:

$$\begin{aligned}
q_i = q &= \max_{1\le k\le r-1} \max_{t\in[t_k,t_{k+1}]} \int_0^1 (-1)^{\gamma_k}G(t,s)b(s)ds,\\
r_i &= \min_{1\le k\le r-1} \min_{t\in I_k} \int_{I_k} (-1)^{\gamma_k}G(t,s)a_i(s)ds,\\
d_{1,i} &= \min_{1\le k\le r-1} \min_{t\in[\tau_{2,k},\tau_{3,k}]} \int_{\tau_{2,k}}^{\tau_{3,k}} (-1)^{\gamma_k}G(t,s)a_i(s)ds,\\
d_{2,i} = d_2 &= \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} \int_{\tau_{1,k}}^{\tau_{4,k}} (-1)^{\gamma_k}G(t,s)b(s)ds,\\
d_{3,i} = d_3 &= \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} \left[\int_0^{\tau_{1,k}} (-1)^{\gamma_k}G(t,s)b(s)ds \right.\\
&\quad \left. + \int_{\tau_{4,k}}^1 (-1)^{\gamma_k}G(t,s)b(s)ds \right].
\end{aligned} \tag{4.17}$$

A modification of the argument in Section 3 yields the following theorem, which improves and extends [30] (for $n = 1$). We refer the reader to [32] for details.

Theorem 4.5. *With the constants defined in (4.17), we have the following:*

(i) *(Theorem 3.1) Let (C4)–(C6) hold, and assume*

(C7)′ *for each $1 \le k \le r-1$ and each $t \in [t_k, t_{k+1}]$, the function $G(t,s)b(s)$ is nonzero on a subset of $[0,1]$ of positive measure;*

(C8)′ *for each $1 \le i \le n$, each $1 \le k \le r-1$ and each $t \in I_k$, the function $G(t,s)a_i(s)$ is nonzero on a subset of I_k of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{\min_{1\le i\le n} \min_{1\le k\le r-1} L_k\rho_i} < w_3$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{w_1}{q}$ *for* $|u_j| \in [0, w_1]$, $1 \le j \le n$;

(Q) *one of the following holds:*

(Q1) $\limsup_{|u_1|,|u_2|,\cdots,|u_n|\to\infty} \frac{f(u_1,u_2,\cdots,u_n)}{|u_j|} < \frac{1}{q}$ *for some* $j \in \{1, 2, \cdots, n\}$;

(Q2) *there exists a number η ($\ge w_3$) such that* $f(u_1, u_2, \cdots, u_n) \le \frac{\eta}{q}$ *for* $|u_j| \in [0, \eta]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_2}{r_i}$ *for* $|u_j| \in [w_2, w_3]$, $1 \le j \le n$.

Then, the system (4.13) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in C$ such that

$$\|u^1\| < w_1; \qquad |u_i^2(t)| > w_2, \;\; t \in I_k, \;\; 1 \le k \le r-1, \;\; 1 \le i \le n;$$
$$\|u^3\| > w_1 \quad \text{and} \quad \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in I_k} |u_i^3(t)| < w_2.$$

(ii) *(Theorem 3.2) Let (C4)–(C6) hold. Assume there exist numbers $\tau_{j,k}$, $1 \le j \le 4$, $1 \le k \le r-1$ with*

$$0 \le t_k \le \tau_{1,k} \le \frac{3t_k + t_{k+1}}{4} \le \tau_{2,k} < \tau_{3,k} \le \frac{t_k + 3t_{k+1}}{4} \le \tau_{4,k} \le t_{k+1} \le 1$$

such that

(C9)′ *for each $1 \le i \le n$, each $1 \le k \le r-1$, and each $t \in [\tau_{2,k}, \tau_{3,k}]$, the function $G(t,s)a_i(s)$ is nonzero on a subset of $[\tau_{2,k}, \tau_{3,k}]$ of positive measure;*

(C10)′ *for each $1 \le k \le r-1$ and each $t \in [\tau_{1,k}, \tau_{4,k}]$, the function $G(t,s)b(s)$ is nonzero on a subset of $[\tau_{1,k}, \tau_{4,k}]$ of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$ *with*

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n}\ \min_{1\le k\le r-1} L_k\rho_i} < w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)$ *for* $|u_j| \in [0, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q}$ *for* $|u_j| \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $|u_j| \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (4.13) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2,\quad t \in [\tau_{1,k}, \tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le i \le n;$$

$$|u_i^2(t)| > w_3,\quad t \in [\tau_{2,k}, \tau_{3,k}],\ 1 \le k \le r-1,\ 1 \le i \le n;$$

$$\max_{1\le i\le n}\ \max_{1\le k\le r-1}\ \max_{t\in[\tau_{1,k},\tau_{4,k}]} |u_i^3(t)| > w_2 \quad \text{and}$$

$$\min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} |u_i^3(t)| < w_3.$$

(iii) (*Theorem 3.3*) *Let* (C4)–(C6) *hold. Assume there exist numbers* $\tau_{j,k}$, $1 \le j \le 4$, $1 \le k \le r-1$ *with*

$$\frac{3t_k + t_{k+1}}{4} \le \tau_{1,k} < \tau_{2,k} < \tau_{3,k} < \tau_{4,k} \le \frac{t_k + 3t_{k+1}}{4}$$

such that (C9)′ *and* (C10)′ *hold. Suppose that there exist numbers* w_i, $1 \le i \le 5$ *with*

$$0 < w_1 < w_2 \cdot \min_{1\le i\le n}\ \min_{1\le k\le r-1} L_k\rho_i < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n}\ \min_{1\le k\le r-1} L_k\rho_i} < w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)$ *for* $|u_j| \in [w_1, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q}$ *for* $|u_j| \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $|u_j| \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (4.13) *has (at least) three constant-sign solutions* $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2,\quad t \in [\tau_{1,k}, \tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le i \le n;$$

$$|u_i^2(t)| > w_3,\quad t \in [\tau_{2,k}, \tau_{3,k}],\ 1 \le k \le r-1,\ 1 \le i \le n;$$

$$\max_{1\le i\le n}\ \max_{1\le k\le r-1}\ \max_{t\in[\tau_{1,k},\tau_{4,k}]} |u_i^3(t)| > w_2 \quad \text{and}$$

$$\min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} |u_i^3(t)| < w_3.$$

Case 4.6. Neumann boundary value problem

Consider the following two systems of Neumann boundary value problems

$$\begin{aligned} -u_i''(t) + cu_i(t) &= P_i(t, u(t)),\ t \in [0,1] \\ u_i'(0) &= u_i'(1) = 0 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} u_i''(t) + ru_i(t) &= P_i(t, u(t)),\ t \in [0,1] \\ u_i'(0) &= u_i'(1) = 0 \end{aligned} \tag{4.19}$$

where $i = 1, 2, \cdots, n$. For each $1 \le i \le n$, assume that $c > 0$ is fixed, $0 < r < \frac{\pi^2}{4}$ is fixed and $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G^{(4.18)}(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} -y''(t) + cy(t) &= 0,\ t \in [0,1] \\ y'(0) &= y'(1) = 0, \end{aligned}$$

and let $G^{(4.19)}(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned} y''(t) + ry(t) &= 0,\ t \in [0,1] \\ y'(0) &= y'(1) = 0. \end{aligned}$$

It is known that [18]

(a) $$G^{(4.18)}(t,s) = \frac{1}{\sqrt{c}\sinh\sqrt{c}} \begin{cases} \cosh(\sqrt{c}(1-t))\cosh(\sqrt{c}s), & 0 \le s \le t \le 1 \\ \cosh(\sqrt{c}(1-s))\cosh(\sqrt{c}t), & 0 \le t \le s \le 1; \end{cases}$$

$$G^{(4.19)}(t,s) = \frac{1}{\sqrt{r}\sin\sqrt{r}} \begin{cases} \cos(\sqrt{r}(1-t))\cos(\sqrt{r}s), & 0 \le s \le t \le 1 \\ \cos(\sqrt{r}(1-s))\cos(\sqrt{r}t), & 0 \le t \le s \le 1; \end{cases}$$

(b) $G^{(4.18)}(t,s) \ge 0$ and $G^{(4.19)}(t,s) \ge 0$, $(t,s) \in [0,1] \times [0,1]$;

(c) $G^{(4.18)}(t,s) > 0$ and $G^{(4.19)}(t,s) > 0$, $(t,s) \in (0,1) \times (0,1)$;

(d) $G^{(4.18)}(t,s) \ge \frac{1}{\sqrt{c}\sinh\sqrt{c}}$ and $G^{(4.19)}(t,s) \ge \frac{\cos^2\sqrt{r}}{\sqrt{r}\sin\sqrt{r}}$, $(t,s) \in [0,1] \times [0,1]$;

(e) $G^{(4.18)}(t,s) \le \frac{\cosh^2\sqrt{c}}{\sqrt{c}\sinh\sqrt{c}}$ and $G^{(4.19)}(t,s) \le \frac{1}{\sqrt{r}\sin\sqrt{r}}$, $(t,s) \in [0,1] \times [0,1]$.

Now, $u = (u_1, u_2, \cdots, u_n)$ is a solution of the system (4.18) if and only if u is a fixed point of the operator $S : (C[0,1])^n \to (C[0,1])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^1 G^{(4.18)}(t,s)P_i(s, u(s))ds, \quad t \in [0,1],\ 1 \le i \le n. \tag{4.20}$$

Likewise, u is a solution of the system (4.19) provided $u = Su$ where

$$Su_i(t) = \int_0^1 G^{(4.19)}(t,s)P_i(s,u(s))ds, \quad t \in [0,1],\ 1 \le i \le n. \tag{4.21}$$

In the context of Section 3, for system (4.18) let

$$\begin{gathered} g_i(t,s) = G^{(4.18)}(t,s),\ 1 \le i \le n, \quad a = 0, \quad b = 1, \\ M_i = \frac{1}{\cosh^2\sqrt{c}} \quad \text{and} \quad H(s) = \frac{\cosh^2\sqrt{c}}{\sqrt{c}\sinh\sqrt{c}},\ 1 \le i \le n \end{gathered} \tag{4.22}$$

whereas for system (4.19), let

$$\begin{gathered} g_i(t,s) = G^{(4.19)}(t,s),\ 1 \le i \le n, \quad a = 0, \quad b = 1, \\ M_i = \cos^2\sqrt{r} \quad \text{and} \quad H(s) = \frac{1}{\sqrt{r}\sin\sqrt{r}},\ 1 \le i \le n. \end{gathered} \tag{4.23}$$

Then, noting (a)–(e), it is clear that $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$ and the conditions (C1)–(C3) are fulfilled for both systems (4.18) and (4.19).

The results in Section 3 reduce to the following theorem, which not only extends the work of Jiang and Lui [18] for the special cases of (4.18) and (4.19) when $n = 1$, but in particular provides the existence of *triple constant-sign* solutions.

Theorem 4.6.

(a) *With g_i, a, b, M_i and H given in (4.22), the statements (i)–(iii) of Theorem 4.1 hold for system (4.18).*

(b) *With g_i, a, b, M_i and H given in (4.23), the statements (i)–(iii) of Theorem 4.1 hold for system (4.19).*

Case 4.7. Sturm-Liouville boundary value problem

Consider the system of Sturm-Liouville boundary value problems

$$\begin{gathered} u_i^{(m_i)}(t) + P_i(t,u) = 0,\ t \in [0,1] \\ u_i^{(j)}(0) = 0,\ 0 \le j \le m_i - 3 \\ \zeta u_i^{(m_i-2)}(0) - \eta u_i^{(m_i-1)}(0) = 0, \qquad \omega u_i^{(m_i-2)}(1) + \delta u_i^{(m_i-1)}(1) = 0 \end{gathered} \tag{4.24}$$

where $i = 1,2,\cdots,n$. For each $1 \le i \le n$, assume that $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, $m_i \ge 2$ is fixed, ζ, η, ω and δ are such that

$$\eta \ge 0, \quad \delta \ge 0, \quad \eta + \zeta > 0, \quad \delta + \omega > 0, \quad \Gamma \equiv \zeta\omega + \zeta\delta + \eta\omega > 0.$$

These assumptions allow ζ and ω to be negative.

Let $h_i(t,s)$ be the Green's function of the boundary value problem

$$-y^{(m_i)}(t)=0,\ t\in[0,1]$$
$$y^{(j)}(0)=0,\ 0\le j\le m_i-3$$
$$\zeta y^{(m_i-2)}(0)-\eta y^{(m_i-1)}(0)=0,\qquad \omega y^{(m_i-2)}(1)+\delta y^{(m_i-1)}(1)=0.$$

It can be verified [24] that $G(t,s)$ where

$$G(t,s)=\frac{\partial^{m_i-2}}{\partial t^{m_i-2}}\,h_i(t,s)=h_i^{(m_i-2)}(t,s) \tag{4.25}$$

is the Green's function of the boundary value problem

$$-y''(t)=0,\ t\in[0,1]$$
$$\zeta y(0)-\eta y'(0)=0;\qquad \omega y(1)+\delta y'(1)=0.$$

Further, it is known that [24]

(a) $G(t,s)=\dfrac{1}{\Gamma}\begin{cases}(\eta+\zeta s)[\delta+\omega(1-t)], & 0\le s\le t\\ (\eta+\zeta t)[\delta+\omega(1-s)], & 0\le t\le s\le 1;\end{cases}$

(b) $G(t,s)\ge 0,\quad (t,s)\in[0,1]\times[0,1]$;

(c) $G(t,s)>0,\quad (t,s)\in(0,1)\times(0,1)$;

(d) $G(t,s)\ge A\,G(s,s),\quad (t,s)\in\left[\frac14,\frac34\right]\times[0,1]$ where $0<A<1$ is given by

$$A=\min\left\{\frac{4\eta+\zeta}{4(\eta+\zeta)},\ \frac{4\delta+\omega}{4(\delta+\omega)},\ \frac{4\eta+3\zeta}{4\eta+\zeta},\ \frac{4\delta+3\omega}{4\delta+\omega}\right\};$$

(e) $G(t,s)\le D\,G(s,s),\quad (t,s)\in[0,1]\times[0,1]$ where $D\ge 1$ is given by

$$D=\max\left\{1,\ \frac{\eta}{\eta+\zeta},\ \frac{\delta}{\delta+\omega}\right\}.$$

In the context of Section 3, let the Banach space

$$B=\left\{u=(u_1,u_2,\cdots,u_n)\in(C^{(m_i)}[0,1])^n\ \middle|\ u_i^{(j)}(0)=0,\ 0\le j\le m_i-3,\ 1\le i\le n\right\} \tag{4.26}$$

be equipped with norm

$$\|u\|=\max_{1\le i\le n}\ \sup_{t\in[0,1]}|u_i^{(m_i-2)}(t)|=\max_{1\le i\le n}|u_i|_0 \tag{4.27}$$

where we let $|u_i|_0=\sup_{t\in[0,1]}|u_i^{(m_i-2)}(t)|$, $1\le i\le n$. Further, define the cone C in B as

$$C=\left\{u=(u_1,u_2,\cdots,u_n)\in B\ \middle|\ \text{for each } 1\le i\le n,\ \theta_i u_i^{(m_i-2)}(t)\ge 0 \text{ for } t\in[0,1],\right.$$

$$\left.\text{and} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \theta_i u_i^{(m_i-2)}(t) \geq M_i\rho_i|u_i|_0\right\} \tag{4.28}$$

where $M_i = \frac{A}{D} \in (0,1)$, $1 \leq i \leq n$. It can be verified that S maps C into C.

If $u = (u_1, u_2, \cdots, u_n) \in C$ is a solution of (4.24), then u is of constant sign (see [24]). Clearly, u is a solution of the system (4.24) if and only if u is a fixed point of the operator $S : B \to B$ defined by (3.3) where

$$Su_i(t) = \int_0^1 h_i(t,s)P_i(s,u(s))ds, \quad t \in [0,1], \ 1 \leq i \leq n \tag{4.29}$$

or equivalently

$$(Su_i)^{(m_i-2)}(t) = \int_0^1 G(t,s)P_i(s,u(s))ds, \quad t \in [0,1], \ 1 \leq i \leq n. \tag{4.30}$$

Now, in the context of Section 3, let

$$\begin{aligned} &g_i(t,s) = G(t,s), \ 1 \leq i \leq n, \qquad a = \frac{1}{4}, \qquad b = \frac{3}{4}, \\ &M_i = \frac{A}{D} \quad \text{and} \quad H(s) = D\, G(s,s), \ 1 \leq i \leq n. \end{aligned} \tag{4.31}$$

Then, noting (a)–(e), we see that $g_i^t(s) \equiv g_i(t,s) \in C[0,1] \subseteq L^\infty[0,1]$ and the conditions (C1)–(C3) are fulfilled. The constants defined earlier in (3.11) are now modified appropriately. For each $1 \leq i \leq n$ and fixed numbers $\tau_j \in [0,1]$, $1 \leq j \leq 4$ we define the following:

$$\begin{aligned}
q_i = q &= \sup_{t\in[0,1]} \int_0^1 G(t,s)b(s)ds, \\
r_i &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)a_i(s)ds, \\
d_{1,i} &= \min_{t\in[\tau_2,\tau_3]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)a_i(s)ds, \\
d_{2,i} = d_2 &= \max_{t\in[\tau_1,\tau_4]} \int_{\tau_1}^{\tau_4} G(t,s)b(s)ds, \\
d_{3,i} = d_3 &= \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\tau_1} G(t,s)b(s)ds + \int_{\tau_4}^1 G(t,s)b(s)ds\right], \\
d_4 &= \max_{t\in[\tau_1,\tau_4]} \int_{\max\{\tau_1,\frac{1}{2}\}}^{\tau_4} G(t,s)b(s)ds, \\
d_5 &= \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\max\{\tau_1,\frac{1}{2}\}} G(t,s)b(s)ds + \int_{\tau_4}^1 G(t,s)b(s)ds\right].
\end{aligned} \tag{4.32}$$

A modification of the argument in Section 3 yields the following theorem (see [27] for details).

Theorem 4.7. *With the constants defined in (4.32), we have the following:*

(i) (Theorem 3.1) *Let (C4)–(C7) hold, and assume*

(C8)″ *for each* $1 \le i \le n$ *and each* $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, *the function* $G(t,s)a_i(s)$ *is nonzero on a subset of* $\left[\frac{1}{2}, \frac{3}{4}\right]$ *of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 *with*

$$0 < w_1 < w_2 < \frac{w_2}{\frac{A}{D}\min_{1\le i\le n}\rho_i} \le w_3$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{w_1}{q}$ *for* $\theta_j u_j \in \left[0, \frac{w_1}{(m_j-2)!}\right]$, $1 \le j \le n$;

(Q) *one of the following holds:*

(Q1) $\limsup_{|u_1|,|u_2|,\cdots,|u_n|\to\infty} \frac{f(u_1,u_2,\cdots,u_n)}{|u_j|} < \frac{1}{q}$ *for some* $j \in \{1, 2, \cdots, n\}$;

(Q2) *there exists a number* η $(\ge w_3)$ *such that* $f(u_1, u_2, \cdots, u_n) \le \frac{\eta}{q}$ *for* $\theta_j u_j \in \left[0, \frac{\eta}{(m_j-2)!}\right]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_2}{\tau_i}$ *for* $\theta_j u_j \in \left[\frac{Aw_2\rho_j}{D4^{m_j-2}(m_j-2)!}, \frac{w_3}{(m_j-2)!}\right]$, $1 \le j \le n$.

Then, the system (4.24) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in C$ *such that*

$$\|u^1\| < w_1; \qquad |(u_i^2)^{(m_i-2)}(t)| > w_2, \quad t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right], \ 1 \le i \le n;$$

$$\|u^3\| > w_1 \quad \text{and} \quad \min_{1\le i\le n} \min_{t\in[\frac{1}{4},\frac{3}{4}]} |(u_i^3)^{(m_i-2)}(t)| < w_2.$$

(ii) (Theorem 3.2) *Let (C4)–(C6) hold. Assume there exist numbers* τ_j, $1 \le j \le 4$ *with*

$$0 \le \tau_1 \le \frac{1}{4} \le \tau_2 < \tau_3 \le \frac{3}{4} \le \tau_4 \le 1$$

such that (C10) holds and

(C9)″ *for each* $1 \le i \le n$ *and each* $t \in [\tau_2, \tau_3]$, *the function* $G(t,s)a_i(s)$ *is nonzero on a subset of* $\left[\frac{1}{2}, \frac{3}{4}\right]$ *of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$ *with*

$$0 < w_2 < w_3 < \frac{w_3}{\frac{A}{D}\min_{1\le i\le n}\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)$ *for* $\theta_j u_j \in \left[0, \frac{w_2 \tau_4^{m_j-2}}{(m_j-2)!}\right]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q}$ *for* $\theta_j u_j \in \left[0, \frac{w_5}{(m_j-2)!}\right]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in \left[\frac{A w_3 \rho_j}{D4^{m_j-2}(m_j-2)!}, \frac{w_4 \tau_4^{m_j-2}}{(m_j-2)!}\right]$, $1 \le j \le n$.

Then, the system (4.24) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|(u_i^1)^{(m_i-2)}(t)| < w_2, \quad t \in [\tau_1, \tau_4], \ 1 \le i \le n;$$

$$|(u_i^2)^{(m_i-2)}(t)| > w_3, \quad t \in [\tau_2, \tau_3], \ 1 \le i \le n;$$

$$\max_{1 \le i \le n} \max_{t \in [\tau_1, \tau_4]} |(u_i^3)^{(m_i-2)}(t)| > w_2 \quad \textit{and} \quad \min_{1 \le i \le n} \min_{t \in [\tau_2, \tau_3]} |(u_i^3)^{(m_i-2)}(t)| < w_3.$$

(iii) *(*Theorem 3.3*) Let (C4)-(C6) hold. Assume there exist numbers* τ_j, $1 \le j \le 4$ *with*

$$\frac{1}{4} \le \tau_1 < \tau_2 < \tau_3 < \tau_4 \le \frac{3}{4}$$

such that (C9)'' holds and

(C10)'' *for each* $t \in [\tau_1, \tau_4]$*, the function* $G(t,s)b(s)$ *is nonzero on a subset of* $\left[\max\left\{\tau_1, \frac{1}{2}\right\}, \tau_4\right]$ *of positive measure.*

Suppose that there exist numbers w_i, $1 \le i \le 5$ *with*

$$0 < w_1 \le w_2 \cdot \frac{A}{D} \min_{1 \le i \le n} \rho_i < w_2 < w_3 < \frac{w_3}{\frac{A}{D} \min_{1 \le i \le n} \rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_4}\left(w_2 - \frac{w_5 d_5}{q}\right)$ *for* $\theta_j u_j \in \left[\frac{A w_1 \rho_j}{D4^{m_j-2}(m_j-2)!}, \frac{w_2 \tau_4^{m_j-2}}{(m_j-2)!}\right]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q}$ *for* $\theta_j u_j \in \left[0, \frac{w_5}{(m_j-2)!}\right]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $\theta_j u_j \in \left[\frac{A w_3 \rho_j}{D4^{m_j-2}(m_j-2)!}, \frac{w_4 \tau_4^{m_j-2}}{(m_j-2)!}\right]$, $1 \le j \le n$.

Then, the system (4.24) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|(u_i^1)^{(m_i-2)}(t)| < w_2, \quad t \in [\tau_1, \tau_4], \ 1 \le i \le n;$$

$$|(u_i^2)^{(m_i-2)}(t)| > w_3, \quad t \in [\tau_2, \tau_3], \ 1 \le i \le n;$$

$$\max_{1 \le i \le n} \max_{t \in [\tau_1, \tau_4]} |(u_i^3)^{(m_i-2)}(t)| > w_2 \quad \textit{and} \quad \min_{1 \le i \le n} \min_{t \in [\tau_2, \tau_3]} |(u_i^3)^{(m_i-2)}(t)| < w_3.$$

Case 4.8. Periodic boundary value problem

Consider the following two systems of periodic boundary value problems

$$\begin{aligned} -u_i''(t) + cu_i(t) &= P_i(t, u(t)),\ t \in [0, 2\pi] \\ u_i(0) = u_i(2\pi), &\quad u_i'(0) = u_i'(2\pi) \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} u_i''(t) + ru_i(t) &= P_i(t, u(t)),\ t \in [0, 2\pi] \\ u_i(0) = u_i(2\pi), &\quad u_i'(0) = u_i'(2\pi) \end{aligned} \tag{4.34}$$

where $i = 1, 2, \cdots, n$. For each $1 \le i \le n$, assume that $c > 0$ is fixed, $0 < r < \frac{1}{4}$ is fixed and $P_i : [0, 2\pi] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

Let $G^{(4.33)}(t, s)$ be the Green's function of the boundary value problem

$$-y''(t) + cy(t) = 0,\ t \in [0, 2\pi]$$
$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi),$$

and let $G^{(4.34)}(t, s)$ be the Green's function of the boundary value problem

$$y''(t) + ry(t) = 0,\ t \in [0, 2\pi]$$
$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

It is known that [17]

(a) $$G^{(4.33)}(t, s) = \frac{1}{2\sqrt{c}\,[\exp(2\pi\sqrt{c}) - 1]} \begin{cases} \exp(\sqrt{c}(t-s)) + \exp(\sqrt{c}(2\pi - t + s)), \\ \qquad 0 \le s \le t \le 2\pi \\ \exp(\sqrt{c}(s-t)) + \exp(\sqrt{c}(2\pi - s + t)), \\ \qquad 0 \le t \le s \le 2\pi; \end{cases}$$

$$G^{(4.34)}(t, s) = \frac{1}{2\sqrt{r}[1 - \cos(2\pi\sqrt{r})]} \begin{cases} \sin(\sqrt{r}(t-s)) + \sin(\sqrt{r}(2\pi - t + s)), \\ \qquad 0 \le s \le t \le 2\pi \\ \sin(\sqrt{r}(s-t)) + \sin(\sqrt{r}(2\pi - s + t)), \\ \qquad 0 \le t \le s \le 2\pi; \end{cases}$$

(b) $G^{(4.33)}(t, s) \ge 0$ and $G^{(4.34)}(t, s) \ge 0$, $(t, s) \in [0, 2\pi] \times [0, 2\pi]$;

(c) $G^{(4.33)}(t, s) > 0$ and $G^{(4.34)}(t, s) > 0$, $(t, s) \in (0, 2\pi) \times (0, 2\pi)$;

(d) for $(t, s) \in [0, 2\pi] \times [0, 2\pi]$,

$$G^{(4.33)}(t, s) \ge \frac{2\exp(\pi\sqrt{c})}{2\sqrt{c}\,[\exp(2\pi\sqrt{c}) - 1]}$$

and

$$G^{(4.34)}(t, s) \ge \frac{\sin(2\pi\sqrt{r})}{2\sqrt{r}[1 - \cos(2\pi\sqrt{r})]};$$

(e) for $(t,s) \in [0,2\pi] \times [0,2\pi]$,

$$G^{(4.33)}(t,s) \le \frac{\exp(2\pi\sqrt{c})+1}{2\sqrt{c}\,[\exp(2\pi\sqrt{c})-1]}$$

and

$$G^{(4.34)}(t,s) \le \frac{\sin(\pi\sqrt{r})}{\sqrt{r}[1-\cos(\pi\sqrt{r})]}.$$

Now, $u=(u_1,u_2,\cdots,u_n)$ is a solution of the system (4.33) if and only if u is a fixed point of the operator $S:(C[0,2\pi])^n \to (C[0,2\pi])^n$ defined by (3.3) where

$$Su_i(t) = \int_0^{2\pi} G^{(4.33)}(t,s)P_i(s,u(s))ds, \quad t\in[0,2\pi],\ 1\le i\le n. \tag{4.35}$$

Likewise, u is a solution of the system (4.34) provided $u=Su$ where

$$Su_i(t) = \int_0^{2\pi} G^{(4.34)}(t,s)P_i(s,u(s))ds, \quad t\in[0,2\pi],\ 1\le i\le n. \tag{4.36}$$

In the context of Section 3 (obviously the interval $[0,1]$ is changed to $[0,2\pi]$), for system (4.33) let

$$g_i(t,s) = G^{(4.33)}(t,s),\ 1\le i\le n, \qquad a=0, \qquad b=2\pi,$$
$$M_i = \frac{2\exp(\pi\sqrt{c})}{\exp(2\pi\sqrt{c})+1} \quad \text{and} \quad H(s) = \frac{\exp(2\pi\sqrt{c})+1}{2\sqrt{c}\,[\exp(2\pi\sqrt{c})-1]},\ 1\le i\le n \tag{4.37}$$

whereas for system (4.34), let

$$g_i(t,s) = G^{(4.34)}(t,s),\ 1\le i\le n, \qquad a=0, \qquad b=2\pi,$$
$$M_i = \cos(\pi\sqrt{r}) \quad \text{and} \quad H(s) = \frac{\sin(\pi\sqrt{r})}{\sqrt{r}[1-\cos(\pi\sqrt{r})]},\ 1\le i\le n. \tag{4.38}$$

Then, noting (a)–(e), it is clear that $g_i^t(s) \equiv g_i(t,s) \in C[0,2\pi] \subseteq L^\infty[0,2\pi]$ and the conditions (C1)–(C3) are fulfilled for both systems (4.33) and (4.34).

The results in Section 3 reduce to the following theorem, which not only extends the work of Jiang [17] for the special cases of (4.33) and (4.34) when $n=1$, but also provides the existence of *triple constant-sign* solutions.

Theorem 4.8.

(a) *With g_i, a, b, M_i and H given in (4.37), and the obvious modification that the interval $[0,1]$ is replaced by $[0,2\pi]$, the statements (i)–(iii) of Theorem 4.1 hold for system (4.33).*

(b) *With g_i, a, b, M_i and H given in (4.38), and the obvious modification that the interval $[0,1]$ is replaced by $[0,2\pi]$, the statements (i)–(iii) of Theorem 4.1 hold for system (4.34).*

5 Triple solutions of (1.2)

This section extends the results in Section 3 to the system of Fredholm integral equations (1.2) on the half-line $[0,\infty)$. To begin, let the Banach space $B = (BC[0,\infty))^n$ be equipped with norm

$$||u|| = \max_{1\le i\le n} \sup_{t\in[0,\infty)} |u_i(t)| = \max_{1\le i\le n} |u_i|_0 \tag{5.1}$$

where we let $|u_i|_0 = \sup_{t\in[0,\infty)} |u_i(t)|,\ 1\le i\le n$.

We shall seek a solution $u = (u_1, u_2, \cdots, u_n)$ of (1.2) in $(C_l[0,\infty))^n$ where

$$(C_l[0,\infty))^n = \left\{ u \in (BC[0,\infty))^n \;\middle|\; \lim_{t\to\infty} u_i(t) \text{ exists, } 1\le i\le n \right\}. \tag{5.2}$$

For the purpose of clarity, we shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1,-1\},\ 1\le i\le n$ are fixed.

$(C1)_\infty$ Let integers p, q be such that $1\le p\le q\le\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. For each $1\le i\le n$, assume that $P_i : [0,\infty)\times \mathbb{R}^n \to \mathbb{R}$ is a L^q-Carathéodory function, and

$$g_i^t(s) \equiv g_i(t,s) \ge 0,\quad t\in[0,\infty),\ a.e.\ s\in[0,\infty),$$

$$g_i^t(s) \in L^p[0,\infty),\ t\in[0,\infty),$$

the map $t\to g_i^t$ is continuous from $[0,\infty)$ to $L^p[0,\infty)$,

there exists $\bar g_i \in L^p[0,\infty)$ such that $\lim_{t\to\infty} \int_0^\infty |g_i^t(s) - \bar g_i(s)|^p ds = 0$

(i.e., $g_i^t \to \bar g_i$ in $L^p[0,\infty)$ as $t\to\infty$).

$(C2)_\infty$ For each $1\le i\le n$, there exists a constant $0<M_i<1$, a function $H\in L^p[0,\infty)$, and an interval $[a,b]\subseteq[0,\infty)$ such that

$$g_i(t,s) \ge M_i H(s) \ge 0,\quad t\in[a,b],\ a.e.\ s\in[0,\infty).$$

$(C3)_\infty$ For each $1\le i\le n$,

$$g_i(t,s) \le H(s),\quad t\in[0,\infty),\ a.e.\ s\in[0,\infty).$$

$(C4)_\infty$ Let $\tilde K$ and K be as in Section 3 with $B = (BC[0,\infty))^n$. For each $1\le i\le n$, assume that

$$\theta_i P_i(t,u) \ge 0,\ u\in\tilde K,\ a.e.\ t\in(0,\infty) \text{ and } \theta_i P_i(t,u) > 0,\ u\in K,\ a.e.\ t\in(0,\infty).$$

$(C5)_\infty$ For each $1\le i\le n$, there exist continuous functions $f,\ a_i,\ b$ with $f:\mathbb{R}^n\to[0,\infty)$ and $a_i, b:(0,\infty)\to[0,\infty)$ such that

$$a_i(t) \le \frac{\theta_i P_i(t,u)}{f(u)} \le b(t),\quad u\in\tilde K,\ a.e.\ t\in(0,\infty).$$

$(C6)_\infty$ For each $1 \le i \le n$, there exists a number $0 < \rho_i \le 1$ such that

$$a_i(t) \ge \rho_i b(t), \text{ a.e. } t \in (0,\infty).$$

Assume $(C1)_\infty$ holds. Let the operator $S : (C_l[0,\infty))^n \to (C_l[0,\infty))^n$ be defined by

$$Su(t) = (Su_1(t), Su_2(t), \cdots, Su_n(t)), \ t \in [0,\infty) \tag{5.3}$$

where

$$Su_i(t) = \int_0^\infty g_i(t,s)P_i(s,u(s))ds, \quad t \in [0,\infty), \ 1 \le i \le n. \tag{5.4}$$

Clearly, a fixed point of the operator S is a solution of the system (1.2). We shall show that S maps $(C_l[0,\infty))^n$ into itself. Let $u \in (C_l[0,\infty))^n$ and $i \in \{1,2,\cdots,n\}$ be fixed. We need to show that $\lim_{t\to\infty} Su_i(t)$ exists. Fix $r > 0$. Since P_i is L^q-Carathéodory, there exists $\mu_{r,i} \in L^q[0,\infty)$ such that $|P_i(s,u(s))| \le \mu_{r,i}(s)$ for $\|u\| \le r$ and *a.e.* $s \in [0,\infty)$. In fact, for a sufficiently large r,

$$\left|\int_0^\infty [g_i(t,s) - \bar g_i(s)]P_i(s,u(s))ds\right| \le \int_0^\infty |g_i(t,s) - \bar g_i(s)|[\mu_{r,i}(s)]ds \to 0$$

as $t \to \infty$. Therefore, as $t \to \infty$ we have

$$Su_i(t) = \int_0^\infty g_i(t,s)P_i(s,u(s))ds \to \int_0^\infty \bar g_i(s)P_i(s,u(s))ds.$$

Hence, S maps $(C_l[0,\infty))^n$ into $(C_l[0,\infty))^n$ if $(C1)_\infty$ holds.

Next, we define a cone in B as

$$C = \left\{ u \in (C_l[0,\infty))^n \ \middle| \ \text{for each } 1 \le i \le n, \ \theta_i u_i(t) \ge 0 \text{ for } t \in [0,\infty), \right.$$
$$\left. \text{and } \min_{t\in[a,b]} \theta_i u_i(t) \ge M_i \rho_i |u_i|_0 \right\} \tag{5.5}$$

where M_i and ρ_i are defined in $(C2)_\infty$ and $(C6)_\infty$ respectively. Note that $C \subseteq \tilde K$. A fixed point of S obtained in C will be a *constant-sign solution* of the system (1.2).

Remark 5.1. Instead of the cone C defined in (5.5), we can also use the cone C' $(\subset C)$ given by

$$C' = \left\{ u \in (C_l[0,\infty))^n \ \middle| \ \text{for each } 1 \le i \le n, \ \theta_i u_i(t) \ge 0 \text{ for } t \in [0,\infty), \right.$$
$$\left. \text{and } \min_{t\in[a,b]} \theta_i u_i(t) \ge M_i \rho_i \|u\| \right\}.$$

The arguments that follow will be similar.

If $(C1)_\infty$, $(C4)_\infty$ and $(C5)_\infty$ hold, then it is clear from (5.4) that for $u \in \tilde{K}$,

$$\int_0^\infty g_i(t,s)a_i(s)f(u(s))ds \leq \theta_i Su_i(t) \leq \int_0^\infty g_i(t,s)b(s)f(u(s))ds, \quad t \in [0,\infty), \quad 1 \leq i \leq n. \tag{5.6}$$

Lemma 5.1. *Let $(C1)_\infty$ hold. Then, the operator S is continuous and completely continuous.*

Proof. As in [21, Theorem 5.2.3], $(C1)_\infty$ ensures that S is continuous and completely continuous. ∎

In what follows we shall only state the results for (1.2) parallel to those in Section 3. The proofs are omitted as the arguments used are similar to those of the corresponding results in Section 3, with the interval $[0,1]$ replaced by $[0,\infty)$.

Lemma 5.2. *Let $(C1)_\infty$–$(C6)_\infty$ hold. Then, the operator S maps C into itself.*

For subsequent results, we define the following constants for each $1 \leq i \leq n$ and fixed numbers $\tau_j \in [0,\infty)$, $1 \leq j \leq 4$:

$$\begin{aligned}
q_i^\infty &= \sup_{t\in[0,\infty)} \int_0^\infty g_i(t,s)b(s)ds,\\
r_i^\infty &= \min_{t\in[a,b]} \int_a^b g_i(t,s)a_i(s)ds,\\
d_{1,i}^\infty &= \min_{t\in[\tau_2,\tau_3]} \int_{\tau_2}^{\tau_3} g_i(t,s)a_i(s)ds,\\
d_{2,i}^\infty &= \max_{t\in[\tau_1,\tau_4]} \int_{\tau_1}^{\tau_4} g_i(t,s)b(s)ds,\\
d_{3,i}^\infty &= \max_{t\in[\tau_1,\tau_4]} \left[\int_0^{\tau_1} g_i(t,s)b(s)ds + \int_{\tau_4}^\infty g_i(t,s)b(s)ds\right].
\end{aligned} \tag{5.7}$$

In view of $(C3)_\infty$ and $(C2)_\infty$, it is clear that for each $1 \leq i \leq n$,

$$q_i^\infty \leq \int_0^\infty H(s)b(s)ds, \; r_i^\infty \geq \int_a^b M_i H(s)a_i(s)ds \text{ and } d_{2,i}^\infty \leq \int_{\tau_1}^{\tau_4} H(s)b(s)ds. \tag{5.8}$$

Lemma 5.3. *Let $(C1)_\infty$–$(C6)_\infty$ hold, and assume*

$(C7)_\infty$ *for each $1 \leq i \leq n$ and each $t \in [0,\infty)$, the function $g_i(t,s)b(s)$ is nonzero on a subset of $[0,\infty)$ of positive measure.*

Suppose that there exists a number $d > 0$ such that for $\theta_j u_j \in [0,d]$, $1 \leq j \leq n$,

$$f(u_1,u_2,\cdots,u_n) < \frac{d}{q_i^\infty}, \quad 1 \leq i \leq n.$$

Then,

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d).$$

Lemma 5.4. *Let (C1)$_\infty$-(C6)$_\infty$ hold. Suppose that there exists a number $d > 0$ such that for $\theta_j u_j \in [0, d]$, $1 \le j \le n$,*

$$f(u_1, u_2, \cdots, u_n) \le \frac{d}{q_i^\infty}, \ 1 \le i \le n.$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

Applying Theorem 2.1, we obtain the following result.

Theorem 5.1. *Let* (C1)$_\infty$-(C7)$_\infty$ hold, and assume

(C8)$_\infty$ for each $1 \le i \le n$ and each $t \in [a, b]$, the function $g_i(t, s)a_i(s)$ is nonzero on a subset of $[a, b]$ of positive measure.

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{\min_{1 \le i \le n} M_i \rho_i} \le w_3$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{w_1}{q_i^\infty}$ for $\theta_j u_j \in [0, w_1]$, $1 \le j \le n$;

(Q) one of the following holds:

(Q1) $\limsup_{|u_1|, |u_2|, \cdots, |u_n| \to \infty} \frac{f(u_1, u_2, \cdots, u_n)}{|u_j|} < \frac{1}{q_i^\infty}$ for some $j \in \{1, 2, \cdots, n\}$ (j depends on i);

(Q2) there exists a number η ($\ge w_3$) such that $f(u_1, u_2, \cdots, u_n) \le \frac{\eta}{q_i^\infty}$ for $\theta_j u_j \in [0, \eta]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_2}{r_i^\infty}$ for $\theta_j u_j \in [w_2, w_3]$, $1 \le j \le n$.

Then, the system (1.2) *has (at least) three constant-sign solutions* $u^1, u^2, u^3 \in C$ *such that*

$$\|u^1\| < w_1; \quad |u_i^2(t)| > w_2, \ \ t \in [a, b], \ 1 \le i \le n;$$
$$\|u^3\| > w_1 \quad \text{and} \quad \min_{1 \le i \le n} \min_{t \in [a,b]} |u_i^3(t)| < w_2.$$

The next two results are derived using Theorem 2.2.

Theorem 5.2. *Let(C1)$_\infty$-(C6)$_\infty$ hold. Assume there exist numbers τ_j, $1 \le j \le 4$ with*

$$0 \le \tau_1 \le a \le \tau_2 < \tau_3 \le b \le \tau_4 \le \infty$$

such that

$(C9)_\infty$ *for each* $1 \le i \le n$ *and each* $t \in [\tau_2, \tau_3]$, *the function* $g_i(t,s)a_i(s)$ *is nonzero on a subset of* $[\tau_2, \tau_3]$ *of positive measure;*

$(C10)_\infty$ *for each* $1 \le i \le n$ *and each* $t \in [\tau_1, \tau_4]$, *the function* $g_i(t,s)b(s)$ *is nonzero on a subset of* $[\tau_1, \tau_4]$ *of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$ *with*

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_{2,i}^\infty}\left(w_2 - \frac{w_5 d_{3,i}^\infty}{q_i^\infty}\right)$ *for* $\theta_j u_j \in [0, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i^\infty}$ *for* $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}^\infty}$ *for* $\theta_j u_j \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (1.2) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2, \quad t \in [\tau_1, \tau_4], \ 1 \le i \le n; \qquad |u_i^2(t)| > w_3, \quad t \in [\tau_2, \tau_3], \ 1 \le i \le n;$$

$$\max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \text{and} \quad \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3.$$

Remark 5.2. Under the special case when $\tau_1 = 0$, $\tau_4 = \infty$, $\tau_2 = a$ and $\tau_3 = b$, we have

$$d_{1,i}^\infty = r_i^\infty, \qquad d_{2,i}^\infty = q_i^\infty \qquad \text{and} \qquad d_{3,i}^\infty = 0.$$

In this case Theorem 5.2 reduces to Theorem 5.1.

Theorem 5.3. *Let* $(C1)_\infty$*-*$(C6)_\infty$ *hold. Assume there exist numbers* τ_j, $1 \le j \le 4$ *with*

$$a \le \tau_1 < \tau_2 < \tau_3 < \tau_4 \le b$$

such that $(C9)_\infty$ *and* $(C10)_\infty$ *hold. Suppose that there exist numbers* w_i, $1 \le i \le 5$ *with*

$$0 < w_1 \le w_2 \cdot \min_{1\le i\le n} M_i\rho_i < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n} M_i\rho_i} \le w_4 \le w_5$$

such that the following hold for each $1 \le i \le n$:

(P) $f(u_1, u_2, \cdots, u_n) < \frac{1}{d_{2,i}^\infty}\left(w_2 - \frac{w_5 d_{3,i}^\infty}{q_i^\infty}\right)$ *for* $\theta_j u_j \in [w_1, w_2]$, $1 \le j \le n$;

(Q) $f(u_1, u_2, \cdots, u_n) \le \frac{w_5}{q_i^\infty}$ *for* $\theta_j u_j \in [0, w_5]$, $1 \le j \le n$;

(R) $f(u_1, u_2, \cdots, u_n) > \frac{w_3}{d_{1,i}^\infty}$ *for* $\theta_j u_j \in [w_3, w_4]$, $1 \le j \le n$.

Then, the system (1.2) has (at least) three constant-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$|u_i^1(t)| < w_2, \quad t \in [\tau_1, \tau_4], \ 1 \le i \le n; \qquad |u_i^2(t)| > w_3, \quad t \in [\tau_2, \tau_3], \ 1 \le i \le n;$$

$$\max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} |u_i^3(t)| > w_2 \quad \text{and} \quad \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} |u_i^3(t)| < w_3.$$

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