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Elliptic operators with infinitely many variables

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ABSTRACT

We present here a review of some recent existence, uniqueness and regularity results for elliptic equations with infinitely many variables. Operators considered here are: the Gross Laplacian, the Ornstein–Uhlenbeck operator and their regular perturbations.

RESUMEN

Presentamos acá algunos resultados recientes acerca de existencia, unicidad y regularidad para ecuaciones elípticas con infinitas variables. Se consideran los operadores Gross Laplacian, Ornstein–Uhlenbeck y sus perturbaciones regulares.

Key words and phrases: *Gross Laplacian, Ornstein–Uhlenbeck operator, Elliptic equations with infinitely many variables.*
Math. Subj. Class.: *35R15, 35B50, 35J15.*

1 Introduction

We are concerned with the following differential operator

$$K_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi(x)] + \langle Ax + F(x), D\varphi(x)\rangle, \quad x \in D(A) \cap D(F), \qquad (1.1)$$

on a separable Hilbert space H. Here $A\colon D(A)\subset H\to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H, $Q\colon H\to H$ is a symmetric nonnegative bounded linear operator in H (possibly the identity operator I), and $F\colon D(F)\subset H\to H$ is nonlinear. Moreover, D_t represents the derivative with respect to t, D the Fréchet derivative with respect to x and Tr the trace.

Let $\{e_k\}$ be a complete orthonormal system in H and set $x_k=\langle x,e_k\rangle$ and $Q_{h,k}=\langle Qe_h,e_k\rangle$ for all $x\in H$, $h,k\in\mathbb{N}$. Then we can write K_0 as

$$K_0\varphi(x)=\sum_{h,k=1}^{\infty}Q_{h,k}D_hD_k\varphi(x)+\sum_{h=1}^{\infty}\langle Ax+F(x),e_h\rangle D_h\varphi(x),$$

where D_h represents derivative with respect to x_h. Therefore K_0 can be seen as an elliptic operator with infinite many variables x_k, $k\in\mathbb{N}$.

If Q is invertible and Q^{-1} is bounded we say that differential operator (1.1) is *strictly elliptic*, otherwise that it is *elliptic degenerate*.

We are interested in the parabolic equation

$$\begin{cases} D_tu(t,x)=K_0u(t,x), & t>0,\ x\in H,\\[2mm] u(0,x)=\varphi(x), & x\in H, \end{cases} \tag{1.2}$$

where $\varphi\in C_b(H)$, the Banach space of all uniformly continuous and bounded mappings $\varphi\colon H\to\mathbb{R}$, endowed with the norm

$$\|\varphi\|_0=\sup_{x\in H}|\varphi(x)|,$$

and in the elliptic equation

$$\lambda\psi-K_0\psi=f, \tag{1.3}$$

where $\lambda>0$ and $f\in C_b(H)$ are given.

One of the main motivation to consider the operator K_0 comes from the following stochastic differential equation,

$$\begin{cases} dX(t,x)=(AX(t,x)+F(X(t,x)))dt+\sqrt{Q}\,dW(t), & t>0,\ x\in H,\\[2mm] X(0,x)=x, & x\in H, \end{cases} \tag{1.4}$$

where $W(t)$ is a cylindrical Wiener process in H and $\mathbb{E}$ represents the expectation, see e. g. [12]. Equation (1.3) is an evolution equation in H perturbed by noise. Several equations in Physics have this form, we mention the reaction–diffusion equations and the Burgers and Navier–Stokes equations. In these cases often H is an L^2 space, A is the Laplacian with suitable boundary conditions and F represents a non linear function describing interactions.

There is in fact a strict connection between the process $X(t,x)$ and the solutions of (1.2),(1.3), given by the formulas (which have to be justified !),

$$u(t,x)=\mathbb{E}[\varphi(X(t,x))],\quad t\geq 0,\ x\in H, \tag{1.5}$$

and

$$\psi(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X(t,x))]dt, \quad x \in H, \tag{1.6}$$

respectively. However, we shall not use these formulas in this paper, but we shall only consider deterministic tools as functional analysis and measure theory, in particular Gaussian measures.

There is an increasing interest in equations with an infinite number of variables, starting from the pionering work of L. Gross [16] and Yu. Daleckij [8], see [14] and references therein. Since the theory is still at the beginning, we shall confine in this paper, to the more understood case of a regular F (but with A being unbounded in general). Also, for the sake of simplicity, we shall look for solutions of (1.2) and (1.3) in spaces of continuous functions. For the important case of irregular nonlinearities and solutions in spaces $L^p(H,\nu)$ where ν is an invariant measure for $X(t,x)$ we refer to [14] and references therein, see also the approach based on Dirichlet forms, [1],[2],[21] and [25].

Let us outline the contents of the paper. Section §2 is devoted to the case when $A = F = 0$, the *heat equation*, section §3 to the case when $F = 0$, the *Ornstein-Uhlenbeck equation*. Finally, in §4 we shall present some results for more general equations. We will follow closely [14] with the exception of §4.3.

We end this section by giving some notation and recalling the definition and some properties of Gaussian probability measures in a Hilbert space H which will play an important rôle in what follows. We shall outline some proofs, for details see e.g. [14, Chapter 1].

We shall fix in all the paper a separable Hilbert space H (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$) and denote by $L(H)$ the Banach algebra of all linear bounded operators in H endowed with the usual norm:

$$\|T\| = \sup\{|Tx| : \ x \in H, \ |x| = 1\}, \quad T \in L(H).$$

Since e^{tA} is a strongly continuous semigroup, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|e^{tA}\| \le Me^{\omega t}, \quad t \ge 0. \tag{1.7}$$

We shall denote by $L^+(H)$ the subset of $L(H)$ of all symmetric, nonnegative operators and by $L_1(H)$ (resp. $L_1^+(H)$) the subset of $L(H)$ (resp. $L^+(H)$) of all operators of trace class. One can show that a linear operator $Q \in L^+(H)$ is of *trace class* if and only if there exists a complete orthonormal system $\{e_k\}$ in H and a sequence of nonnegative numbers $\{\lambda_k\}$ such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}, \tag{1.8}$$

and

$$\operatorname{Tr} Q := \sum_{k=1}^{\infty} \lambda_k < +\infty.$$

For any $a \in H$ and $Q \in L_1^+(H)$ we define the Gaussian probability measure $N_{a,Q}$ in H by identifing H with ℓ^2 ([1]), and setting

$$N_{a,Q} = \prod_{k=1}^{\infty} N_{a_k,\lambda_k}, \quad a_k = \langle a, e_k \rangle,\ k \in \mathbb{N}. \tag{1.9}$$

Obviously, the measure $N_{a,Q}$ is defined on $\mathbb{R}^\infty$, the product space of all real sequences, but it is concentrated on ℓ^2 (that is $\mu(\ell^2) = 1$) since, in view of the monotone convergence theorem, we have

$$\int_{\mathbb{R}^\infty} |x|^2_{\ell^2} N_{a,Q}(dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} x_k^2 N_{a_k,\lambda_k}(dx_k) = \sum_{k=1}^{\infty} (\lambda_k + a_k^2) < +\infty.$$

If $a = 0$ we shall write $N_{a,Q} = N_Q$ for brevity. We shall always assume Ker $Q = \{0\}$ in what follows.

If H is n–dimensional, $n \in \mathbb{N}$, we have (since det $Q > 0$)

$$N_{a,Q}(dx) = (2\pi)^{-n/2}(\det Q)^{-1/2}\, e^{-\frac{1}{2}\langle Q^{-1}(x-a),x-a\rangle}\, dx. \tag{1.10}$$

Let us list some useful identities about integrals with respect to the measure $\mu := N_{a,Q}$. They are straightforward when H is n–dimensional and they can be proved in the general case letting $n \to \infty$. We have

$$\int_H |x|^2 \mu(dx) = \operatorname{Tr} Q + |a|^2, \tag{1.11}$$

$$\int_H \langle x, h\rangle \mu(dx) = a, \quad h \in H, \tag{1.12}$$

$$\int_H \langle x-a, h\rangle\langle x-a, k\rangle \mu(dx) = \langle Qh, k\rangle, \quad h, k \in H. \tag{1.13}$$

$$\int_H e^{i\langle x,h\rangle} \mu(dx) = e^{i\langle a,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle}, \quad h \in H. \tag{1.14}$$

The range $Q^{1/2}(H)$ of $Q^{1/2}$ is called the *Cameron–Martin* space of N_Q. If H is infinite dimensional $Q^{1/2}(H)$ is dense in H but different from H and it is important to notice that

$$N_Q(Q^{1/2}(H)) = 0. \tag{1.15}$$

Let us introduce the *Cameron–Martin* formula. Consider a measure N_Q and the translated measure $N_{a,Q}$ with $a \in Q^{1/2}(H)$. If H is finite dimensional, it follows from (1.10) that $N_{a,Q}$ and N_Q are equivalent and,

$$\frac{dN_{a,Q}}{dN_Q}(x) = e^{-\frac{1}{2}|Q^{-1/2}a|^2 + \langle Q^{-1/2}a, Q^{-1/2}x\rangle}, \quad x \in H. \tag{1.16}$$

[1] ℓ^2 is the space of all sequences $\{x_k\}$ of real numbers such that $|x|^2_{\ell^2} := \sum_{k=1}^\infty |x_k|^2 < +\infty$.

This formula does not generalize immediately in infinite dimensions. In fact in this case the term $\langle Q^{-1/2}a, Q^{-1/2}x\rangle$ is only meaningful when x belongs to $Q^{1/2}(H)$ which, however, is a set having N_Q measure 0 by (1.15).

To give a meaning to formula (1.16) in infinite dimensions, it is convenient to introduce the *white noise* function W. Let us start with the function

$$W^0 : Q^{1/2}(H) \subset H \to L^2(H,\mu),\ f \to W_f^0,$$

where

$$W_f^0(x) = \langle x, Q^{-1/2}f\rangle, \quad x \in H. \tag{1.17}$$

In view of (1.13) we have

$$\int_H W_f^0(x)W_g^0(x)\mu(dx) = \langle QQ^{-1/2}f, Q^{-1/2}g\rangle = \langle f,g\rangle, \quad f,g \in H.$$

Thus, W^0 is an isomorphism and, since $Q^{1/2}(H)$ is dense in H, it can be uniquely extended to a mapping W from H into $L^2(H,\mu)$.

If $f \in H$ it is usual to write in the literature ("par abus de language")

$$W_f(x) = \langle x, Q^{-1/2}f\rangle, \quad x \in H,$$

even if this is meaningful only when $f \in Q^{1/2}(H)$. We shall also follow this convention.

Now the following result can proved by a straighforward limit procedure, see e.g. [14, Theorem 1.3.6] for details.

Theorem 1.1 *Let $Q \in L_1^+(H)$ and $a \in Q^{1/2}(H)$. Then the measures $N_{a,Q}$ and N_Q are equivalent* ([2]) *and*

$$\frac{dN_{a,Q}}{dN_Q}(x) = \exp\left\{-\frac{1}{2}|Q^{-1/2}a|^2 + \langle Q^{-1/2}a, Q^{-1/2}x\rangle\right\},\ x \in H. \tag{1.18}$$

We stress the fact that the term $\langle Q^{-1/2}a, Q^{-1/2}x\rangle$ in the exponential above, should be intended more precisely as $W_{Q^{-1/2}a}(x)$.

2 The Heat equation

2.1 Introduction

We are here concerned with the following problem

$$\begin{cases} D_t u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2u(t,x)], \quad t>0,\ x\in H, \\ u(0,x) = \varphi(x), \quad x \in H,\ \varphi \in C_b(H). \end{cases} \tag{2.1}$$

where $Q \in L^+(H)$.

[2] If $a \notin Q^{1/2}(H)$ then $N_{a,Q}$ and N_Q are singular.

A function $u : [0, +\infty) \times H \to \mathbb{R}$ is said to be a *strict* (resp. *classical*) solution to (2.1) if the derivatives $D_t u(t,x)$ and $D^2 u(t,x)$ exist for all $t \geq 0$ (resp. $t > 0$) and $x \in H$, are continuous and bounded on $[0,+\infty) \times H$ (resp. $(0,+\infty) \times H$) and u satisfies (2.1).

When H is finite dimensional ($H = \mathbb{R}^d$, $d \in \mathbb{N}$), problem (2.1) can be written as

$$\begin{cases} D_t u(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} Q_{ij} D_i D_j u(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = \varphi(x), & \varphi \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d, \end{cases} \tag{2.2}$$

where $Q_{ij} = \langle Qe_j, e_i \rangle$ and $\{e_i\}$ is an orthonormal basis in $\mathbb{R}^d$. In this case it is well known (recall that $\det Q > 0$) that there exists a unique classical solution of (2.2), given by

$$\begin{aligned} u(t,x) &= (2\pi)^{-d/2} (\det Q)^{-1/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q^{-1}y,y\rangle} \varphi(x+y)dy \\ &= \int_{\mathbb{R}^d} \varphi(x+y) N_{tQ}(dy), \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.3}$$

Moreover, setting $S_t\varphi(x) = u(t,x)$, S_t, is a strongly continuous semigroup of linear bounded operator in $C_b(\mathbb{R}^d)$ ([3]).

Assume now that H is infinite dimensional and let $Q \in L^+(H)$. Then the last integral in (2.3) is still meaningful provided Q is of trace class. This is in fact a necessary condition if one want to solve the problem for "enough" functions, see [14, Proposition 3.1.2].

From now on, we shall assume in this section that $\operatorname{Tr} Q < +\infty$. Then we define $S_0 = I$ and, for any $t > 0$,

$$S_t\varphi(x) = \int_H \varphi(x+y) N_{tQ}(dy), \quad x \in H,\ \varphi \in C_b(H). \tag{2.4}$$

It is easy to see, see [14, Proposition 3.5.1] that S_t is a strongly continuous semigroup of linear bounded operators on $C_b(H)$ and that

$$\|S_t\varphi\|_0 \leq \|\varphi\|_0, \quad t \geq 0,\ \varphi \in C_b(H).$$

We shall denote by $\mathcal{A}$ its infinitesimal generator. For any $\varphi \in C_b(H)$ the function $u(t,x) = S_t\varphi(x)$ is called a *generalized* solution of (2.1).

S_t is called the *heat semigroup*, it has been introduced in a different setting by L. Gross [16], see also Yu. Daleckij [8].

[3] Notice that if one replace $C_b(H)$ with the space of all *continuous* and bounded functions on H, then S_t is not strongly continuous.

It is important to understand how the generator $\mathcal{A}$ looks like. The usual way is to find a *core* (4) $Y_\mathcal{A}$ of $\mathcal{A}$, where $\mathcal{A}$ has an explicit differential expression. When H is finite dimensional, a core is provided by $C_b^2(H)$ (5). However, when H is infinite dimensional $C_b^2(H)$ is not dense in $C_b(H)$, see [22] (one can show, however, that $C_b^{1,1}(H)$ (6) is, see [19]).

To define a core we shall proceed as follows. First we shall introduce, following L. Gross [16] the concept of Q–derivative (or derivative in the directions of the Cameron–Martin space $Q^{1/2}(H)$). A mapping $\varphi : H \to \mathbb{R}$ is called *Q-differentiable* if for any $x \in H$ the function $F(y) = \varphi(x + Q^{1/2}y)$, $y \in H$, is differentiable at 0. In this case we set $D_Q\varphi(x) = DF(0)$ and call $D_Q\varphi(x)$ the Q–derivative of φ at x. If $\varphi \in C_b^1(H)$ then it is Q–differentiable and we have $D_Q\varphi(x) = Q^{1/2}D\varphi(x)$. We shall denote by $C_Q^1(H)$ the set of all $\varphi \in C_b(H)$ that possess uniformly continuous Q-derivatives. In a similar way we define second order Q–derivatives and the space $C_Q^2(H)$.

Now, the following subspace is a core for $\mathcal{A}$, see [23].

$$Y_\mathcal{A} = \{\varphi \in C_Q^2(H) : \; D_Q^2\varphi \in C_b(H, L_1(H))\}. \tag{2.5}$$

Moreover, if $\varphi \in Y_\mathcal{A}$ we have

$$\mathcal{A}\varphi = \frac{1}{2}\,\mathrm{Tr}\,[D_Q^2\varphi].$$

Before proving in §2.3 existence and uniqueness for equation (2.1), we shall present in §2.2 the *maximum principle*. This result will be useful to obtain uniqueness. In §2.4 we consider the elliptic equation (1.3) and present *Schauder estimates*. Finally, in §2.5 we study a generalization of equation (2.1), taking $Q = C(t)$ depending in time. This will be used in §3 to study the Ornstein–Uhlenbeck equation.

2.2 The maximum principle

We shall prove the maximum principle for more general equations of the form

$$\left\{\begin{array}{l} D_t u(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[Q(t,x)D^2u(t,x)], \quad t > 0,\ x \in H, \\[2ex] u(0,x) = \varphi(x), \quad x \in H,\ \varphi \in C_b(H). \end{array}\right. \tag{2.6}$$

where $Q : [0, +\infty) \times H \to L_1^+(H)$ is continuous.

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4 That is a dense subspace Y of $C_b(H)$ which is also dense for the domain $D(\mathcal{A})$ of $\mathcal{A}$ endowed with its graph norm.

5 For any $k \in \mathbb{N}$, $C_b^k(H)$ is the subspace of $C_b(H)$ of all functions $\varphi : H \to \mathbb{R}$ which are k times Fréchet differentiable on H with uniformly continuous and bounded derivatives $D^h\varphi$ with h less than or equal to k.

6 $C_b^{1,1}(H)$ is the space of all functions $\varphi \in C_b^1(H)$ such that $D\varphi$ is Lipschitz continuous.

Let $T > 0$ be fixed. Assume that u, not identically equal to 0, is a strict solution of (2.6). Setting $v(t,x) = e^{-t}u(t,x)$, we have

$$D_t v(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[Q(t,x)D^2 v(t,x)] - v(t,x), \quad t \in [0,T],\ x \in H. \tag{2.7}$$

If v attains a maximum on $(t_0, x_0) \in [0,T] \times H$ then $t_0 = 0$, otherwise (2.7) will yield a contradiction. Consequently, in this case we have

$$\sup_{t\in[0,T]} e^{-t}\|u(t,\cdot)\|_0 \le \|\varphi\|_0. \tag{2.8}$$

We are going to show that (2.8) is always true (maximum principle), the problem in proving this fact is that v does not attain a maximum in general. To overcome this difficulty, we shall use the following Asplund lemma, see e.g. [3]. Roughly speaking it says that, given a continuous and bounded function u defined on a bounded subset K of a Hilbert space X, it is possible to change "slightly" u by a linear function in several ways so that it attains a maximum.

Lemma 2.1 *Let X be a Hilbert space, K a closed bounded subset of X, and ζ a bounded real continuous function on K. Then there exists a dense subset Σ of X such that the mapping $K \to \mathbb{R}$, $x \to \zeta(x) + \langle x, y\rangle$, attains a maximum in K for all $y \in \Sigma$.*

Notice that we cannot apply the Asplund lemma to our function $v(t,x)$ defined above, since it is defined on $[0,T] \times H$ which is not a bounded subset of the Hilbert space $X = \mathbb{R} \times H$. However, it is not diffucult to find a ball $B \in H$ and a function $\tilde v$ close to v such that

$$\sup\{\tilde v(t,x) :\ (t,x) \in [0,T] \times H\} = \sup\{\tilde v(t,x) :\ (t,x) \in [0,T] \times B\}, \tag{2.9}$$

so that we will able to apply the Asplund lemma to the function $\tilde v(t,x)$. More precisely the following lemma holds, see [14, Lemma 3.2.6].

Lemma 2.2 *Assume that K is a closed subset of a Hilbert space X, and that u is a bounded and continuous function on K. Then, for any $\varepsilon > 0$ there exists $p \in C_0^\infty(X)$ and $C > 0$ such that*

(i) $u + p$ attains its maximum on K,

(ii) $\|p\|_0 + \|Dp\|_0 + \|D^2 p\|_0 \le C\varepsilon$.

Now the proof of the following maximum principle is straightforward, for details see [14, Theorem 3.2.7].

Theorem 2.3 *Let $\varphi \in C_b^2(H)$ and let u be a strict solution of (2.1). Then*

$$\sup_{t\in[0,T]} e^{-t}\|u(t,\cdot)\|_0 \le \|\varphi\|_0.$$

2.3 Strict solutions

In this subsection we show that if the function φ is sufficiently regular then (2.3) defines a strict solution to (2.1). Notice first that, by a straightforward change of variables, we can write

$$S_t\varphi(x) = \int_H \varphi(x + \sqrt{t}\, y) N_Q(dy), \quad t > 0,\ x \in H. \tag{2.10}$$

Now, we can prove the following result.

Theorem 2.4 *If $\varphi \in C_b^2(H)$, then the function $u(t,\cdot) = S_t\varphi$ is the unique strict solution of* (2.1).

Sketch of the Proof. Let $\varphi \in C_b^2(H)$. By (2.10) it follows that $u(t,\cdot) \in C_b^2(H)$ for all $t \geq 0$ and,

$$\langle Du(t,x), h\rangle = \int_H \langle D\varphi(x + \sqrt{t}\, y), h\rangle N_Q(dy), \quad t \geq 0,\ x \in H, \tag{2.11}$$

$$\langle D^2u(t,x)\cdot h, h\rangle = \int_H \langle D^2\varphi(x + \sqrt{t}\, y)\cdot h, h\rangle N_Q(dy), \quad t \geq 0,\ x \in H. \tag{2.12}$$

It remains to show that u is differentiable with respect to t and (2.1) holds.

By the Taylor formula we have that

$$u(t,x) = \varphi(x) + \sqrt{t}\int_H \langle D\varphi(x), y\rangle N_Q(dy) + \frac{1}{2}\, t\int_H \langle D^2\varphi(x)\cdot y, y\rangle N_Q(dy) + r(t,x),$$

where $r(t,x)$ is a "small" remainder. Since N_Q has mean 0, the second term of the right hand side vanishes by (1.12). Moreover, by (1.13)

$$\int_H \langle D^2\varphi(x)\cdot y, y\rangle N_Q(dy) = \sum_{h,k=1}^{\infty}\int_H D_hD_k\varphi(x) y_h y_k N_Q(dy) = \mathrm{Tr}\ [QD^2\varphi(x)],$$

where $D_h\varphi(x) = \langle D\varphi(x), e_h\rangle$, $y_h = \langle y, e_h\rangle$ and $\{e_h\}$ is defined by (1.8). Consequently

$$u(t,x) = \varphi(x) + \frac{1}{2}\, t\ \mathrm{Tr}\ [QD^2\varphi(x)] + r(t,x),$$

and we deduce that

$$D_t^+u(0,x) = \frac{1}{2}\ \mathrm{Tr}[QD^2u(0,x)], \quad x \in H,$$

so that u fulfills (2.1) for $t = 0$. Using the fact that S_t is a semigroup, it is standard to see that u fulfills (2.1) for any $t \geq 0$. The existence is proved. Uniqueness follows from the maximum principle. ∎

We end this subsection by proving some regularity results of $R_t\varphi$. We notice that if H is infinite dimensional the operator Q is compact and consequently its inverse is not bounded. So, the operator $\frac{1}{2}$ Tr $[QD^2\varphi]$ is never strictly elliptic in this case. As a consequence, if $\varphi \in C_b(H)$ and $t > 0$ we cannot expect in general that $S_t\varphi$ is more regular than φ as in the finite dimensional case. As proved by L. Gross, see [16], $S_t\varphi$ is differentiable infinitely many times along the direction of the Cameron–Martin space $Q^{1/2}(H)$. Let us give an idea of this interesting fact.

Theorem 2.5 *Let $\varphi \in C_b(H)$ and $u(t,\cdot) = S_t\varphi$. Then for all $t > 0$, and $x \in H$ we have $u(t,\cdot) \in C_Q^2(H)$ and* ([7]),

$$\langle D_Q u(t,x), h\rangle = \frac{1}{\sqrt{t}} \int_H \langle (tQ)^{-1/2}y, h\rangle \varphi(x+y) N_{tQ}(dy), \quad h \in H, \tag{2.13}$$

$$\langle D_Q^2 u(t,x)\cdot h, h\rangle = \frac{1}{t} \int_H \langle (tQ)^{-1/2}y, h\rangle^2 N_{tQ}(dy), \quad h \in H. \tag{2.14}$$

Sketch of the proof. Let $x, g \in H$, $t > 0$ and $\alpha \in \mathbb{R}$. We have

$$u(t, x + \alpha Q^{1/2}g) = \int_H \varphi(x+y) N_{\alpha Q^{1/2}g, tQA}(dy).$$

By the Cameron-Martin formula (1.16) it follows that

$$\frac{dN_{\alpha Q^{1/2}g,tQ}}{dN_{tQ}}(y) = e^{-\frac{\alpha^2}{2t}|g|^2 + \frac{\alpha}{\sqrt{t}}\langle g,(tQ)^{-1/2}y\rangle}.$$

Therefore

$$u(t, x + \alpha Q^{1/2}g) = \int_H \varphi(x+y) e^{-\frac{\alpha^2}{2t}|g|^2 + \frac{\alpha}{\sqrt{t}}\langle g,(tQ)^{-1/2}y\rangle} N_Q(dy).$$

Taking the derivative with respect to α at $\alpha = 0$ yields (2.13). Equation (2.14) can be proved similarly. ■

2.4 Elliptic equations

We are here concerned with the elliptic equation (1.3) which we write in the more convenient form,

$$\lambda\psi(x) - \frac{1}{2}\mathrm{Tr}[D_Q^2\psi(x)] = f(x), \; x \in H, \tag{2.15}$$

where $\lambda > 0$ and $f \in C_b(H)$.

We say that ψ defined by

$$\psi = (\lambda - \mathcal{A})^{-1} f = \int_0^{+\infty} e^{-\lambda t} S_t f(x) dt, \; x \in H, \tag{2.16}$$

[7] In formulas (2.13) and (2.14) we have to read $\langle (tQ)^{-1/2}y, h\rangle = W_h(y)$ where W is the white noise function related to the Gaussian measure N_{tQ}.

(which is well defined by the Hille–Yosida theorem) is a *generalized solution* of (2.15). ψ is said to be a *strict solution* if $\psi \in C_Q^2(H)$, $D_Q^2\psi \in C_b(H, L_1(H))$ and fulfills (2.15) [8].

There is a misprint in the second line of Page 89: $\langle Qx_h, e_k\langle$ should be replaced by $\langle Qe_h, e_k\langle$.

It is well known that, even if H is finite dimensional (with dimension greater than 1), a strict solution of (2.15) does not exist in general since the domain of $\mathcal{A}$ is not $C_b^2(H)$. However, several regularity results for the solution ψ can be proved, see [14, §4.2]. We recall in particular a maximal regularity result which generalizes the classical Schauder estimates, proved in [4].

We need the following notation. For any $\theta \in (0,1)$ we set

$$C_Q^\theta(H) = \left\{\psi \in C_Q(H) : \psi(Q^{1/2}\cdot) \in C_b^\theta(H)\right\}.$$

We define $C_Q^{k+\theta}(H)$, $k \in \mathbb{N}$ similarly.

Theorem 2.6 *Let $\lambda > 0$, $\theta \in (0,1)$ and $f \in C_Q^\theta(H)$. Then*

$$\psi = (\lambda - \mathcal{A})^{-1} f \in C_Q^{2+\theta}(H)$$

and there exists $C > 0$ such that

$$\|\psi\|_{2+\theta,Q} \leq C\|g\|_{\theta,Q}. \tag{2.17}$$

Remark 2.7 It is not known whether $D^2\psi(x) \in L_1(H)$. However, one can show, see [24], that $D_Q^2\psi(x) \in L_2(H)$ (the space of all Hilbert–Schmidt operators) and there exists $C_{\alpha,\lambda}^1 > 0$ such that

$$\|D_Q^2\psi(Q^{1/2}x) - D_Q^2\psi(Q^{1/2}y)\|_{L^2(H)} \leq C_{\alpha,\lambda}^1|(x-y)|^\theta, \quad x, y \in H. \tag{2.18}$$

Remark 2.8 Theorem 2.6 can be used to solve, by using maximum principle and the continuity method, the following heat equation with variables coefficients:

$$\lambda\psi(x) - \mathcal{A}\psi(x) - \frac{1}{2}\mathrm{Tr}\,[F(x)D_Q^2\psi(x)] = g(x), \quad x \in H. \tag{2.19}$$

In fact, the following result was proved in [4]. For a more general equation, involving lower order terms see [28].

Theorem 2.9 *Let $\theta \in (0,1)$, $\lambda > 0$, $g \in C_Q^\theta(H)$, and $F \in C_b^\theta(H; L_1(H))$ be such that $I + F(x) \in L^+(H)$ for all $x \in H$. Then there exists a unique strict solution ψ to the equation* (2.19).

[8] That is if ψ belongs to the core $Y_{\mathcal{A}}$ defined by (2.5)

2.4.1 Potential

It is well known that if $H = \mathbb{R}^n$, $n \geq 3$, and $f \in C_b(H)$ has a bounded support then there exists a unique function (up to an additive constant) ψ, called the *potential* of g, such that

$$-\frac{1}{2}\,\Delta\psi = g. \tag{2.20}$$

Moreover, $\psi \in C_b^{1+\theta}(H)$ for all $\theta \in (0,1)$ and it is given by

$$\varphi(x) = \int_0^\infty P_t g(x)dt = C_n \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-2}}\,dy, \quad x \in H, \tag{2.21}$$

where C_n is a positive constant.

This result was generalized in infinite dimensions by L. Gross, [16]. We have in fact, see [14, §4.3].

Proposition 2.10 *Let $g \in C_b^1(H)$ with bounded support and set*

$$\psi(x) = \int_0^{+\infty} P_t g(x)\,dt,\ x \in H. \tag{2.22}$$

Then $\psi \in Y_A$ (the core of A defined by (2.5)) and

$$-\frac{1}{2}\,\mathrm{Tr}\,[D_Q^2 u(x)] = g(x),\ x \in H. \tag{2.23}$$

2.4.2 The Liouville theorem

We say that function $\psi \in C_b(H)$ is *harmonic* if it belongs to $D(A)$ and $A\psi = 0$, or, equivalently, if $S_t\psi = \psi$ for all $t \geq 0$.

The following result is a generalization of the classical *Liouville* theorem, see [14, Theorem 4.3.4].

Theorem 2.11 *Any harmonic function in $C_b(H)$ is constant.*

Sketch of the proof. Let $\varphi \in C_b(H)$ be such that $P_t\varphi = \varphi$, $t \geq 0$. Then by Theorem 2.5 it follows that $\varphi \in C_Q^1(H)$. Moreover, from (2.13) it follows, using the Hölder inequality, that

$$|D_Q\varphi(x)| \leq \frac{1}{\sqrt{t}}\,\|\varphi\|_0, \quad t \geq 0,\ x \in H.$$

Letting $t \to \infty$ we see that $D_Q\varphi(x) = 0$ for all $x \in H$. This implies that φ is constant in $Q^{1/2}(H)$. Since $Q^{1/2}(H)$ is dense in H, it follows that φ is a constant as required. ■

2.5 A generalization to time dependent coefficients

We consider here the problem

$$\begin{cases} D_t u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[C(t)D^2u(t,x)], & t>0,\ x\in H,\\ u(0,x)=\varphi(x), & x\in H, \end{cases} \tag{2.24}$$

where C is a mapping from $[0,T]$ into $L(H)$ such that $C(\cdot)x$ is continuous for all $x \in H$.

When $H = \mathbb{R}^d$, $d \in \mathbb{N}$, (2.24) can be written as

$$\begin{cases} D_t u(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}(t)D_iD_ju(t,x), \quad t>0,\ x\in\mathbb{R}^d,\\ u(0,x) &= \varphi(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{2.25}$$

where $C_{ij}(t) = \langle C(t)e_j, e_i\rangle$ and $\{e_i\}$ is an orthonormal basis in $\mathbb{R}^d$.

Notice that equation (2.25) is elliptic and its coefficients depend on t but not on x. Then, if

$$\langle C(t)x, x\rangle \ge \nu|x|^2, \quad x\in\mathbb{R}^d,$$

for some $\nu > 0$ and $\varphi \in C_b(\mathbb{R}^d)$, there exists a unique classical solution of (2.25), given by

$$\begin{aligned} u(t,x) &= (2\pi)^{-d/2}(\det Q_t)^{-1/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle}\varphi(x+y)dy\\ &= \int_{\mathbb{R}^d}\varphi(x+y)N_{Q_t}(dy), \quad x\in\mathbb{R}^d \end{aligned} \tag{2.26}$$

where

$$Q_t x = \int_0^t C(s)x\,ds, \quad x\in H. \tag{2.27}$$

If H is infinite dimensional, formula (2.26) is still meaningful provided Q_t is of trace class for all $t \in [0,T]$.

Proceeding as before we can prove the following result.

Theorem 2.12 *Assume that*

$$Q_t \in L_1^+(H) \quad \text{for all } t\ge 0. \tag{2.28}$$

If $\varphi \in C_b^2(H)$, there exists a unique strict solution u to (2.24), *given by*

$$u(t,x) = \int_H \varphi(x+y)N_{Q_t}(dy), \quad x\in H,\ t\in[0,T]. \tag{2.29}$$

Remark 2.13 In order that condition (2.28) is fulfilled it is not necessary that $C(t)$ is of trace class for some $t \in [0,T]$. We shall see an example of this situation in the next section.

3 The Ornstein–Uhlenbeck equation

We are here concerned with the following equation

$$\begin{cases} D_t u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2u(t,x)] + \langle Ax, Du(t,x)\rangle, & t>0,\ x\in D(A), \\ u(0,x) = \varphi(x), & x \in H, \end{cases} \tag{3.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H fulfilling (1.7) and $Q \in L^+(H)$. We set

$$L_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi(x)] + \langle Ax, D\varphi(x)\rangle, \quad t>0,\ x\in D(A).$$

We call (3.1) *Ornstein–Uhlenbeck equation* because it is the Kolmogorov equation corresponding to the Ornstein–Uhlenbeck process $X(t,x)$, which is the solution of the following differential stochastic equation,

$$\begin{cases} dX(t,x) = AX(t,x)dt + \sqrt{Q}\,dW(t), & t>0,\ x\in H, \\ X(0,x) = x, & x\in H. \end{cases} \tag{3.2}$$

A function $u : [0,+\infty)\times H \to \mathbb{R}$ is said to be a *strict solution* to (3.1) if the derivatives $D_t u(t,x)$ and $D^2u(t,x)$ exist for all $t\ge 0$ and $x \in D(A)$, are continuous and bounded on $[0,+\infty)\times H$ and u satisfies (3.1).

In order to solve equation (3.1), we make a change of variables, see [8] and [4], setting $u(t,x) = v(t, e^{tA}x)$. Then v satisfies the following problem

$$\begin{cases} D_t v(t,x) &= \frac{1}{2}\,\mathrm{Tr}\,[e^{tA}Qe^{tA^*}D^2v(t,x)], \quad t>0,\ x\in H, \\ v(0,x) &= \varphi(x), \quad x\in H, \end{cases} \tag{3.3}$$

which is of the form (2.24). Thus, in order to apply Theorem 2.5, we have to assume that the operator Q_t,

$$Q_t x = \int_0^t e^{sA}Qe^{sA^*}x\,ds, \quad x\in H, \tag{3.4}$$

is of trace class for all $t>0$. In this case, if $\varphi \in C^2_b(H)$ by Theorem 2.5 it follows that problem (3.3) has a unique strict solution given by

$$v(t,x) = \int_H \varphi(x+y)N_{Q_t}(dy), \quad x\in H,\ t\ge 0. \tag{3.5}$$

Coming back to u we find the following result.

Theorem 3.1 *Assume that Q_t is of trace class for all $t > 0$. Let $\varphi \in C_b^2(H)$ be such that $QD^2\varphi \in C_b(H; L_1(H))$. Then problem* (3.1) *has a unique strict solution u given by*

$$u(t,x) = \int_H \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad t \geq 0,\ x \in H. \tag{3.6}$$

Now we define the *Ornstein-Uhlenbeck semigroup* setting

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad x \in H,\ t \geq 0, \quad \varphi \in C_b(H). \tag{3.7}$$

We shall always assume from now on that Tr $Q_t < +\infty$ for all $t > 0$. Note that this condition does not imply that Q is of trace class. So, it can happens that the operator L_0 is strictly elliptic as the following example shows.

Example 3.2 Assume that A and Q are such that

$$Ae_k = -\alpha_k e_k, \quad Qe_k = \lambda_k e_k, \quad k \in \mathbb{N},$$

where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H and $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\lambda_k\}_{k\in\mathbb{N}}$ are sequence of positive numbers.

Then we have

$$Q_t\, e_k = \frac{\lambda_k}{2\alpha_k}\, \left(1 - e^{-2\alpha_k t}\right) e_k, \quad k \in \mathbb{N}.$$

Thus, the condition Tr $Q_t < +\infty$ is equivalent to

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\alpha_k} < +\infty.$$

For instance, it is fulfilled if $Q = I$ and $\alpha_k = k^2$, $k \in \mathbb{N}$. ■

The semigroup R_t is not strongly continuous in $C_b(H)$, unless $A = 0$. In fact, if $\varphi_h(x) = e^{i\langle x,h\rangle}$, $x \in H$ with $h \in H$ different from 0, we have, by a direct computation, that

$$R_t\varphi_h = \varphi_{e^{tA^*}h}, \quad t > 0.$$

Now, it is easy to see that $R_t\varphi_h$ does not converge to φ_h in $C_b(H)$ as $t \to 0$.

3.1 The case when L_0 is strictly elliptic

Let us assume that the operator L_0 is strictly elliptic, that is $Q^{-1} \in L(H)$. Then R_t is *smoothing* (as in finite dimensions), that is it maps $C_b(H)$ into $C_b^\infty(H)$ for all $t > 0$. Let us give an idea of this fact.

Proposition 3.3 *Assume that $Q^{-1} \in L(H)$ and $\varphi \in C_b(H)$. Then for all $t > 0$ we have $R_t\varphi \in C_b^\infty(H)$ and, in particular* [9],

$$\langle DR_t\varphi, h\rangle = \int_H \langle \Gamma(t)h, Q_t^{-1/2}y\rangle \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad h \in H, \tag{3.8}$$

[9] In formula (3.8) we have to read $\langle \Gamma(t)h, Q_t^{-1/2}y\rangle = W_{\Gamma(t)h}(y)$ where W is the white noise function related to the Gaussian measure N_{Q_t}.

where the operator

$$\Gamma(t) = Q_t^{-1/2} e^{tA}, \quad t > 0, \tag{3.9}$$

is well defined and bounded ([10]). *Moreover, there exists* $c > 0$ *such that* ([11])

$$\|DR_t\varphi\|_0 \le ct^{-1/2} e^{\omega t} \, \|\varphi\|_0. \tag{3.10}$$

Sketch of the proof. By a straightforward change of variables we can write

$$R_t\varphi(x) = \int_H \varphi(y) N_{e^{tA}x,Q_t}(dy), \quad t \ge 0,\ x \in H, \varphi \in B_b(H).$$

In order to differentiate $R_t\varphi(x)$ with respect to x we shall use the Cameron–Martin formula (1.18), by replacing integration with respect to $N_{e^{tA}x,Q_t}$ with integration with respect to N_{Q_t}. To apply (1.18) we need that

$$e^{tA}(H) \subset Q_t^{1/2}(H), \quad t > 0. \tag{3.11}$$

In fact (3.11) always holds when $Q^{-1} \in L(H)$, see the discussion below. Now, by Theorem 1.1 it follows that

$$\frac{dN_{e^{tA}x,Q_t}}{dN_{Q_t}}(y) = e^{-\frac{1}{2}|\Gamma(t)x|^2 + \langle \Gamma(t)x, Q_t^{-1/2}y\rangle}, \quad y \in H.$$

Therefore, we can write

$$R_t\varphi(x) = \int_H \varphi(y) N_{e^{tA}x,Q_t}(dy) = \int_H e^{-\frac{1}{2}|\Gamma(t)x|^2 + \langle \Gamma(t)x, Q_t^{-1/2}y\rangle} \varphi(y) N_{Q_t}(dy)$$

and, differentiating with respect to x, (3.11) follows. For (3.10) see next comment. ■

In order to understand the meaning of condition (3.11), it is convenient to consider the following deterministic controlled equation in $[0,T]$,

$$y'(t) = Ay(t) + \sqrt{Q}\, u(t), \quad y(0) = x, \tag{3.12}$$

where $x \in H$ and $u \in L^2(0,T;H)$. Here $y(t)$ represents is the *state* and $u(t)$ the *control* of system (3.12). Moreover $E(u) = \int_0^T |u(s)|^2 ds$ is called the *energy* of u. The mild solution of (3.12) is given by

$$y(t;u) = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{Q}\, u(s) ds. \tag{3.13}$$

System (3.12) is said to be *null controllable* if for any $T > 0$ there exists $u \in L^2(0,T;H)$ such that $y(T;u) = 0$; in this case u is called a control *driving* the state

[10] It is easy to see that Ker $Q_t = \{0\}$ so that $Q_t^{-1/2}$ is well defined but it is not bounded in general. By saying that $\Gamma(t)$ is well defined we mean that $e^{tA}(H) \subset Q_t^{1/2}(H)$, $t > 0$. See the discussion below.

[11] Recall that ω is defined by (1.7).

y to 0 in time T. One can show, see [27], that system (3.12) is null controllable if and only if condition (3.11) is fulfilled. In this case, the minimal energy for driving y from x to 0 in time T is precisely $|\Gamma(t)x|^2$ where $\Gamma(t)$ is defined by (3.9).

When Q is continuously invertible it is easy to see that (3.12) is null controllable so that (3.11) is fulfilled. In fact in this case, fixing $T > 0$ and choosing the following control ([12]),

$$u(t) = -\frac{1}{T}\, e^{tA}Q^{-1/2}x, \quad t \in [0,T],$$

we find by (3.13) that $y(T,u) = 0$. Moreover, for the minimal energy $|\Gamma(T)x|^2$ we have, recalling (1.7), that

$$|\Gamma(T)x|^2 \le E(u) \le M^2T^{-2}\|Q^{-1/2}\|^2 \int_0^T e^{2\omega t}dt, \quad T > 0.$$

Therefore the following useful estimate holds

$$\|\Gamma(t)\| \le ct^{-1/2}e^{\omega t}, \quad t > 0, \tag{3.14}$$

for some $c > 0$. By using the Hölder inequality in (3.8), it implies (3.10).

Remark 3.4 Condition (3.11) may be fulfilled even if L_0 is not strictly elliptic. In this case we say that L_0 is ***hypoelliptic*** because when H is finite dimensional, condition (3.11) reduces precisely to the Hörmander's hypoellipticity condition for the operator L_0. In this case the conclusions of Proposition 3.3 still hold.

3.2 The infinitesimal generator of R_t

As we have noticed before, the semigroup R_t is not strongly continuous in $C_b(H)$ when $A \neq 0$. R_t belongs to a class of semigroups, called ***π-semigroups***, extensively studied in [23], see also [15]. However, a notion of ***infinitesimal generator*** of R_t can be defined as follows, see [6]. Consider the Laplace transform of R_t,

$$F(\lambda)f(x) := \int_0^{+\infty} e^{-\lambda t}R_tf(x)dt, \quad f \in C_b(H),\ \lambda > 0,\ x \in H.$$

Then, it is easy to see that $F(\lambda)$ is one–to–one and that fulfills the resolvent identity,

$$F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \lambda, \mu > 0.$$

Consequently, there exists a unique closed operator L in $C_b(H)$ such that $F(\lambda) = (\lambda - L)^{-1}$ for any $\lambda > 0$. L is clearly m–dissipative in $C_b(H)$ ([13]); it is called the infinitesimal generator of R_t.

---

[12] u is not the control of minimal energy in general.

[13] That is the resolvent set of L includes $(0, +\infty)$ and its resolvent fulfills $\|(\lambda - L)^{-1}\|_{L(C_b(H))} \le \lambda^{-1}$ for all $\lambda > 0$. If $A \neq 0$ then $D(L)$ is not dense in $C_b(H)$ and L is not the infinitesimal generator of a strongly continuous semigroup in $C_b(H)$.

Remark 3.5 By (3.10) it follows easily, taking Laplace transform, that

$$D(L) \subset C_b^1(H).$$

It is useful to define a subspace Y of $D(L)$, that plays the rôle of a core, where the expression of L coincides with L_0. Following [7], we denote by Y_L the set of all $\varphi \in C_b(H)$ such that

(i) $\varphi \in C_b^2(H)$,

(ii) $D\varphi(x) \in D(A^*)$ for all $x \in H$ and the mapping $H \to \mathbb{R}, x \to \langle x, A^* D\varphi(x)\rangle$, belongs to $C_b(H)$.

(iii) $QD^2\varphi \in C_b(H, L_1(H))$.

If $\varphi \in Y_L$ we have

$$L\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi(x)] + \langle x, A^* D\varphi(x)\rangle, \quad x \in H.$$

Moreover, the set Y_L is pointwise dense in $C_b(H)$ in the following sense
For arbitrary $\varphi \in C_b^1(H)$ there exists a sequence $\{\varphi_n\} \subset Y_L$ such that

(i) $\|\varphi_n\|_0 \le 2\|\varphi\|_0$, $n \in \mathbb{N}$,

(ii) $\varphi_n \to \varphi$ uniformly on any compact subset of H.

3.3 Elliptic equations

We are here concerned with the elliptic equation

$$\lambda\varphi(x) - \frac{1}{2}\,\mathrm{Tr}[QD^2\varphi(x)] - \langle Ax, D\varphi(x)\rangle = f(x), \quad x \in D(A), \tag{3.15}$$

where $\lambda > 0$ and $f \in C_b(H)$.

We say that φ defined by

$$\varphi = (\lambda - L)^{-1} f = \int_0^{+\infty} e^{-\lambda t} R_t f(x) dt, \; x \in H, \tag{3.16}$$

is a *generalized solution* of (3.15). φ is said to be a ***strict solution*** whenever $\varphi \in Y_L$.

As in the case of the heat equation, there is no hope to give a simple characterization of the domain of L. The situation could be better in the space $C_b^\theta(H)$, $\theta \in (0,1)$, the space of all θ-Hölder continuous and bounded real functions on H. It is easy to see that $C_b^\theta(H)$ is invariant for R_t. Let us denote by R_t^θ the restriction of R_t to $C_b^\theta(H)$, and by L^θ the part of L in $C_b^\theta(H)$:

$$L^\theta\varphi = L\varphi, \quad \forall\,\varphi \in D(L^\theta) = \{\varphi \in D(L) \cap C_b^\theta(H) : \; L\varphi \in C_b^\theta(H)\}.$$

Also the characterization of the domain of L^θ is still an open problem. However two *maximal regularity* results are known when L_0 is strictly elliptic. The first one, generalizes the classical Schauder estimates, see [11] and [5].

Proposition 3.6 *Assume that $Q^{-1} \in L(H)$. Let $f \in C_b^\theta(H)$, with $\theta \in (0,1)$ and $\lambda > 0$. Set $\varphi = (\lambda - L^\theta)^{-1} f$. Then we have $\varphi \in C_b^{2+\theta}(H)$ and there exists $N > 0$ (independent on λ and on f) such that*

$$\|\varphi\|_{C_b^{2+\theta}(H)} \le N \|f\|_{C_b^\theta(H)}. \tag{3.17}$$

The second one is the following, see [10]

Proposition 3.7 *Assume that $Q^{-1} \in L(H)$. Let $f \in C_b^\theta(H)$, with $\theta \in (0,1)$ and $\lambda > 0$. Set $\varphi = (\lambda - L^\theta)^{-1} f$. Then $D\varphi(x)$ belongs to $D((-A)^{1/2})$ for any $x \in H$ and that $(-A)^{1/2} D\varphi \in C_b^\theta(H)$.*

Both results can be used to study more general equations with variable coefficients.

4 The case when F is nonlinear

Here we still assume that the operator

$$Q_t x = \int_0^t e^{sA} Q e^{sA^*} x ds, \quad x \in H,$$

is of trace class for any $t > 0$, so that the Ornstein–Uhlenbeck semigroup R_t in $C_b(H)$ is well defined. We still denote by L its infinitesimal generator. We are given in addition a uniformly continuous and bounded function $F\colon H \to H$.

We shall consider the linear operator

$$N_0\varphi(x) = L\varphi(x) + \langle F(x), D\varphi(x)\rangle, \quad \varphi \in D(L) \cap C_b^1(H),\ x \in H. \tag{4.1}$$

In §4.1 and §4.2 we shall assume in addition that $Q = I$ so that N_0 is strictly elliptic. In this case we know by Remark 3.5 that $D(L) \subset C_b^1(H)$, so that the term $\langle F(x), D\varphi\rangle$ is well defined for any $\varphi \in D(L)$. In this case we say that the operator N_0 is a perturbation of L. Then it is not difficult to solve the parabolic equation concerning N_0 by a fixed point argument, see §4.1. Moreover, one can show, see §4.2, that the operator N_0 is m–dissipative ([14]), that is it resolvent set includes $(0, +\infty)$ and the following estimate for the resolvent holds

$$\|(\lambda - N)^{-1}\varphi\|_0 \le \frac{1}{\lambda} \|\varphi\|_0, \quad \varphi \in C_b(H).$$

More difficult is the situation, treated in §4.3, when the operator Q is general, since in this case we do not know whether $D(L)$ is included in $C_b^1(H)$ or not. We can only show that N_0 is *essentially m–dissipative* in $C_b(H)$, that is N_0 is dissipative and its closure (a-priori multi-valued) is m–dissipative.

[14] N does not generate a strongly continuous semigroup because its domain $D(L)$ is not dense in $C_b(H)$.

4.1 Parabolic equations when $Q = I$

We assume here that $Q = I$ and consider the problem,

$$\begin{cases} D_t u(t,x) &= \frac{1}{2}\,\mathrm{Tr}\,[QD^2u(t,x)] + \langle Ax + F(x), Du(t,x)\rangle, \quad t \geq 0,\ x \in D(A), \\ u(0,x) &= \varphi(x), \quad x \in H, \end{cases} \tag{4.2}$$

where $\varphi \in C_b(H)$.

We shall write problem (4.2) in the following more abstract form

$$\begin{cases} D_t u(t,\cdot) = Lu(t,\cdot) + \langle F(\cdot), Du(t,\cdot)\rangle, \quad t \geq 0,\ x \in H, \\ u(0,\cdot) = \varphi. \end{cases} \tag{4.3}$$

We say that u is a *mild solution* of (4.3) if it fulfills the following integral equation

$$u(t,\cdot) = R_t\varphi + \int_0^t R_{t-s}(\langle F(\cdot), Du(s,\cdot)\rangle)ds, \quad t \geq 0. \tag{4.4}$$

Since, by Proposition 3.3, R_t maps $C_b(H)$ into $C_b^1(H)$, it is natural to try to solve (4.4) by a fixed point argument in a space of differentiable functions. So, let us consider the following space Z_T consisting of all functions $u : [0,T] \times H \to \mathbb{R}$ such that

(i) u is continuous in $(0,T] \times H$.

(ii) $u(t,\cdot) \in C_b^1(H)$ for all $t > 0$.

(iii) $\sup_{t\in(0,T]} t^{1/2}\|u(t,\cdot)\|_1 < +\infty$.

We notice that condition (iii) is ispired by estimate (3.10). It is easy to check that Z_T, endowed with the norm

$$\|u\|_{Z_T} := \|u\|_0 + \sup_{t\in(0,T]} t^{1/2}\|u(t,\cdot)\|_1,$$

is a Banach space.

Now, the proof of the following result is a straightforward application of the contractions principle, see [14, Proposition 6.5.1] for details.

Proposition 4.1 *For any $\varphi \in C_b(H)$ there is a unique mild solution of equation* (4.2).

4.2 m–dissipativity of N_0

Here we make the same assumptions as in §4.1 and consider the operator N_0 with domain $D(L)$.

Proposition 4.2 *N is m-dissipative in $C_b(H)$.*

Sketch of the proof.

Step 1. There exists $\lambda_0 > 0$ such that $(\lambda_0, +\infty)$ belongs to the resolvent set of N_0.

Let us consider in fact the equation

$$\lambda\varphi - N_0\varphi = \lambda\varphi - L\varphi - \langle F(x), D\varphi\rangle = f, \tag{4.5}$$

where $\lambda > 0$ and $f \in C_b(H)$ are given.

Setting $\psi = \lambda\varphi - L\varphi$, equation (4.5) becomes

$$\psi - T_\lambda\psi = f, \tag{4.6}$$

where T_λ is defined by

$$T_\lambda\psi(x) = \langle F(x), DR(\lambda, L)\psi(x)\rangle, \quad \psi \in C_b(H), \ x \in H. \tag{4.7}$$

Taking the Laplace transform of (3.10), we see that

$$\|T_\lambda\psi\|_0 \le c\sqrt{\frac{\pi}{\lambda - \omega}}\ \|F\|_0\ \|\psi\|_0.$$

Therefore, if $\lambda > \lambda_0$ where

$$\lambda_0 := \omega + \pi c^2\|F\|_0^2, \tag{4.8}$$

T_λ is a contraction in $C_b(H)$ and so, equation (4.6) has a unique solution φ. Consequently, equation (4.5) has a unique solution too $\varphi \in D(L)$ given by

$$\varphi = (\lambda - L)^{-1}(1 - T_\lambda)^{-1}f$$

and Step 1 follows.

It remains to show that N_0 is dissipative ([15]).

Step 2 If F is in addition Lipschitz continuous, N_0 is dissipative.

It is convenient to introduce for any $\varepsilon > 0$ an operator N_ε approximating N_0,

$$N_\varepsilon\varphi = L\varphi + \mathcal{F}_\varepsilon\varphi, \quad \varphi \in D(L),$$

where

$$\mathcal{F}_\varepsilon\varphi(x) = \frac{1}{\varepsilon}\left(\varphi(\eta(\varepsilon, x)) - \varphi(x)\right), \tag{4.9}$$

and η is the solution to the initial value problem

$$\eta_t(t, x) = F(\eta(t, x)), \quad \eta(0, x) = x \in H. \tag{4.10}$$

[15] In fact, it is well known that a dissipative operator is m-dissipative if and only if its resolvent set contains a positive number.

Clearly for any $\varphi \in C_b^1(H)$ we have

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon \varphi = \langle F, D\varphi \rangle \quad \text{in } C_b(H). \tag{4.11}$$

Now, given $\lambda > 0$ and $f \in C_b(H)$, we consider the equation

$$\lambda \varphi_\varepsilon - L\varphi_\varepsilon - \mathcal{F}_\varepsilon \varphi_\varepsilon = f, \tag{4.12}$$

which can be solved as before by a standard fixed point argument depending on the parameter ε. We have clearly

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = \varphi \quad \text{in } C_b(H). \tag{4.13}$$

Now by (4.12) we find that

$$\left(\lambda + \frac{1}{\varepsilon}\right) \varphi_\varepsilon - L\varphi_\varepsilon = f + \frac{1}{\varepsilon}\, \varphi(\eta(h,x)), \tag{4.14}$$

and so, by the dissipativity of L, it follows that

$$\|\varphi_\varepsilon\|_0 \le \frac{1}{\lambda + \frac{1}{\varepsilon}} \left(\|f\|_0 + \frac{1}{\varepsilon}\, \|\varphi_\varepsilon\|_0\right),$$

which implies $\|\varphi_\varepsilon\|_0 \le \frac{1}{\lambda}\, \|f\|_0$. Consequently, letting ε tend to 0 yields

$$\|\varphi\|_0 \le \frac{1}{\lambda}\, \|f\|_0.$$

Therefore N is dissipative as required.

Step 3. Conclusion.

By [26] there exists a sequence $\{F_n\}$ of Lipschitz bounded functions from H into H which converges to F in $C_b(H, H)$. Set

$$N_n\varphi = L\varphi + \langle F_n(x), D\varphi \rangle, \quad \varphi \in D(L),\ n \in \mathbb{N}.$$

Given $\lambda \ge \lambda_0$ (defined by (4.8)) and $f \in C_b(H)$, consider the equation

$$\lambda \varphi_n - N_n \varphi_n = f, \tag{4.15}$$

which can be solved as before by successive approximations. Due to the uniformity in n of the estimates, we have that

$$\lim_{n \to \infty} \varphi_n = (\lambda - N)^{-1} f \quad \text{in } C_b(H; H).$$

Moreover, by Step 2, N_n is dissipative so that $\|\varphi_n\|_0 \le \frac{1}{\lambda}\, \|f\|_0$. As $n \to \infty$ we find

$$\|\varphi\|_0 \le \frac{1}{\lambda}\, \|f\|_0, \quad \forall x \in H,$$

and consequently N_0 is m-dissipative. ■

4.3 Essential m–dissipativity of N_0

We are again concerned with the linear operator N_0 defined by (4.1), where now we assume that $F \in C_b^2(H;H)$. Let us write N_0 in the following form

$$N_0\varphi = L\varphi + \mathcal{F}\varphi, \quad \varphi \in D(L) \cap C_b^1(H),$$

where $\mathcal{F}$ is the linear bounded operator

$$\mathcal{F}\varphi = \langle F(\cdot), D\varphi\rangle, \quad \varphi \in C_b^1(H).$$

Notice that $\mathcal{F}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $e^{t\mathcal{F}}$ in $C_b(H)$ given by

$$e^{t\mathcal{F}}\varphi(x) = \varphi(\eta(t,x)), \quad t \geq 0,\ x \in H,\ \varphi \in C_b(H),$$

where η is the solution to the initial value problem (4.10). Therefore $\mathcal{F}$ is m–dissipative in $C_b(H)$.

It is useful to consider the sequence $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ approximating $\mathcal{F}$ defined by (4.9). Clearly, F_ε is also m–dissipative. The following result is proved in [9].

Theorem 4.3 *N_0 is dissipative and its closure $\overline{N_0}$ is m–dissipative in $C_b(H)$.*

Sketch of the proof. We claim that N_0 is dissipative in $C_b(H)$ and that the range of $\lambda - N_0$ is dense in $C_b(H)$ for λ large. This will imply the conclusion by the Lumer-Phillips theorem, see [20]. To prove the claim, we introduce the following approximating operators

$$N_\varepsilon\varphi = L\varphi + \mathcal{F}_\varepsilon\varphi, \quad \varphi \in D(L),\ \varepsilon > 0.$$

We know from Step 2 of Proposition 4.2 that N_ε is m–dissipative. This easily implies that N_0 is dissipative.

It remains to prove that that the range of $\lambda - N_0$ is dense in $C_b(H)$ for λ large. To this purpose, fix $f \in C_b^{1,1}(H)$, $\lambda > 0$, and consider the solution $\varphi_\varepsilon \in D(L) \cap C_b^1(H)$ of the equation

$$\lambda\varphi_\varepsilon - L\varphi_\varepsilon - \mathcal{F}_\varepsilon(\varphi_\varepsilon) = f, \tag{4.16}$$

which is equivalent to

$$\lambda\varphi_\varepsilon - N_0\varphi_\varepsilon = f + \mathcal{F}_\varepsilon(\varphi_\varepsilon) - \mathcal{F}(\varphi_\varepsilon). \tag{4.17}$$

We claim now that

$$\lim_{\varepsilon\to 0}[\mathcal{F}_\varepsilon(\varphi_\varepsilon) - \mathcal{F}(\varphi_\varepsilon)] = 0 \quad \text{in } C_b(H). \tag{4.18}$$

This follows from the estimates, see [9, equations 2.11 and 3.2],

$$\|\mathcal{F}\varphi_\varepsilon - \mathcal{F}_\varepsilon\varphi_\varepsilon\|_0 \leq \frac{\varepsilon}{2}\left(\|F\|_0^2\|\varphi_\varepsilon\|_{1,1} + \|F\|_0\|F\|_1\|\varphi_\varepsilon\|_1\right),$$

and
$$\|\varphi_\varepsilon\|_{1,1} \le c_1,$$
for a suitable constant $c_1 > 0$. From (4.18) it follows that
$$\lim_{\varepsilon\to 0}[\lambda\varphi_\varepsilon - N_0\varphi_\varepsilon] = f \quad \text{in } C_b(H).$$

Therefore the closure of the range of $\lambda - N_0$ includes $C_b^{1,1}(H)$ which is dense in $C_b(H)$ by [19] and the result follows. ■

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